Problems on adic spaces and perfectoid spaces
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1 Topological rings and valuations

Notation For a valuation \( v: A \to \Gamma \cup \{ 0 \} \) on a ring \( A \), we write \( \Gamma_v \) for the subgroup of \( \Gamma \) generated by \( \{ v(a) \mid a \in A \} \setminus \{ 0 \} \), and call it the value group of \( v \). A subgroup \( H \) of \( \Gamma_v \) is said to be convex if \( a_1, a_2, a_3 \in \Gamma_v \) with \( a_1 \leq a_2 \leq a_3 \) and \( a_1, a_3 \in H \) implies \( a_2 \in H \). The height of \( v \) means the supremum of the length \( r \) of a chain of convex subgroups \( \{ 1 \} = H_0 \subset H_1 \subset \cdots \subset H_r = \Gamma_v \). We write \( \text{supp} v \) for the prime ideal \( v^{-1}(0) \), and call it the support of \( v \).

1.1 Let \( A \) be a ring and \( v: A \to \Gamma \cup \{ 0 \} \) be a valuation on \( A \). Prove that the height of \( v \) is 1 if and only if \( \Gamma_v \neq 1 \) and there exists an order-preserving injective group homomorphism \( \Gamma_v \to \mathbb{R}_{>0} \).

1.2 Let \( V \) be a valuation ring with valuation \( v: V \to \Gamma \cup \{ 0 \} \), and \( K = \text{Frac} V \) its fraction field. Consider the valuation topology on \( K \), i.e., the topology generated by the subsets \( \{ x \in K \mid v(x) \leq a \} \) with \( a \in \Gamma_v \). Prove that the following are equivalent:

\( - \) \( K \) is a Tate ring (i.e., a Huber ring which has a topologically nilpotent unit).

\( - \) \( V \) has a prime ideal of height 1.

In [Hub96, Definition 1.1.4], such a valuation ring is said to be microbial.

1.3 Let \( A \) be a ring, and \( \text{Spv} A \) the set of equivalence classes of valuations on \( A \). Consider the topology of \( \text{Spv} A \) generated by the subsets \( \{ v \in \text{Spv} A \mid v(a) \leq v(b) \neq 0 \} \) with \( a, b \in A \). Prove that \( \text{Spv} A \) is quasi-compact.

Hint: consider the map \( \phi: \text{Spv} A \to \prod_{A \times A} \{ 0, 1 \} = \text{Map}(A \times A, \{ 0, 1 \}) \) defined by

\[
\phi(v)(a, b) = \begin{cases} 
1 & \text{if } v(a) \leq v(b), \\
0 & \text{if } v(a) > v(b).
\end{cases}
\]

Observe that \( \text{Im} \phi \) is a closed subset of \( \prod_{A \times A} \{ 0, 1 \} \) with respect to the product topology of the discrete topology on \( \{ 0, 1 \} \).

1.4 Let the notation be as in 1.3. Let \( v: A \to \Gamma \cup \{ 0 \} \) be a valuation on \( A \).

(i) For a convex subgroup \( H \subset \Gamma_v \) containing \( \{ v(a) \mid a \in A, v(a) \geq 1 \} \), let \( v|_H: A \to H \cup \{ 0 \} \) be a map defined by

\[
a \mapsto \begin{cases} 
v(a) & \text{if } v(a) \in H, \\
0 & \text{if } v(a) \notin H.
\end{cases}
\]
Prove that $v|_H$ is a valuation of $A$, and it is a specialization of $v$ in $\text{Spv} A$. Such a specialization of $v$ is called a primary specialization.

(ii) For a convex subgroup $H \subset \Gamma_v$, let $v/H : A \to \Gamma_v/H \cup \{0\}$ be a map defined by

$$a \mapsto \begin{cases} v(a) \mod H & \text{if } v(a) \neq 0, \\
0 & \text{if } v(a) = 0. \end{cases}$$

Prove that $v/H$ is a valuation of $A$, and $v$ is a specialization of $v/H$ (i.e., $v$ lies in the closure of $v/H$) in $\text{Spv} A$. A valuation $v \in \text{Spv} A$ is said to be a secondary specialization of $w \in \text{Spv} A$ if there exists a convex subgroup $H$ of $\Gamma_v$ such that $w = v/H$.

(iii) Let $w \in \text{Spv} A$ be a specialization of $v \in \text{Spv} A$ such that $\text{supp } v = \text{supp } w$. Observe that $w$ is a secondary specialization of $v$. (In fact, if $A$ is not necessarily Tate, $w$ is known to be a primary specialization of a secondary specialization.)

(iv) We put $k_v = \text{Frac}(A/\text{supp } v)$. The valuation $v$ on $A$ induces that on $k_v$, by which $k_v$ becomes a valuation field. We write $k_v^\sim$ for the residue field of $k_v$. Construct a natural continuous map $\text{Spv} k_v^\sim \to \text{Spv} A$ which sends the trivial valuation to $v$, and prove that it induces a homeomorphism between $\text{Spv} k_v^\sim$ and the subset of $\text{Spv} A$ consisting of all secondary specializations of $v$.

1.5 Let $A$ be a Huber ring. Let $v, w \in \text{Cont} A$ be continuous valuations such that $w$ is a specialization of $v$. Suppose that $\text{supp } w$ is not open (note that this condition is satisfied if $A$ is Tate). Prove that $\text{supp } v = \text{supp } w$ (hence 1.4 (iii) tells us that $w$ is a secondary specialization of $v$).

Hint: for $a, b \in A$ with $w(a) = 0$, show that $v(b) < v(a) \neq 0$ implies $w(b) = 0$.

