Problems on adic spaces and perfectoid spaces

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1 Topological rings and valuations

Notation For a valuation $v\colon A \to \Gamma \cup \{0\}$ on a ring A, we write Γ_v for the subgroup of Γ generated by $\{v(a) \mid a \in A\} \setminus \{0\}$, and call it the value group of v. A subgroup H of Γ_v is said to be convex if $a_1, a_2, a_3 \in \Gamma_v$ with $a_1 \leq a_2 \leq a_3$ and $a_1, a_3 \in H$ implies $a_2 \in H$. The height of v means the supremum of the length r of a chain of convex subgroups $\{1\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_r = \Gamma_v$. We write $\operatorname{supp} v$ for the prime ideal $v^{-1}(0)$, and call it the support of v.

1.1 Let A be a ring and $v\colon A \to \Gamma \cup \{0\}$ be a valuation on A. Prove that the height of v is 1 if and only if $\Gamma_v \neq 1$ and there exists an order-preserving injective group homomorphism $\Gamma_v \hookrightarrow \mathbb{R}_{>0}$.

1.2 Let V be a valuation ring with valuation $v\colon V \to \Gamma \cup \{0\}$, and $K = \operatorname{Frac} V$ its fraction field. Consider the valuation topology on K, i.e., the topology generated by the subsets $\{x \in K \mid v(x) \leq a\}$ with $a \in \Gamma_v$. Prove that the following are equivalent:

- K is a Tate ring (i.e., a Huber ring which has a topologically nilpotent unit).
- V has a prime ideal of height 1.

In [Hub96, Definition 1.1.4], such a valuation ring is said to be microbial.

1.3 Let A be a ring, and $\operatorname{Spv} A$ the set of equivalence classes of valuations on A. Consider the topology of $\operatorname{Spv} A$ generated by the subsets $\{v \in \operatorname{Spv} A \mid v(a) \leq v(b) \neq 0\}$ with $a, b \in A$. Prove that $\operatorname{Spv} A$ is quasi-compact.

Hint: consider the map $\phi\colon \operatorname{Spv} A \to \prod_{A\times A}\{0,1\} = \operatorname{Map}(A \times A, \{0,1\})$ defined by

$$\phi(v)(a,b) = \begin{cases} 1 & \text{if } v(a) \leq v(b), \\ 0 & \text{if } v(a) > v(b). \end{cases}$$

Observe that $\operatorname{Im} \phi$ is a closed subset of $\prod_{A\times A}\{0,1\}$ with respect to the product topology of the discrete topology on $\{0,1\}$.

1.4 Let the notation be as in 1.3. Let $v\colon A \to \Gamma \cup \{0\}$ be a valuation on A.

(i) For a convex subgroup $H \subset \Gamma_v$ containing $\{v(a) \mid a \in A, v(a) \geq 1\}$, let $v|_H\colon A \to H \cup \{0\}$ be a map defined by

$$a \mapsto \begin{cases} v(a) & \text{if } v(a) \in H, \\ 0 & \text{if } v(a) \notin H. \end{cases}$$

Prove that $v|_H$ is a valuation of A, and it is a specialization of v in $\operatorname{Spv} A$. Such a specialization of v is called a primary specialization.

(ii) For a convex subgroup $H \subset \Gamma_v$, let $v/_H\colon A \to \Gamma_v/H \cup \{0\}$ be a map defined by

$$a \mapsto \begin{cases} v(a) \bmod H & \text{if } v(a) \neq 0, \\ 0 & \text{if } v(a) = 0. \end{cases}$$

Prove that $v/_H$ is a valuation of A, and v is a specialization of $v/_H$ (i.e., v lies in the closure of $v/_H$) in $\operatorname{Spv} A$. A valuation $v \in \operatorname{Spv} A$ is said to be a secondary specialization of $w \in \operatorname{Spv} A$ if there exists a convex subgroup H of Γ_v such that $w = v/_H$.

(iii) Let $w \in \operatorname{Spv} A$ be a specialization of $v \in \operatorname{Spv} A$ such that $\operatorname{supp} v = \operatorname{supp} w$. Observe that w is a secondary specialization of v. (In fact, if A is not necessarily Tate, w is known to be a primary specialization of a secondary specialization.)

(iv) We put $k_v = \operatorname{Frac}(A/\operatorname{supp} v)$. The valuation v on A induces that on k_v, by which k_v becomes a valuation field. We write $k_v^\sim$ for the residue field of k_v. Construct a natural continuous map $\operatorname{Spv} k_v^\sim \to \operatorname{Spv} A$ which sends the trivial valuation to v, and prove that it induces a homeomorphism between $\operatorname{Spv} k_v^\sim$ and the subset of $\operatorname{Spv} A$ consisting of all secondary specializations of v.

