# Project.

Let $K$ be an imaginary quadratic field, and $p$ a rational prime which splits in $K$ into two distinct primes $\wp, \wp^*$. By class field theory, there is a unique $\mathbb{Z}_p$-extension $K_\infty/K$ which is unramified outside of $\wp$. Assume now that $F$ is an arbitrary finite extension of $K$. We call

$$F_\infty = FK_\infty$$

the "split prime" $\mathbb{Z}_p$-extension of $F$. It seems probable that this split prime $\mathbb{Z}_p$-extension $F_\infty/F$ has many properties in close analogy with those of the cyclotomic $\mathbb{Z}_p$-extension of $F$. The aim of the project is to discuss several of these analogies, and establish a few rather limited theoretical and numerical examples in support of them.

## Part I

## Analogues of the Leopoldt and weak Leopoldt conjectures.

We assume from now on that $K$ is an imaginary quadratic in which $p$ splits into $\wp, \wp^*$, and that $F$ is an arbitrary finite extension of $K$. For each prime $v$ of $F$ lying above $\wp$, write $U_v$ for the group of local units in the completion of $F$ at $v$ which are $\equiv 1 \bmod v$. Put $U_F = \prod_{v|\wp} U_v$. Thus $U_F$ is a $\mathbb{Z}_p$-module of rank equal to $r_2$, where $r_2$ denotes the number of complex primes of $F$ $(= [F:K])$. Let $E_F$ be the group of all global units of $F$ which are $\equiv 1 \bmod v$ for all $v|\wp$. By Dirichlet's theorem, $E_F$ has $\mathbb{Z}$-rank equal to $r_2 - 1$. Now we have the obvious embedding of $E_F$ into $U_F$, and we define $\overline{E}_F$ to be the closure of the image in $U_F$ under the $\wp$-adic topology (equivalently, $\overline{E}_F$ is the $\mathbb{Z}_p$-submodule of $U_F$

which is generated by the image of $E_F$). Thus $\overline{E}_F$ must have $\mathbb{Z}_p$-rank equal to $r_2 - 1 - \delta_{F,\wp}$ for some integer $\delta_{F,\wp} \geq 0$.

<u>$\wp$-adic Leopoldt conjecture</u>. $\delta_{F,\wp} = 0$.

Again global class field theory gives a Galois-theoretic interpretation of this conjecture. Let $L$ be the $p$-Hilbert class field of $F$, and let $M$ be the maximal abelian $p$-extension of $F$, which is unramified outside the set of primes of $F$ lying above $\wp$. Then the Artin map induces an isomorphism

$$U_F / \overline{E}_F \xrightarrow{\sim} \mathrm{Gal}(M/L),$$

whence we obtain:–

<u>Theorem</u> 1.1. Let $M$ be the maximal abelian $p$-extension of $F$ which is unramified outside the primes of $F$ lying above $\wp$. Then $\mathrm{Gal}(M/F)$ is a finitely generated $\mathbb{Z}_p$-module of $\mathbb{Z}_p$-rank equal to $1 + \delta_{F,\wp}$.

<u>Corollary 1.2</u>. $\delta_{F,\wp} = 0$ if and only if $\mathrm{Gal}(M/F_\infty)$ is finite.

Let $\sigma_1, \dots, \sigma_{r_2}$ be the embeddings of $F$ into $\overline{\mathbb{Q}}_p$ extending the embedding of $K$ into $\mathbb{Q}_p$ given by $\wp$. Let $\varepsilon_1, \dots, \varepsilon_{r_2-1}$ be a $\mathbb{Z}$-basis of $E_F$ modulo torsion. Note that the series $\log x$ converges on all principal units of $\overline{\mathbb{Q}}_p$. Define

$$R_\wp(F) = \det\left(\log \sigma_i(\varepsilon_j)\right)_{i,j=1,\dots,r_2-1}.$$

<u>Ex</u> 1.1. Prove that $\delta_{F,\wp} \neq 0$ if and only if $R_\wp(F) \neq 0$. If $F$ is an abelian extension of $K$, use Baker's theorem that $\log \varepsilon_1, \dots, \log \varepsilon_{r_2-1}$ are linearly independent over the field of algebraic numbers to show that $R_\wp(F) \neq 0$.