1.6 Let $A$ be a Huber ring.

(i) Prove that a subring $A_0$ of $A$ is a ring of definition if and only if it is open and bounded.

(ii) Assume that $A$ is Tate and $A_0$ is a ring of definition of $A$. Prove that there exists a topologically nilpotent unit $\varpi$ of $A$ belonging to $A_0$. Further, observe that $A = A_0[1/\varpi]$ and $\varpi A_0$ is an ideal of definition of $A_0$.

1.7 Let $A$ be a Huber ring. We write $A^\circ$ for the subset of $A$ consisting of power-bounded elements, and $\hat{A}$ for the completion of $A$.

(i) Check that $A^\circ$ is an integrally closed open subring of $A$.

(ii) Prove that $\hat{A}$ is a Huber ring.

(iii) Prove that $(\hat{A})^\circ = \hat{A^\circ}$.

(iv) Let $A^+$ be a ring of integral elements; in other words, $(A, A^+)$ forms a Huber pair. Show that $(\hat{A}, \hat{A^+})$ is a Huber pair.

**Notation** A non-archimedean field $k$ is a complete topological field whose topology is induced from a height 1 valuation $|-| : k \to \mathbb{R}_{\geq 0}$. Note that our convention that $k$ is complete is different from [Hub96, Definition 1.1.3].
It can be easily seen that $k^\circ$ equals the set \( \{a \in k \mid |a| \leq 1\} \), where \(|-|\) is any height 1 valuation inducing the topology of $k$.

1.8 Let $k$ be a non-archimedean field. We write $k\langle T_1, \ldots, T_n \rangle$ for the subring of $k[[T_1, \ldots, T_n]]$ consisting of convergent power series

$$\sum_{I \in \mathbb{Z}_{\geq 0}^n} a_I T^I$$ such that \( \lim_{|I| \to \infty} a_I \to 0. \)

Here, for $I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$, we put $T^I = T_1^{i_1} \cdots T_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$.

Further, we write $k^\circ \langle T_1, \ldots, T_n \rangle$ for the subring $k\langle T_1, \ldots, T_n \rangle \cap k^\circ[[T_1, \ldots, T_n]]$ of $k\langle T_1, \ldots, T_n \rangle$. Take a topologically unipotent unit $\varpi$ of $k$, and consider the topology on $k\langle T_1, \ldots, T_n \rangle$ such that \( \{\varpi^m k^\circ \langle T_1, \ldots, T_n \rangle\}_{m \geq 0} \) is a fundamental system of open neighborhoods of 0.

(i) Check that $k\langle T_1, \ldots, T_n \rangle$ is a complete Huber ring.

(ii) Prove that $k^\circ \langle T_1, \ldots, T_n \rangle$ coincides with $k^\circ \langle T_1, \ldots, T_n \rangle$.

(iii) Check that $k\langle T_1, \ldots, T_n \rangle$ satisfies the following universal property: for any complete Huber $k$-algebra $A$ and its power-bounded elements $a_1, \ldots, a_n \in A$, there exists a unique continuous $k$-algebra homomorphism $\phi : k\langle T_1, \ldots, T_n \rangle \to A$ such that $\phi(T_i) = a_i$.

1.9 Let $k$ be a non-archimedean field, and fix a norm $|-| : k \to \mathbb{R}_{\geq 0}$. We consider the lexicographic order on $\mathbb{Z}_{\geq 0}^n$. For a non-zero $f = \sum_{I \in \mathbb{Z}_{\geq 0}^n} a_I T^I \in k^\circ \langle T_1, \ldots, T_n \rangle$, we write $\nu(f)$ for the maximal element $\nu \in \mathbb{Z}_{\geq 0}^n$ such that $|a_\nu| = \max_I |a_I|$. We put $\text{LT}(f) = a_{\nu(f)} T^{\nu(f)}$, and call it the leading term of $f$.

(i) Let $g_1, \ldots, g_m$ be non-zero elements of $k^\circ \langle T_1, \ldots, T_n \rangle$ whose leading terms are monic (i.e., $\text{LT}(g_i) = T^{\nu(g_i)}$). We put $M = \bigcup_{1 \leq i \leq m} (\nu(g_i) + \mathbb{Z}_{\geq 0}^n)$. For every $f \in k^\circ \langle T_1, \ldots, T_n \rangle$, find $h_1, \ldots, h_m \in k^\circ \langle T_1, \ldots, T_n \rangle$ such that $f - (h_1 g_1 + \cdots + h_m g_m)$ has no exponent in $M$.

Hint: choose $a \in k^\circ$ so that the leading term of $g_i$ mod $ak^\circ$ equals $T^{\nu(a)}$ for every $i$, and consider the division in $(k^\circ/ak^\circ)[T_1, \ldots, T_n]$.

(ii) Let $I$ be an ideal of $k^\circ \langle T_1, \ldots, T_n \rangle$. We write $\text{LT}(I)$ for the ideal of $k^\circ \langle T_1, \ldots, T_n \rangle$ generated by $\text{LT}(f)$ for all $f \in I \setminus \{0\}$. Suppose that there exist non-zero elements $g_1, \ldots, g_m \in I$ whose leading terms are monic such that $\text{LT}(I) = (\text{LT}(g_1), \ldots, \text{LT}(g_m))$. Prove that $I$ is generated by $g_1, \ldots, g_m$.

(iii) Let $I$ be a non-zero ideal of $k^\circ \langle T_1, \ldots, T_n \rangle$. We assume that $I$ is saturated for a topologically nilpotent unit $\varpi$ of $k$, that is, $k^\circ \langle T_1, \ldots, T_n \rangle/I$ is $\varpi$-torsion free. Prove that there exist non-zero elements $g_1, \ldots, g_m \in I$ as in (ii), hence $I$ is finitely generated.