1.5 Let A be a Huber ring. Let $v, w \in \operatorname{Cont} A$ be continuous valuations such that w is a specialization of v. Suppose that $\operatorname{supp} w$ is not open (note that this condition is satisfied if A is Tate). Prove that $\operatorname{supp} v = \operatorname{supp} w$ (hence 1.4 (iii) tells us that w is a secondary specialization of v).

Hint: for $a, b \in A$ with $w(a) = 0$, show that $v(b) < v(a) \neq 0$ implies $w(b) = 0$.

1.6 Let A be a Huber ring.

(i) Prove that a subring A_0 of A is a ring of definition if and only if it is open and bounded.

(ii) Assume that A is Tate and A_0 is a ring of definition of A. Prove that there exists a topologically nilpotent unit ϖ of A belonging to A_0. Further, observe that $A = A_0[1/\varpi]$ and ϖA_0 is an ideal of definition of A_0.

1.7 Let A be a Huber ring. We write A° for the subset of A consisting of power-bounded elements, and $\widehat{A}$ for the completion of A.

(i) Check that A° is an integrally closed open subring of A.

(ii) Prove that $\widehat{A}$ is a Huber ring.

(iii) Prove that $(\widehat{A}\,)^\circ = \widehat{A^\circ}$.

(iv) Let A^+ be a ring of integral elements; in other words, (A, A^+) forms a Huber pair. Show that $(\widehat{A}, \widehat{A}^+)$ is a Huber pair.

Notation A non-archimedean field k is a complete topological field whose topology is induced from a height 1 valuation $|-|\colon k \to \mathbb{R}_{\geq 0}$. Note that our convention that k is complete is different from [Hub96, Definition 1.1.3].

It can be easily seen that k° equals the set $\{a \in k \mid |a| \leq 1\}$, where $|-|$ is any height 1 valuation inducing the topology of k.

1.8 Let k be a non-archimedean field. We write $k\langle T_1, \ldots, T_n \rangle$ for the subring of $k[[T_1, \ldots, T_n]]$ consisting of convergent power series

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} a_I T^I \quad \text{such that } \lim_{|I| \to \infty} a_I \to 0.$$

Here, for $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$, we put $T^I = T_1^{i_1} \cdots T_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$. Further, we write $k^\circ\langle T_1, \ldots, T_n \rangle$ for the subring $k\langle T_1, \ldots, T_n \rangle \cap k^\circ[[T_1, \ldots, T_n]]$ of $k\langle T_1, \ldots, T_n \rangle$. Take a topologically unipotent unit ϖ of k, and consider the topology on $k\langle T_1, \ldots, T_n \rangle$ such that $\{\varpi^m k^\circ\langle T_1, \ldots, T_n \rangle\}_{m \geq 0}$ is a fundamental system of open neighborhoods of 0.

(i) Check that $k\langle T_1, \ldots, T_n \rangle$ is a complete Huber ring.

(ii) Prove that $k\langle T_1, \ldots, T_n \rangle^\circ$ coincides with $k^\circ\langle T_1, \ldots, T_n \rangle$.

(iii) Check that $k\langle T_1, \ldots, T_n \rangle$ satisfies the following universal property: for any complete Huber k-algebra A and its power-bounded elements $a_1, \ldots, a_n \in A$, there exists a unique continuous k-algebra homomorphism $\phi\colon k\langle T_1, \ldots, T_n \rangle \to A$ such that $\phi(T_i) = a_i$.

1.9 Let k be a non-archimedean field, and fix a norm $|-|\colon k \to \mathbb{R}_{\geq 0}$. We consider the lexicographic order on $\mathbb{Z}^n_{\geq 0}$. For a non-zero $f = \sum_{I \in \mathbb{Z}^n_{\geq 0}} a_I T^I \in k^\circ\langle T_1, \ldots, T_n \rangle$, we write $\nu(f)$ for the maximal element $\nu \in \mathbb{Z}^n_{\geq 0}$ such that $|a_\nu| = \max_I |a_I|$. We put $\mathrm{LT}(f) = a_{\nu(f)} T^{\nu(f)}$, and call it the leading term of f.