Exc 1.2. With the help of SAGE or MAGMA, one can often check numerically that $R_{\wp}(F) \neq 0$ even when $F$ is not an abelian extension of $K$. Here is one example. Take $K = \mathbb{Q}(i)$, $p = 5$, and $\wp = (1-2i)\mathbb{Z}[i]$. Let $w = \frac{1-\sqrt{5}}{2}$, and take

$$F = K(w, \beta^{1/4}), \text{ where } \beta = w(1-2i)^3.$$

Show that $F = \mathbb{Q}(\delta)$, where $\delta$ is a root of $x^8 - 4x^6 + 9x^4 + 10x^2 + 5 = 0$. Using one of the above programmes, find the group of global units of $F$, and check that $\mathrm{ord}_{\wp}(R_{\wp}(F)) = 3/2$.

We now turn to the weak $\wp$-adic Leopoldt conjecture for $F_\infty/F$. For each $n \geqslant 0$, let $F_n$ be the unique extension of $F$ contained in $F_\infty$ with $[F_n : F] = p^n$. Let $\delta_{F_n,\wp}$ denote the $\wp$-adic default of Leopoldt for $F_n$.

Weak $\wp$-adic Leopoldt conjecture for $F_\infty/F$.

$$\delta_{F_n,\wp} \text{ is bounded as } n \to \infty.$$

Of course, the analogue of this statement for the cyclotomic $\mathbb{Z}_p$-extension of $F$ was proven by Iwasawa, but unfortunately his proof does not seem to extend to $F_\infty/F$.

There is an equivalent formulation of this conjecture purely in terms of an Iwasawa module. Let $M(F_\infty)$ be the maximal abelian $p$-extension of $F_\infty$, which is unramified outside the set of primes of $F_\infty$ lying above $\wp$, and put

$$X(F_\infty) = \mathrm{Gal}(M(F_\infty)/F_\infty).$$

Clearly $M(F_\infty)$ is Galois over $F$, and so $\Gamma = \mathrm{Gal}(F_\infty/F)$ acts on $X(F_\infty)$ in the usual fashion. It follows that $X(F_\infty)$ is a module over the Iwasawa algebra $\Lambda(\Gamma)$ of $\Gamma$, and it is easily seen to be finitely generated over $\Lambda(\Gamma)$. Moreover, we have

$$(X(F_\infty))_{\Gamma_n} = \mathrm{Gal}(M_n/F_\infty),$$

where $M_n$ is the maximal abelian $p$-extension of $F_n$, which is unramified outside the primes of $F_n$ lying above $\mathfrak{p}$.

<u>Theorem 1.3</u>. $X(F_\infty)$ is $\Lambda(\Gamma)$-torsion if and only if $\delta_{F_n,\mathfrak{p}}$ is bounded as $n \to \infty$.

<u>Corollary 1.4</u> If $\delta_{F,\mathfrak{p}} = 0$, then $\delta_{F_n,\mathfrak{p}}$ is bounded as $n \to \infty$.

Of course, one can use Corollary 1.4 to prove the weak $\mathfrak{p}$-adic Leopoldt conjecture in numerical examples (e.g. in the example of Ex 1.2).

There are two other important aspects of the weak $\mathfrak{p}$-adic Leopoldt conjecture for $F_\infty/F$ which we mention briefly. Firstly, there is an exact formula for $\#(\mathrm{Gal}(M/F_\infty))$ when $R_\mathfrak{p}(F) \neq 0$, which is a first hint that there may be a "main conjecture" for $X(F_\infty)$. Let $h(F)$ be the class number of $F$, $w(F)$ the number of roots of unity in $F$, and $\Delta(F/K)$ any generator of the discriminant ideal of $F/K$. If $v$ is a finite place of $F$, $Nv$ will denote the cardinality of the residue field of $v$.

<u>Ex 1.3</u> (see [CW1]). Assume that $R_\mathfrak{p}(F) \neq 0$. Then

$$[M : F_\infty] = \left| \frac{p^{e(F)+1} h(F) R_\mathfrak{p}(F)}{w(F)\sqrt{\Delta(F/K)}} \prod_{v \mid \mathfrak{p}} \left(1 - \frac{1}{Nv}\right) \right|_p^{-1},$$

where the integer $e(F)$ is defined by $F \cap K_\infty = K_{e(F)}$.

Here the $p$-adic valuation on $\bar{\mathbb{Q}}_p$ is normalized by $\left|\frac{1}{p}\right|_p = p$.

Secondly, the weak Leopoldt conjecture for $F_\infty/F$ is closely related to the Iwasawa theory for $F_\infty/F$ of elliptic curves with complex multiplication by the full ring of integers of $K$ (see [C2]).