Hint: let $L$ be the subset \( \{\nu(f) \mid f \in I \setminus \{0\}\} \) of $\mathbb{Z}_{\geq 0}^n$, which is an ideal of the monoid $\mathbb{Z}_{\geq 0}^n$. Use the fact that any ideal of the monoid $\mathbb{Z}_{\geq 0}^n$ is finitely generated.

(iv) Prove that $k\langle T_1, \ldots, T_n \rangle$ is Noetherian.
1.10 A non-archimedean field $K$ is said to be spherically complete if every decreasing sequence $D_1 \supset D_2 \supset \cdots$ of closed disks in $K$ has non-empty intersection.

(i) Prove that every $p$-adic field (that is, a finite extension of $\mathbb{Q}_p$) is spherically complete.

(ii) Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Prove that $\mathbb{C}_p$ is not spherically complete.

2 Underlying spaces of adic spaces

2.1 Let $(A, A^+)$ be a Tate Huber pair.

(i) Fix a topologically nilpotent unit $\varpi$ of $A$. For $v \in \text{Spv} A$ with $v(\varpi) < 1$, we write $\Gamma_v^\varpi$ for the largest convex subgroup of $\Gamma_v$ such that $v(\varpi)$ is cofinal in $\Gamma_v^\varpi$ (i.e., for any $\gamma \in \Gamma_v^\varpi$, there exists $n \geq 0$ such that $v(\varpi)^n < \gamma$).

Prove that the map $\phi: \{v \in \text{Spv} A \mid v(a) < 1 \ (a \in A^\varpi)\} \to \text{Spv} A; \ v \mapsto v|_{\Gamma_v^\varpi}$ (see 1.4 (i)) is well-defined and continuous.

(ii) Observe that the image of $\phi$ in (i) equals $\text{Cont} A$. Deduce that $\text{Spa}(A, A^+)$ is quasi-compact.

(iii) Recall that a rational subset of $\text{Spa}(A, A^+)$ is a subset of the form $U\left(\frac{f_1, \ldots, f_n}{g}\right) = \left\{v \in \text{Spa}(A, A^+) \mid v(f_i) \leq f(g) \neq 0\right\}$, where $f_1, \ldots, f_n, g \in A$ such that $f_1A + \cdots + f_nA = A$. Prove that rational subsets form an open basis of $\text{Spa}(A, A^+)$. 

2.2 Let $(A, A^+)$ be a Tate Huber pair. Pick a point $x$ of $\text{Spa}(A, A^+)$, and denote by $G(x)$ the set of all generalizations of $x$.

(i) Prove that $G(x)$ forms a chain; namely, for $y, z \in G(x)$, either $y$ specializes to $z$ or $z$ specializes to $y$.

Hint: use 1.5.

(ii) Prove that $G(x)$ contains a point $y$ which is a generalization of every point in $G(x)$. Such a point is called the maximal generalization of $x$.

Hint: use 1.2.

(iii) Let $f \in A$ be an element and $Y = \{v \in \text{Spa}(A, A^+) \mid v(f) = 0\}$ the closed subset defined by $f$. Prove that $Y$ is stable under generalization.

2.3 Fix a norm $|−|: \mathbb{C}_p \to \mathbb{R}_{\geq 0}$ of $\mathbb{C}_p$. For a closed disk $D$ in $\mathcal{O}_{\mathbb{C}_p}$, we write $v_D: \mathbb{C}_p\langle T \rangle \to \mathbb{R}_{\geq 0}$ for the map $f \mapsto \sup_{x \in D} f(x)$. Further, for a collection $\mathcal{E}$ of closed disks in $\mathcal{O}_{\mathbb{C}_p}$ such that every $D, D' \in \mathcal{E}$ satisfy either $D \subseteq D'$ or $D \supset D'$, we put $v_{\mathcal{E}} = \inf_{D \in \mathcal{E}} v_D$.

(i) Check that $v_{\mathcal{E}}$ gives a point of $D^1 = \text{Spa}(\mathbb{C}_p\langle T \rangle, \mathcal{O}_{\mathbb{C}_p}\langle T \rangle)$.

(ii) Observe that $\bigcap_{D \in \mathcal{E}} D$ is one of the following:
one point \( a \in \mathcal{O}_{C_p} \),
- a disk \( \{ z \in \mathcal{O}_{C_p} \mid |z - a| \leq r \} \) with \( r \in |\mathcal{O}_{C_p}^\times| \),
- a disk \( \{ z \in \mathcal{O}_{C_p} \mid |z - a| \leq r \} \) with \( r \in \mathbb{R}_{>0} \setminus |\mathcal{O}_{C_p}^\times| \), or
- empty.

In each of the first three cases, describe \( v_E \) concretely.

(iii) In each of the cases above, determine all specializations of \( v_E \) by using 1.4 (iv).

(iv) Let \( v : C_p\langle T \rangle \to \mathbb{R}_{\geq 0} \) be a point with height 1 of \( \mathbb{D}^1 \). For \( a \in \mathcal{O}_{C_p} \), we write \( D_a \) for the closed disk \( \{ z \in \mathcal{O}_{C_p} \mid |z - a| \leq v(T - a) \} \). Prove that \( v = v_E \) for \( E = \{ D_a \mid a \in \mathcal{O}_{C_p} \} \).

(v) Find all points in \( \mathbb{D}^1 \).

