

Project.

Let K be an imaginary quadratic field, and p a rational prime which splits in K into two distinct primes $\mathfrak{p}, \mathfrak{p}^*$. By class field theory, there is a unique \mathbb{Z}_p -extension K_∞/K which is unramified outside of \mathfrak{p} . Assume now that F is an arbitrary finite extension of K . We call

$$F_\infty = FK_\infty$$

the "split prime" \mathbb{Z}_p -extension of F . It seems probable that this split prime \mathbb{Z}_p -extension F_∞/F has many properties in close analogy with those of the cyclotomic \mathbb{Z}_p -extension of F . The aim of the project is to discuss several of these analogies, and establish a few rather limited theoretical and numerical examples in support of them.

Part IAnalogues of the Leopoldt and weak Leopoldt conjectures.

We assume from now on that K is an imaginary quadratic in which p splits into $\mathfrak{p}, \mathfrak{p}^*$, and that F is an arbitrary finite extension of K . For each prime v of F lying above \mathfrak{p} , write U_v for the group of local units in the completion of F at v which are $\equiv 1 \pmod{v}$. Put $U_F = \prod U_v$. Thus U_F is a \mathbb{Z}_p -module of rank equal to $\tau_2^{v|p}$, where τ_2 denotes the number of complex primes of F ($= [F:K]$). Let E_F be the group of all global units of F which are $\equiv 1 \pmod{v}$ for all $v|\mathfrak{p}$. By Dirichlet's theorem, E_F has \mathbb{Z} -rank equal to $\tau_2 - 1$. Now we have the obvious embedding of E_F into U_F , and we define \overline{E}_F to be the closure of the image in U_F under the p -adic topology (equivalently, \overline{E}_F is the \mathbb{Z}_p -submodule of U_F

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which is generated by the image of E_F). Thus \overline{E}_F must have \mathbb{Z}_p -rank equal to $r_2 - 1 - \delta_{F, p}$ for some integer $\delta_{F, p} \geq 0$.

p -adic Leopoldt conjecture. $\delta_{F, p} = 0$.

Again global class field theory gives a Galois-theoretic interpretation of this conjecture. Let L be the p -Hilbert class field of F , and let M be the maximal abelian p -extension of F , which is unramified outside the set of primes of F lying above p . Then the Artin map induces an isomorphism

$$U_F / \overline{E}_F \xrightarrow{\sim} \text{Gal}(M/L),$$

whence we obtain: -

Theorem 1.1. Let M be the maximal abelian p -extension of F which is unramified outside the primes of F lying above p . Then $\text{Gal}(M/F)$ is a finitely generated \mathbb{Z}_p -module of \mathbb{Z}_p -rank equal to $1 + \delta_{F, p}$.

Corollary 1.2. $\delta_{F, p} = 0$ if and only if $\text{Gal}(M/F_\infty)$ is finite.

Let $\sigma_1, \dots, \sigma_{r_2}$ be the embeddings of F into $\overline{\mathbb{Q}}_p$ extending the embedding of K into \mathbb{Q}_p given by \mathfrak{p} . Let $\varepsilon_1, \dots, \varepsilon_{r_2-1}$ be a \mathbb{Z} -basis of E_F modulo torsion. Note that the series $\log x$ converges on all principal units of $\overline{\mathbb{Q}}_p$.

Define

$$R_{\mathfrak{p}}(F) = \det \left(\log \sigma_i(\varepsilon_j) \right)_{i, j=1, \dots, r_2-1}.$$

Exc 1.1. Prove that $\delta_{F, p} \neq 0$ if and only if $R_{\mathfrak{p}}(F) \neq 0$.

If F is an abelian extension of K , use Baker's theorem that $\log \varepsilon_1, \dots, \log \varepsilon_{r_2-1}$ are linearly independent over the field of algebraic numbers to show that $R_{\mathfrak{p}}(F) \neq 0$.

Exc 1.2. With the help of SAGE or MAGMA, one can often check numerically that $R_{\mathfrak{p}}(F) \neq 0$ even when F is not an abelian extension of K . Here is one example. Take $K = \mathbb{Q}(i)$, $\mathfrak{p} = 5$, and $\mathfrak{p} = (1-2i)\mathbb{Z}[i]$. Let $w = \frac{1-\sqrt{5}}{2}$, and take

$$F = K(w, \beta^{1/4}), \text{ where } \beta = w(1-2i)^3.$$

Show that $F = \mathbb{Q}(\delta)$, where δ is a root of $x^8 - 4x^6 + 9x^4 + 10x^2 + 5 = 0$. Using one of the above programmes, find the group of global units of F , and check that $\text{ord}_{\mathfrak{p}}(R_{\mathfrak{p}}(F)) = 3/2$.