(i) Let $g_1, \ldots, g_m$ be non-zero elements of $k^\circ\langle T_1, \ldots, T_n \rangle$ whose leading terms are monic (i.e., $\mathrm{LT}(g_i) = T^{\nu(g_i)}$). We put $M = \bigcup_{1 \leq i \leq m} (\nu(g_i) + \mathbb{Z}^n_{\geq 0})$. For every $f \in k^\circ\langle T_1, \ldots, T_n \rangle$, find $h_1, \ldots, h_m \in k^\circ\langle T_1, \ldots, T_n \rangle$ such that $f - (h_1 g_1 + \cdots + h_m g_m)$ has no exponent in M.
Hint: choose $a \in k^\circ$ so that the leading term of g_i mod ak° equals $T^{\nu(g_i)}$ for every i, and consider the division in $(k^\circ/ak^\circ)[T_1, \ldots, T_n]$.

(ii) Let I be an ideal of $k^\circ\langle T_1, \ldots, T_n \rangle$. We write $\mathrm{LT}(I)$ for the ideal of $k^\circ\langle T_1, \ldots, T_n \rangle$ generated by $\mathrm{LT}(f)$ for all $f \in I \setminus \{0\}$. Suppose that there exist non-zero elements $g_1, \ldots, g_m \in I$ whose leading terms are monic such that $\mathrm{LT}(I) = (\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_m))$. Prove that I is generated by $g_1, \ldots, g_m$.

(iii) Let I be a non-zero ideal of $k^\circ\langle T_1, \ldots, T_n \rangle$. We assume that I is saturated for a topologically nilpotent unit ϖ of k, that is, $k^\circ\langle T_1, \ldots, T_n \rangle / I$ is ϖ-torsion free. Prove that there exist non-zero elements $g_1, \ldots, g_m \in I$ as in (ii), hence I is finitely generated.
Hint: let L be the subset $\{\nu(f) \mid f \in I \setminus \{0\}\}$ of $\mathbb{Z}^n_{\geq 0}$, which is an ideal of the monoid $\mathbb{Z}^n_{\geq 0}$. Use the fact that any ideal of the monoid $\mathbb{Z}^n_{\geq 0}$ is finitely generated.

(iv) Prove that $k\langle T_1, \ldots, T_n \rangle$ is Noetherian.

1.10 A non-archimedean field K is said to be spherically complete if every decreasing sequence $D_1 \supset D_2 \supset \cdots$ of closed disks in K has non-empty intersection.

(i) Prove that every p-adic field (that is, a finite extension of $\mathbb{Q}_p$) is spherically complete.

(ii) Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Prove that $\mathbb{C}_p$ is not spherically complete.

2 Underlying spaces of adic spaces

2.1 Let (A, A^+) be a Tate Huber pair.

(i) Fix a topologically nilpotent unit ϖ of A. For $v \in \operatorname{Spv} A$ with $v(\varpi) < 1$, we write Γ_v^{ϖ} for the largest convex subgroup of Γ_v such that $v(\varpi)$ is cofinal in Γ_v^{ϖ} (i.e., for any $\gamma \in \Gamma_v^{\varpi}$, there exists $n \geq 0$ such that $v(\varpi)^n < \gamma$).
Prove that the map $\phi\colon \{v \in \operatorname{Spv} A \mid v(a) < 1\ (a \in A^{\circ\circ})\} \to \operatorname{Spv} A;\ v \mapsto v|_{\Gamma_v^{\varpi}}$ (see 1.4 (i)) is well-defined and continuous.

(ii) Observe that the image of ϕ in (i) equals $\operatorname{Cont} A$. Deduce that $\operatorname{Spa}(A, A^+)$ is quasi-compact.

(iii) Recall that a rational subset of $\operatorname{Spa}(A, A^+)$ is a subset of the form

$$U\Big(\frac{f_1, \ldots, f_n}{g}\Big) = \Big\{v \in \operatorname{Spa}(A, A^+) \;\Big|\; v(f_i) \leq f(g) \neq 0\Big\},$$

where $f_1, \ldots, f_n, g \in A$ such that $f_1 A + \cdots + f_n A = A$. Prove that rational subsets form an open basis of $\operatorname{Spa}(A, A^+)$.

2.2 Let (A, A^+) be a Tate Huber pair. Pick a point x of $\operatorname{Spa}(A, A^+)$, and denote by $G(x)$ the set of all generalizations of x.

(i) Prove that $G(x)$ forms a chain; namely, for $y, z \in G(x)$, either y specializes to z or z specializes to y.
Hint: use 1.5.

(ii) Prove that $G(x)$ contains a point y which is a generalization of every point in $G(x)$. Such a point is called the maximal generalization of x.
Hint: use 1.2.

(iii) Let $f \in A$ be an element and $Y = \{v \in \operatorname{Spa}(A, A^+) \mid v(f) = 0\}$ the closed subset defined by f. Prove that Y is stable under generalization.