2.4 An admissible blow-up of a formal scheme \( \mathcal{X} \) means the blow-up along a finitely generated open ideal sheaf of \( \mathcal{O}_X \). For example, if \( \mathcal{X} = \text{Spf} \mathbb{Z}_p\langle T \rangle \), an admissible blow-up is the formal completion along the special fiber of a blow-up \( \mathcal{X}' \to A^1_{\mathbb{Z}_p} \) along a closed subscheme which is set-theoretically contained in the special fiber of \( A^1_{\mathbb{Z}_p} \). We write \( \Phi_\mathcal{X} \) for the set of admissible blow-ups of \( \mathcal{X} \), and put \( \langle \mathcal{X}^{\text{rig}} \rangle = \varprojlim_{(\mathcal{X}' \to \mathcal{X}) \in \Phi_\mathcal{X}} \mathcal{X}' \).

(i) Assuming \( \mathcal{X} \) is quasi-compact, deduce that \( \langle \mathcal{X}^{\text{rig}} \rangle \) is quasi-compact.

Hint: use the following general result due to Stone: if \( \{ Y_i \}_{i \in I} \) is a filtered projective system of quasi-compact \( T_0 \) topological spaces with closed transition maps, the limit space \( \varprojlim Y_i \) is quasi-compact.

(ii) Let \( \mathcal{X} = \text{Spf} \mathcal{O}_{C_p}(T) \). Construct a natural map \( \mathbb{D}^1 = \text{Spa}(C_p\langle T \rangle, \mathcal{O}_{C_p}\langle T \rangle) \to \langle \mathcal{X}^{\text{rig}} \rangle \).

Hint: use the valuative criterion.

(iii) Describe the image under the map in (ii) of each point of \( \mathbb{D}^1 \) found in 2.3 (v).

(iv) Prove that the map in (ii) is a homeomorphism.

3 Structure (pre)sheaves of adic spaces

3.1 (i) Prove that \( \mathbb{D}^1 = \text{Spa}(C_p\langle T \rangle, \mathcal{O}_{C_p}\langle T \rangle) \) is connected.

(ii) Let \( x \) be a point of \( \mathbb{D}^1 \). When is \( \mathbb{D}^1 \setminus \{ x \} \) non-connected?

3.2 Let \( (A,A^+) \) be a Huber pair. For a rational subset \( U \) of \( \text{Spa}(A,A^+) \), prove that the natural map \( \text{Spa}(\mathcal{O}(U), \mathcal{O}^+(U)) \to \text{Spa}(A,A^+) \) induces a homeomorphism between \( \text{Spa}(\mathcal{O}(U), \mathcal{O}^+(U)) \) and \( U \). (Together with 2.1, we conclude that every rational subset is quasi-compact.)

Hint: first prove that \( \text{Spa}(\widehat{A}, \widehat{A}^+) \to \text{Spa}(A,A^+) \) is a homeomorphism.

3.3 (i) Let \( X = \text{Spa}(A,A^+) \) be an affinoid adic space with complete Huber pair \( (A,A^+) \) and \( B \) a ring. Prove that morphisms of locally ringed spaces \( (X, \mathcal{O}_X) \to \text{Spec} B \) are in bijection with ring homomorphisms \( B \to A \).

Hint: the map \( X \to \text{Spec} B \) corresponding to \( \phi : B \to A \) is given by \( v \mapsto \{ b \in B \mid v(\phi(b)) = 0 \} \).
(ii) Let $k$ be a non-archimedean field, and $\varpi \in k$ a topologically nilpotent unit. We put $A^{1,\text{ad}} = \bigcup_{m \geq 1} \text{Spa}(k\langle \varpi^mT \rangle, k^\circ\langle \varpi^mT \rangle)$. Check that $A^{1,\text{ad}}$ fits into a commutative diagram

$$
\begin{array}{ccc}
A^{1,\text{ad}} & \rightarrow & A^1 \\
\downarrow & & \downarrow \\
\text{Spa}(k, k^\circ) & \rightarrow & \text{Spec } k,
\end{array}
$$

where the horizontal arrows are morphisms of locally ringed spaces. Further, prove that $A^{1,\text{ad}}$ satisfies the following universal property:

For an adic space $S$ over $\text{Spa}(k, k^\circ)$ and a morphism of locally ringed spaces $f: S \rightarrow A^1$ which makes the following diagram commute, there exists a unique morphism of adic spaces $g: S \rightarrow A^{1,\text{ad}}$ that makes the diagram commute:

$$
\begin{array}{ccc}
S & \xrightarrow{g} & A^{1,\text{ad}} \\
\downarrow & & \downarrow \\
\text{Spa}(k, k^\circ) & \rightarrow & \text{Spec } k.
\end{array}
$$

(iii) By extending the construction in (ii), find a definition of the adic space $X^{\text{ad}}$ attached to an algebraic variety $X$ over $k$.

3.4 Let $A$ be a ring and $I$ a finitely generated ideal of $A$. Assume that $A$ is $I$-adically complete, and consider the formal scheme $\mathcal{X} = \text{Spf } A$.

(i) Let $Y = \text{Spa}(B, B^+)$ be an affinoid adic space with complete Huber pair $(B, B^+)$. Prove that morphisms of locally topologically ringed spaces $(Y, \mathcal{O}_Y^+) \rightarrow \mathcal{X}$ are in bijection with continuous ring homomorphisms $A \rightarrow B^+$. Hint: the map $Y \rightarrow \mathcal{X}$ corresponding to $\phi: A \rightarrow B^+$ is given by $v \mapsto \{a \in A \mid v(\phi(a)) < 1\}$.