We now turn to the weak \mathfrak{p} -adic Leopoldt conjecture for F_{∞}/F . For each $n \geq 0$, let F_n be the unique extension of F contained in F_{∞} with $[F_n : F] = \mathfrak{p}^n$. Let $\delta_{F_n, \mathfrak{p}}$ denote the \mathfrak{p} -adic defect of Leopoldt for F_n .

Weak \mathfrak{p} -adic Leopoldt conjecture for F_{∞}/F .

$\delta_{F_n, \mathfrak{p}}$ is bounded as $n \rightarrow \infty$.

Of course, the analogue of this statement for the cyclotomic $\mathbb{Z}_{\mathfrak{p}}$ -extension of F was proven by Iwasawa, but unfortunately his proof does not seem to extend to F_{∞}/F .

There is an equivalent formulation of this conjecture purely in terms of an Iwasawa module. Let $M(F_{\infty})$ be the maximal abelian \mathfrak{p} -extension of F_{∞} , which is unramified outside the set of primes of F_{∞} lying above \mathfrak{p} , and put

$$X(F_{\infty}) = \text{Gal}(M(F_{\infty})/F_{\infty}).$$

Clearly $M(F_{\infty})$ is Galois over F , and so $\Gamma = \text{Gal}(F_{\infty}/F)$ acts on $X(F_{\infty})$ in the usual fashion. It follows that $X(F_{\infty})$ is a module over the Iwasawa algebra $\Lambda(\Gamma)$ of Γ , and it is easily seen to be finitely generated over $\Lambda(\Gamma)$. Moreover, we have

$$\left(X(F_{\infty}) \right)_{\Gamma_n} = \text{Gal}(M_n/F_{\infty}),$$

where M_n is the maximal abelian p -extension of F_n , which is unramified outside the primes of F_n lying above \mathfrak{p}_0 .

Theorem 1.3. $X(F_\infty)$ is $\Lambda(\Gamma)$ -torsion if and only if $\delta_{F_n, \mathfrak{p}_0}$ is bounded as $n \rightarrow \infty$.

Corollary 1.4 If $\delta_{F, \mathfrak{p}_0} = 0$, then $\delta_{F_n, \mathfrak{p}_0}$ is bounded as $n \rightarrow \infty$.

Of course, one can use Corollary 1.4 to prove the weak \mathfrak{p}_0 -adic Leopoldt conjecture in numerical examples (e.g. in the example of Ex 1.2).

There are two other important aspects of the weak \mathfrak{p}_0 -adic Leopoldt conjecture for F_∞/F which we mention briefly. Firstly, there is an exact formula for $\#(\text{Gal}(M/F_\infty))$ when $R_{\mathfrak{p}_0}(F) \neq 0$, which is a first hint that there may be a "main conjecture" for $X(F_\infty)$. Let $h(F)$ be the class number of F , $w(F)$ the number of roots of unity in F , and $\Delta(F/K)$ any generator of the discriminant ideal of F/K . If v is a finite place of F , Nv will denote the cardinality of the residue field of v .

Ex 1.3 (see [CW1]). Assume that $R_{\mathfrak{p}_0}(F) \neq 0$. Then

$$[M : F_\infty] = \left| \frac{p^{e(F)+1} h(F) R_{\mathfrak{p}_0}(F)}{w(F) \sqrt{\Delta(F/K)}} \prod_{v|\mathfrak{p}_0} \left(1 - \frac{1}{Nv}\right) \right|_p^{-1},$$

where the integer $e(F)$ is defined by $F \cap K_\infty = K_{e(F)}$.

Here the p -adic valuation on \mathbb{Q}_p is normalized by $|\frac{1}{p}|_p = \frac{1}{p}$.

Secondly, the weak Leopoldt conjecture for F_∞/F is closely related to the Iwasawa theory for F_∞/F of elliptic curves with complex multiplication by the full ring of integers of K (see [C2]).

Exc 1.4. In the specific numerical example discussed in Example 1.2, use the formula of Example 1.3 to prove that $\chi(F_\infty) = 0$.