2.3 Fix a norm $|{-}|\colon \mathbb{C}_p \to \mathbb{R}_{\geq 0}$ of $\mathbb{C}_p$. For a closed disk D in $\mathcal{O}_{\mathbb{C}_p}$, we write $v_D\colon \mathbb{C}_p\langle T\rangle \to \mathbb{R}_{\geq 0}$ for the map $f \mapsto \sup_{x \in D} f(x)$. Further, for a collection $\mathcal{E}$ of closed disks in $\mathcal{O}_{\mathbb{C}_p}$ such that every $D, D' \in \mathcal{E}$ satisfy either $D \subset D'$ or $D \supset D'$, we put $v_{\mathcal{E}} = \inf_{D \in \mathcal{E}} v_D$.

(i) Check that $v_{\mathcal{E}}$ gives a point of $\mathbb{D}^1 = \operatorname{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle)$.

(ii) Observe that $\bigcap_{D \in \mathcal{E}} D$ is one of the following:

– one point $a \in \mathcal{O}_{\mathbb{C}_p}$,
– a disk $\{z \in \mathcal{O}_{\mathbb{C}_p} \mid |z-a| \leq r\}$ with $r \in |\mathcal{O}_{\mathbb{C}_p}^\times|$,
– a disk $\{z \in \mathcal{O}_{\mathbb{C}_p} \mid |z-a| \leq r\}$ with $r \in \mathbb{R}_{>0} \setminus |\mathcal{O}_{\mathbb{C}_p}^\times|$, or
– empty.

In each of the first three cases, describe $v_{\mathcal{E}}$ concretely.

(iii) In each of the cases above, determine all specializations of $v_{\mathcal{E}}$ by using 1.4 (iv).

(iv) Let $v\colon \mathbb{C}_p\langle T\rangle \to \mathbb{R}_{\geq 0}$ be a point with height 1 of $\mathbb{D}^1$. For $a \in \mathcal{O}_{\mathbb{C}_p}$, we write D_a for the closed disk $\{z \in \mathcal{O}_{\mathbb{C}_p} \mid |z-a| \leq v(T-a)\}$. Prove that $v = v_{\mathcal{E}}$ for $\mathcal{E} = \{D_a \mid a \in \mathcal{O}_{\mathbb{C}_p}\}$.

(v) Find all points in $\mathbb{D}^1$.

2.4 An admissible blow-up of a formal scheme $\mathcal{X}$ means the blow-up along a finitely generated open ideal sheaf of $\mathcal{O}_{\mathcal{X}}$. For example, if $\mathcal{X} = \operatorname{Spf} \mathbb{Z}_p\langle T\rangle$, an admissible blow-up is the formal completion along the special fiber of a blow-up $X' \to \mathbb{A}^1_{\mathbb{Z}_p}$ along a closed subscheme which is set-theoretically contained in the special fiber of $\mathbb{A}^1_{\mathbb{Z}_p}$. We write $\Phi_{\mathcal{X}}$ for the set of admissible blow-ups of $\mathcal{X}$, and put $\langle \mathcal{X}^{\mathrm{rig}}\rangle = \varprojlim_{(\mathcal{X}'\to\mathcal{X})\in\Phi_{\mathcal{X}}} \mathcal{X}'$.

(i) Assuming $\mathcal{X}$ is quasi-compact, deduce that $\langle \mathcal{X}^{\mathrm{rig}}\rangle$ is quasi-compact.
Hint: use the following general result due to Stone: if $\{Y_i\}_{i\in I}$ is a filtered projective system of quasi-compact T_0 topological spaces with closed transition maps, the limit space $\varprojlim_i Y_i$ is quasi-compact.

(ii) Let $\mathcal{X} = \operatorname{Spf} \mathcal{O}_{\mathbb{C}_p}\langle T\rangle$. Construct a natural map $\mathbb{D}^1 = \operatorname{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle) \to \langle \mathcal{X}^{\mathrm{rig}}\rangle$.
Hint: use the valuative criterion.

(iii) Describe the image under the map in (ii) of each point of $\mathbb{D}^1$ found in 2.3 (v).

(iv) Prove that the map in (ii) is a homeomorphism.

3 Structure (pre)sheaves of adic spaces

3.1 (i) Prove that $\mathbb{D}^1 = \operatorname{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle)$ is connected.

(ii) Let x be a point of $\mathbb{D}^1$. When is $\mathbb{D}^1 \setminus \{x\}$ non-connected?