(ii) Assume that $(A, A)$ is sheafy (this is the case if $A$ is Noetherian), and put $t(\mathcal{X}) = \text{Spa}(A, A)$. Check that the morphism of locally topologically ringed spaces $\lambda: (t(\mathcal{X}), \mathcal{O}_{t(\mathcal{X})}^+) \rightarrow \mathcal{X}$ corresponding to $\text{id}: A \rightarrow A$ satisfies the following universal property: for every adic space $Y$ and a morphism of locally topologically ringed spaces $\mu: (Y, \mathcal{O}_Y^+) \rightarrow \mathcal{X}$, there exists a unique morphism of adic spaces $f: Y \rightarrow t(\mathcal{X})$ such that $\mu = \lambda \circ f$.

By this property, we can attach to locally Noetherian formal scheme $\mathcal{X}$ an adic space $t(\mathcal{X})$ by gluing.

3.5 Let $V$ be a discrete valuation ring and $\mathcal{X}$ a locally Noetherian formal scheme over $\text{Spf } V$. We put $F = \text{Frac } V$.

(i) Prove that $t(\text{Spf } V) = \text{Spa}(V, V)$ consists of two points $s$ and $\eta$, where $s$ is closed and $\eta$ is open.
We write $\mathcal{X}^{\mathrm{ad}}_\eta$ for the fiber of $t(\mathcal{X}) \to \text{Spf} V$ at $\eta$, and call it the rigid generic fiber of $\mathcal{X}$. The composite map $\text{sp}_X : \mathcal{X}^{\mathrm{ad}}_\eta \to t(\mathcal{X}) \to \mathcal{X} = \mathcal{X}_{\text{red}}$ is called the specialization map.

(ii) Prove that $(\text{Spf} V(T))^{\text{ad}}_\eta = \text{Spa}(F(T), V(T))$.

(iii) Observe that $(\text{Spf} V[[T]])^{\text{ad}}_\eta$ can be regarded as an open disk.

Hint: $(\text{Spf} V[[T]])^{\text{ad}}_\eta \subset t(\text{Spf} V[[T]])$ is not a rational subset. Write it as an increasing union of rational subsets.

(iv) Let $X$ be a scheme of finite type over $V$, and $Y$ a closed subscheme of the special fiber of $X$. We write $X$ (resp. $Y$) for the formal completion of $X$ along the special fiber (resp. $Y$). Prove that $\mathcal{Y}^{\text{ad}}_\eta$ is isomorphic to the open adic subspace of $\mathcal{X}^{\text{ad}}_\eta$ whose underlying space is the interior of $\text{sp}_X(Y)$ in $\mathcal{X}^{\text{ad}}_\eta$.

When $V = \mathcal{O}_{\mathbb{C}_p}$, I do not know whether $t(\mathcal{X})$ can be defined or not. Nevertheless, for a formal scheme $\mathcal{X}$ locally formally of finite type over $\mathcal{O}_{\mathbb{C}_p}$, one can define its rigid generic fiber $\mathcal{X}^{\text{ad}}_\eta$, which is an adic space locally of finite type over $\text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$.

3.6 Let $k$ be a non-archimedean field. We put $A = k(T)$, and let $A'$ be the integral closure of $k^e[A^\infty]$ in $A$ (recall that $A^\infty$ denotes the set of topologically nilpotent elements in $A$).

(i) Show that $A'$ equals \( \{ \sum_{n=0}^\infty a_n T^n \in k^e(T) \mid a_n \in k^e \ (n \geq 1) \} \).

(ii) Observe that $\text{Spa}(A, A')$ is partially proper over $\text{Spa}(k, k^e)$, and contains $D^1 = \text{Spa}(A, A^e)$ as an open subset.

(iii) Prove that $\overline{D^1} = \text{Spa}(A, A')$ is the universal compactification of $D^1$ in the following sense: for every partially proper adic space $Y$ over $\text{Spa}(k, k^e)$, a $k$-morphism $f : D^1 \to Y$ extends uniquely to $\overline{f} : \overline{D^1} \to Y$.

(iv) Check that $A^{1,\text{ad}}$ is partially proper over $\text{Spa}(k, k^e)$. Determine the image of the induced map $\overline{f} : \overline{D^1} \to A^{1,\text{ad}}$.

(v) Consider the questions (ii), (iii) for more general topologically finitely generated $k$-algebras.

3.7 Let $(A, A^+)$ is a Tate Huber pair such that $A$ is uniform (i.e., $A^e$ is bounded). We put $X = \text{Spa}(A, A^+)$. Let $t \in A$, and consider rational subsets $U = \{ v \in X \mid v(t) \leq 1 \}$ and $V = \{ v \in X \mid v(t) \geq 1 \}$. We want to prove the exactness of $0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \to 0$.

Take a ring of definition $A_0$ of $A$ and a topologically nilpotent unit $\varpi$ of $A$ belonging to $A_0$. We put $B_0 = A_0[t]$ and write $\mathcal{B}$ for the ring $A$ with the topology induced from the $\varpi$-adic topology on $B_0$. We put $C = A[1/t], C_0 = A_0[1/t]$ and equip $C$ with the topology induced from the $\varpi$-adic topology on $C_0$. Finally, we put $D = A[1/t], D_0 = A_0[1/t]$ and equip $D$ with the topology induced from the $\varpi$-adic topology on $D_0$. Note that we have $\tilde{A} = \mathcal{O}_X(X), \tilde{B} = \mathcal{O}_X(U), \tilde{C} = \mathcal{O}_X(V),$ and $\tilde{D} = \mathcal{O}_X(U \cap V)$.

(i) We write $\phi : A \to A[1/t]$ for the natural map. Prove that $B_0 \cap \phi^{-1}(C_0) \subset A^e$.