Part II

Analogue of Iwasawa's $\mu = 0$ conjecture.

We first recall Iwasawa's $\mu = 0$ conjecture. Let F be any finite extension of \mathbb{Q} , p any prime number, and F_∞^{cyc}/F the cyclotomic \mathbb{Z}_p -extension of F .

Iwasawa's $\mu = 0$ conjecture. Let L_∞ be the maximal abelian p -extension of F_∞^{cyc} , which is unramified everywhere. Then $\text{Gal}(L_\infty/F_\infty^{\text{cyc}})$ is a finitely generated \mathbb{Z}_p -module.

Iwasawa (see [IW2]) has given examples of fields F and primes p for which the analogue of this conjecture is false for certain non-cyclotomic \mathbb{Z}_p -extensions of F . The best result to date in support of Iwasawa's conjecture is the following:—

Theorem (Ferrero - Washington, Sinnott). Iwasawa's $\mu = 0$ conjecture is valid for all finite abelian extensions F of \mathbb{Q} , and all primes p .

Now assume F is a finite extension of an imaginary quadratic field K , and F_∞/F is the split prime \mathbb{Z}_p -extension of K . Let \mathfrak{p} be the degree 1 prime of K giving rise to F_∞/F .

Split prime analogue of Iwasawa's $\mu = 0$ conjecture.

Let F_∞/F be the split prime \mathbb{Z}_p -extension, and let $M(F_\infty)$ be the maximal abelian p -extension of F_∞ which is unramified outside the primes of F_∞ lying above \mathfrak{p} . Then $\chi(F_\infty) = \text{Gal}(M(F_\infty)/F_\infty)$ is a finitely generated \mathbb{Z}_p -module.

Ex 1.5. Let J/F be a finite Galois extension, whose Galois group is cyclic of order p . Let J_∞/J and F_∞/F be the split prime \mathbb{Z}_p -extensions, so that $J_\infty = F_\infty J$. If $X(F_\infty)$ is a finitely generated \mathbb{Z}_p -module, prove that $X(J_\infty)$ is a finitely generated \mathbb{Z}_p -module.

There is a considerable body of literature showing how Sinnott's beautiful proof, by analytic means, of the Iwasawa $\mu = 0$ conjecture for the cyclotomic \mathbb{Z}_p -extension of a finite abelian extension F of \mathbb{Q} can be generalized to the split prime \mathbb{Z}_p -extension $\mu = 0$ conjecture given above for F a finite abelian extension of K . However, it must be said that the analytic arguments needed in the split prime case are not fully worked out in the existing literature, especially for the primes $p = 2, 3$. There would be real interest in writing a fully detailed and comprehensible proof showing that Sinnott's method proves in complete generality the above $\mu = 0$ conjecture for the split prime \mathbb{Z}_p -extension of any finite abelian extension F of K .

REFERENCES FOR THE PROJECT

REFERENCES

- [1] J. Coates, *Infinite descent on elliptic curves with complex multiplication*, Arithmetic and geometry, Vol. I, 107-137, Progr. Math. 35, Birkhauser Boston, Boston, MA, 1983.
- [2] J.Coates; A.Wiles, *Kummer's criterion for Hurwitz numbers*, Algebraic number theory (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976), pp. 9-23. Japan Soc. Promotion Sci. Tokyo, 1977.
- [3] J.Choi, Y.Kezuka, Y.Li *Analogues of Iwasawa's $\mu = 0$ conjecture and weak Leopoldt theorem for certain non-cyclotomic \mathbb{Z}_2 -extensions*, Arxiv:1711.01697v1
- [4] E.de Shalit, *The Iwasawa theory of elliptic curves with complex multiplication*, Perspect. Math. Vol.3 (1987).
- [5] R. Gillard, *Fonctions L p -adiques des des corps quadratiques imaginaires et de leurs extensions abéliennes*, J. Reine Angew. Math. 358 (1985), 76–91.
- [6] R. Gillard, *Transformation de Mellin-Leopoldt des fonctions elliptiques*, J. Number Theory 25 (1987), no. 3, 379–393.
- [7] H.Oukhaba; S.Vigui, *On the μ -invariant of Katz p -adic L -functions attached to imaginary quadratic fields*, Forum Math. 28 (2016), no. 3, 507-525
- [8] L. Schneps, *On the μ -invariant of p -adic L -functions attached to elliptic curves with complex multiplication*, J. Number Theory 25 (1987), no. 1, 20–33.