(In this step we do not need to assume that $A$ is uniform.)
Let \( a \in B_0 \cap \phi^{-1}(C_0) \), find \( f(T), g(T) \in A_0[T] \) and \( c \geq \deg g \) such that \( a = f(t) \) and \( t^ca = g(t) \). Let \( d = \deg f + c \), and \( n \geq 0 \) be an integer such that \( \varpi^nt \in A_0 \). Prove that \( \varpi^{nd_i}a^m \in A_0 \) for every \( m \geq 0 \) and \( 0 \leq i \leq d \) by the induction on \( m \).

(ii) By (i), there exists an integer \( n \geq 0 \) such that \( \varpi^n(B_0 \cap \phi^{-1}(C_0)) \subset A_0 \). By using this fact, prove that the exact sequence \( 0 \to A \to B \oplus C \to D \to 0 \) remains exact after completion. This means that the sequence \( 0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \to 0 \) is exact.

3.8 Let \((A, A^+)\) be a stably uniform Tate Huber pair. Put \( X = \text{Spa}(A, A^+) \).

(i) Let \( t_1, \ldots, t_n \in A \). For a subset \( I \subset \{1, \ldots, n\} \), we put \( U_I = \{ v \in X \mid v(t_i) \leq 1 \text{ (} i \in I, v(t_i) \geq 1 \text{ (} i \notin I) \} \). They form an open covering \( \{U_I\}_{I \subset \{1, \ldots, n\}} \) (such an open covering is called a Laurent covering). Prove that \( \mathcal{O}_X \) satisfies the sheaf condition with respect to this covering.

Hint: use 3.7.

(ii) Let \( a_1, \ldots, a_n \in A \). For \( 1 \leq i \leq n \), we put \( U_i = \{ v \in X \mid v(a_i) \leq v(a_j) \neq 0 \text{ (} 1 \leq j \leq n \} \). They form an open covering \( \{U_i\}_{1 \leq i \leq n} \) (such an open covering is called a rational covering). Assume moreover that \( a_1, \ldots, a_n \in A^\times \). Prove that there exists a Laurent covering refining \( \{U_i\}_{1 \leq i \leq n} \), and deduce from this fact that \( \mathcal{O}_X \) satisfies the sheaf condition with respect to \( \{U_i\}_{1 \leq i \leq n} \).

(iii) Let \( \{U_i\}_{1 \leq i \leq n} \) be as in (ii), but we do not assume that \( a_1, \ldots, a_n \) are units. Prove that there exists a Laurent covering \( V = \{V_j\} \) such that \( \{U_i \cap V_j\}_{1 \leq i \leq n} \) is a rational covering of \( V_j \) of the type considered in (ii) for every \( J \).

(iv) Prove that every open covering of \( X \) can be refined by a rational covering.

(v) Conclude that \( \mathcal{O}_X \) is a sheaf.

3.9 Let \( k \) be a non-archimedean field, and \( \varpi \in k \) a topologically nilpotent unit. We put \( A = k[T, T^{-1}, Z]/(Z^2) \). Let \( A_0 \) be the \( k^\times \)-submodule of \( A \) generated by \( \varpi^nT^{\pm n}, \varpi^{-n}T^{\pm n}Z \) with \( n \geq 0 \).

(i) Check that \( A_0 \) is a \( k^\times \)-subalgebra of \( A \) and \( A = A_0[1/\varpi] \).

(ii) We equip \( A \) with the topology such that \( \{\varpi^nA_0\}_{n \geq 0} \) is a fundamental system of open neighborhoods of \( 0 \), and consider \( X = \text{Spa}(A, A^\times) \). Let \( U = \{v \in X \mid v(T) \leq 1\} \) and \( V = \{v \in X \mid v(T) \geq 1\} \), which are rational subsets of \( X \). Prove that \( Z \in \mathcal{O}_X(X) \) is non-zero, and the image of \( Z \) under the restriction map \( \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \) is zero. This means that the presheaf \( \mathcal{O}_X \) on \( X \) is not a sheaf.

Hint: consider the intersection of \( kZ \) with \( A_0, A_0[T] \) and \( A_0[T^{-1}] \).

This problem is taken from [BV16, Proposition 12].

3.10 Let \( k \) and \( \varpi \) be as in 3.9. Let \( A_0 \) be a \( k^\times \)-submodule of \( k[T, T^{-1}, Z] \) generated by \( (\varpi T)^a(\varpi Z)^b \) with \( b \geq 0 \) and \( a \geq -b^2 \).

(i) Check that \( A_0 \) is a \( k^\times \)-subalgebra of \( k[T, T^{-1}, Z] \).
(ii) We put \( A = A_0[1/\varpi] \) and consider the topology on it such that \( \{ \varpi^n A_0 \}_{n \geq 0} \) is a fundamental system of open neighborhoods of 0. Prove that the natural \( \mathbb{Z}^2 \)-grading on \( k[T, T^{-1}, Z] \) induces that on \( A^\circ \).

Hint: the crucial point is that the ring of definition \( A_0 \) is also graded.

(iii) By using (ii), show that \( A^\circ = A_0 \), hence \( A \) is uniform.

(iv) For a rational subset \( U = \{ v \in \text{Spa}(A, A^\circ) \mid v(T) \leq 1 \} \), prove that \( \mathcal{O}(U) \) is not uniform. This means that \( (A, A^\circ) \) is not stably uniform.

Hint: observe that \( \varpi^{-1} \mathcal{O} \notin A_0[T] \) and \( (\varpi^{-n} \mathcal{O})^{n+1} \in A_0[T] \) for every \( n \geq 1 \).

This problem is taken from [BV16, Proposition 17]. By slight modification, one can also give a uniform Tate ring \( A \) such that \( \text{Spa}(A, A^\circ) \) is not sheafy. See [BV16, Proposition 18].

## 4 Perfectoid spaces

### 4.1 Let \( F \) be a non-archimedean local field. We fix a uniformizer \( \varpi \) of \( F \). Let \( \mathbb{X} \) be the Lubin-Tate formal group (= 1-dimensional formal \( \mathcal{O}_F \)-module of height 1) over \( \mathcal{O}_F \) such that \( [\varpi]_{\mathbb{X}}(T) = \varpi T + T^q \), where \( q \) is the cardinality of the residue field of \( F \). We write \( F_m \) for the extension field of \( F \) obtained by adjoining all roots of \( [\varpi^m]_{\mathbb{X}}(T) = 0 \). Let \( \hat{F}_\infty \) be the completion of \( \lim_{\rightarrow m} F_m \). Prove that \( \hat{F}_\infty \) is a perfectoid field.

### 4.2 Let \( K \) be a perfectoid field. Prove that if \( K^\circ \) is algebraically closed, so is \( K \).

Hint: take an irreducible polynomial \( P(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \in K^\circ[T] \). By changing the variable, we may assume that \( a_0 \) is a unit (why?). Take \( Q(T) = T^d + b_{d-1}T^{d-1} + \cdots + b_0 \in K^\circ[T] \) such that the image of \( Q(T) \) in \( (K^\circ/\varpi)[T] \) is equal to that of \( P(T) \) in \( (K^\circ/\varpi)[T] \). Pick a root \( y \) of \( Q(T) \) and approximate a root of \( P(T) \) by \( y^p \).

### 4.3 Let \( A \) be a uniform complete Tate ring, and \( p \) a prime number.

(i) For a topologically nilpotent unit \( \varpi \) of \( A \) such that \( p \in \varpi^p A^\circ \), prove that the \( p \)-th power map \( \Phi: A^\circ/\varpi A^\circ \to A^\circ/\varpi^p A^\circ \) is injective.

(ii) Show that the condition that \( \Phi: A^\circ/\varpi A^\circ \to A^\circ/\varpi^p A^\circ \) is surjective is independent of the choice of a topologically nilpotent unit \( \varpi \in A \) with \( p \in \varpi^p A^\circ \).

### 4.4 Let \( R \) be a perfect \( \mathbb{F}_p \)-algebra, and \( W(R) \) the ring of Witt vectors with coefficients in \( R \). Let \( S \) be a \( p \)-adically complete ring. Let \( t: R \to S \) be a multiplicative map such that the composite \( R \overset{t}{\to} S \to S/pS \) is a ring homomorphism. Prove that the map \( T: W(R) \to S \) defined by \( T\left( \sum_{n=0}^{\infty} p^n [a_n] \right) = \sum_{n=0}^{\infty} p^n t(a_n) \quad (a_n \in R) \) becomes a ring homomorphism. Check also that \( T \) is surjective if the composite \( R \overset{t}{\to} S \to S/pS \) is.
4.5 Let $R$ be a perfectoid $\mathbb{F}_p$-algebra, and $\xi = \sum_{n=0}^{\infty} p^n [a_n]$ ($a_n \in R^\circ$) be an element of $W(R^\circ)$. We say that $\xi$ is primitive of degree 1 if $a_0$ is topologically nilpotent and $a_1$ is a unit of $R^\circ$.

(i) Prove that a primitive element of degree 1 is a non-zero-divisor.

(ii) Prove that $\xi \in W(R^\circ)$ is primitive of degree 1 if and only if there exist $u \in W(R^\circ)^\times$, $\alpha \in W(R^\circ)$ and a topologically nilpotent unit $\varpi \in R^\circ$ such that $u^\xi = p + \alpha [\varpi]$.

Hint: note that $(\xi - [a_0])/p$ is a unit of $W(R^\circ)$.

4.6 Let $A$ be a perfectoid ring.

(i) Use 4.4 to construct a surjective ring homomorphism $\theta: W(A^\circ) \to A^\circ$.

(ii) Check that $\theta([x]) = x^\#$ for $x \in A^\circ$.

(iii) Take topologically nilpotent units $\varpi \in A$ and $\varpi^\beta \in A^\circ$ such that $p \in \varpi^p A^\circ$ and $(\varpi^\beta)^\ell = \varpi$. Pick $\alpha \in W(A^\circ)$ such that $\theta(\alpha) = p/\varpi$ and put $\xi = p - \alpha [\varpi^\beta]$. Prove that $\xi$ generates $\text{Ker} \theta$.

4.7 (i) Let $K$ be the completion of $\mathbb{Q}_p(\mu_{p^n})$, which is a perfectoid field of characteristic 0. Prove that $K^\circ$ is isomorphic to the completion of $\mathbb{F}_p((T^{p^{-\infty}}))$. Find a generator of $\text{Ker} \theta$ (see 4.6) in this case.

(ii) Answer the same question for the completion of $\mathbb{Q}_p(p^{p^{-\infty}})$.

4.8 Let $\{X_i\}$ be a filtered projective system of adic spaces whose transition maps are quasi-compact and quasi-separated. For a perfectoid space $X$, we write $X \sim \varprojlim X_i$ if the following conditions are satisfied:

- A compatible family of morphisms $\phi_i: X \to X_i$ is given and the induced map $|X| \to \varprojlim |X_i|$ on the underlying spaces is a homeomorphism.

- For each $x \in X$, there exists an affinoid open neighborhood $U$ of $x$ such that the image of $\varprojlim_{i: (U_i \subset X_i)} \mathcal{O}_{X_i}(U_i) \to \mathcal{O}_X(U)$ is dense. Here $U_i$ runs through affinoid open subsets of $X_i$ which contain $\phi_i(U)$.

(i) For a perfectoid Huber pair $(B, B^+)$, show that the map

$$\text{Hom} (\text{Spa}(B, B^+), X) \to \varprojlim \text{Hom} (\text{Spa}(B, B^+), X_i)$$

is bijective. Conclude that a perfectoid space $X$ satisfying $X \sim \varprojlim X_i$ is unique up to isomorphism.

(ii) Let $K$ be a perfectoid field of residue characteristic $p$. Let us consider the projective system $(\cdots \phi^{\pi} \to A^{n,\mathrm{ad}} \to A^{n,\mathrm{ad}} \to \cdots)$, where $\phi: A^n \to A^n$ is given by $(x_1, \ldots, x_n) \mapsto (x_p, \ldots, x_p)$. Check that there exists a perfectoid space $X$ over $K$ such that $X \sim \varprojlim_{\phi^{\pi}} A^{n,\mathrm{ad}}$.

4.9 Let $X = \text{Spa}(A, A^+)$ be an affinoid perfectoid space, and $Z$ a closed subset of $X$ defined by $f_1 = \cdots = f_n = 0$ for $f_1, \ldots, f_n \in A$. 

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(i) Fix a topologically nilpotent unit \( \varpi \in A \). For \( m \geq 0 \), let \( U_m \) be the open neighborhood of \( Z \) defined by \( \{ v \in X \mid v(f_i) \leq v(\varpi^m) \} \). Prove that there exists a perfectoid space \( \tilde{Z} \) such that \( \tilde{Z} \sim \lim_\leftarrow \bigcup_m U_m \).

(ii) Let \( Y \) be a perfectoid space and \( \phi: Y \to X \) a morphism whose set-theoretic image is contained in \( Z \). Prove that \( \phi \) uniquely factors through \( Y \to \tilde{Z} \). In particular, \( \tilde{Z} \) is independent of the choice of \( f_1, \ldots, f_n \) and \( \varpi \).

4.10 Let \( K \) be a perfectoid field of characteristic 0, and \( p \) the residue characteristic of \( K \).

(i) Let \( A \) be a complete Tate \( K \)-algebra satisfying the following conditions:
   (a) Every element of \( 1 + A^{p\infty} \) has a \( p \)-th root in \( A \).
   (b) \( A \) is uniform.

   Prove that \( A \) is a perfectoid \( K \)-algebra.
   Hint: first observe that a \( p \)-th root of \( a \in 1 + A^{p\infty} \) can be taken from \( 1 + A^{p\infty} \).

(ii) Let \( A \) be a Tate \( K \)-algebra satisfying the condition (a) in (i). Take a topologically nilpotent unit \( \varpi \) of \( K \) and equip \( A \) with the new topology such that \( \{ \varpi^m A^{p\infty} \} \) is a fundamental system of open neighborhoods of 0. Let \( \hat{A} \) denote the completion of \( A \) with respect to this topology. Prove that \( \hat{A} \) satisfies the conditions (a), (b) in (i), hence is a perfectoid \( K \)-algebra.

(iii) Let \( X = \text{Spa}(B, B^{p\infty}) \) be an affinoid adic space of finite type over \( \text{Spa}(K, K^{p\infty}) \).

   Prove that there exist a filtered projective system \( \{ X_i \} \) of finite \( \acute{e} \text{tale} \) covers of \( X \) and a perfectoid space \( X_\infty \) over \( K \) such that \( X_\infty \sim \lim_\leftarrow i X_i \).

This problem is taken from [Col02, §2.8] and [Sch13, Proposition 4.8].

4.11 Let \( K \) be a perfectoid field of characteristic 0, and \( G \) a finite group acting on \( K \). Let us prove that \( K^G \) is a perfectoid field. Note that the surjection \( \theta: W(K^{p\infty}) \to K^{p\infty} \) in 4.6 is \( G \)-equivariant.

(i) Prove that for every integer \( m \geq 0 \) there exists a topologically nilpotent unit \( \varpi \) in \( K^G \) such that \( p \in \varpi^{p^m+1} K^{p\infty} \).

   Hint: find \( \varpi \) of the form \( \theta([u]) \) with \( u \in K^{p\infty} \).

(ii) Assume first that \( |G| = p^m \). Take \( \varpi \) as in (i). For \( x \in K^{G^{p\infty}} \), pick \( y \in K^{p\infty} \) such that \( \theta([y]) \equiv x \mod pK^{p\infty} \) and put \( z = \prod_{g \in G} g(y)^{1/p^{m+1}} \). Check that \( \theta([z]) \in K^{G^{p\infty}} \) and \( x \equiv \theta([z])^p \mod \varpi^p K^{G^{p\infty}} \). This shows that \( K^G \) is a perfectoid field.

   Hint: use 4.3.

(iii) Prove that \( K^G \) is a perfectoid field for general \( G \).

(iv) Repeat the argument above to prove the following claim: for a perfectoid \( K \)-algebra \( A \) and a finite group \( G \) acting on \( A \), \( A^G \) is a perfectoid \( K^G \)-algebra.

This problem is taken from [KL16, Theorem 3.3.25].

4.12 Let \( K \) be a perfectoid field of characteristic \( p > 0 \). Modify 3.9 to construct a Huber \( K \)-algebra \( A \) satisfying the following condition: \( X = \text{Spa}(A, A^{p\infty}) \) is covered by affinoid perfectoid spaces, but \( \mathcal{O}_X \) is not a sheaf.
This problem is taken from [BV16, Proposition 13].

References