3.2 Let (A, A^+) be a Huber pair. For a rational subset U of $\operatorname{Spa}(A, A^+)$, prove that the natural map $\operatorname{Spa}(\mathcal{O}(U), \mathcal{O}^+(U)) \to \operatorname{Spa}(A, A^+)$ induces a homeomorphism between $\operatorname{Spa}(\mathcal{O}(U), \mathcal{O}^+(U))$ and U. (Together with 2.1, we conclude that every rational subset is quasi-compact.)

Hint: first prove that $\operatorname{Spa}(\widehat{A}, \widehat{A}^+) \to \operatorname{Spa}(A, A^+)$ is a homeomorphism.

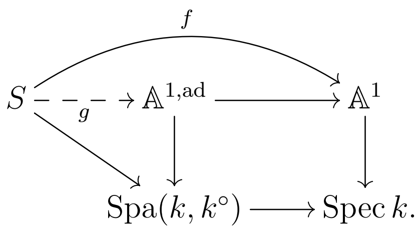
3.3 (i) Let $X = \operatorname{Spa}(A, A^+)$ be an affinoid adic space with complete Huber pair (A, A^+) and B a ring. Prove that morphisms of locally ringed spaces $(X, \mathcal{O}_X) \to \operatorname{Spec} B$ are in bijection with ring homomorphisms $B \to A$.
Hint: the map $X \to \operatorname{Spec} B$ corresponding to $\phi\colon B \to A$ is given by $v \mapsto \{b \in B \mid v(\phi(b)) = 0\}$.

(ii) Let k be a non-archimedean field, and $\varpi \in k$ a topologically nilpotent unit. We put $\mathbb{A}^{1,\mathrm{ad}} = \bigcup_{m \geq 1} \operatorname{Spa}(k\langle \varpi^m T\rangle, k^\circ\langle \varpi^m T\rangle)$. Check that $\mathbb{A}^{1,\mathrm{ad}}$ fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{1,\mathrm{ad}} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \operatorname{Spa}(k,k^\circ) & \longrightarrow & \operatorname{Spec} k, \end{array}$$

where the horizontal arrows are morphisms of locally ringed spaces. Further, prove that $\mathbb{A}^{1,\mathrm{ad}}$ satisfies the following universal property:

> For an adic space S over $\operatorname{Spa}(k, k^\circ)$ and a morphism of locally ringed spaces $f\colon S \to \mathbb{A}^1$ which makes the following diagram commute, there exists a unique morphism of adic spaces $g\colon S \to \mathbb{A}^{1,\mathrm{ad}}$ that makes the diagram commute:



(iii) By extending the construction in (ii), find a definition of the adic space X^{ad} attached to an algebraic variety X over k.

3.4 Let A be a ring and I a finitely generated ideal of A. Assume that A is I-adically complete, and consider the formal scheme $\mathcal{X} = \operatorname{Spf} A$.

(i) Let $Y = \operatorname{Spa}(B, B^+)$ be an affinoid adic space with complete Huber pair (B, B^+). Prove that morphisms of locally topologically ringed spaces $(Y, \mathcal{O}_Y^+) \to \mathcal{X}$ are in bijection with continuous ring homomorphisms $A \to B^+$.
Hint: the map $Y \to \mathcal{X}$ corresponding to $\phi\colon A \to B^+$ is given by $v \mapsto \{a \in A \mid v(\phi(a)) < 1\}$.

(ii) Assume that (A, A) is sheafy (this is the case if A is Noetherian), and put $t(\mathcal{X}) = \operatorname{Spa}(A, A)$. Check that the morphism of locally topologically ringed spaces $\lambda\colon (t(\mathcal{X}), \mathcal{O}^+_{t(\mathcal{X})}) \to \mathcal{X}$ corresponding to $\mathrm{id}\colon A \to A$ satisfies the following universal property: for every adic space Y and a morphism of locally topologically ringed spaces $\mu\colon (Y, \mathcal{O}_Y^+) \to \mathcal{X}$, there exists a unique morphism of adic spaces $f\colon Y \to t(\mathcal{X})$ such that $\mu = \lambda \circ f$.
By this property, we can attach to locally Noetherian formal scheme $\mathcal{X}$ an adic space $t(\mathcal{X})$ by gluing.

3.5 Let V be a discrete valuation ring and $\mathcal{X}$ a locally Noetherian formal scheme over $\operatorname{Spf} V$. We put $F = \operatorname{Frac} V$.

(i) Prove that $t(\operatorname{Spf} V) = \operatorname{Spa}(V, V)$ consists of two points s and η, where s is closed and η is open.

We write $\mathcal{X}^{\mathrm{ad}}_\eta$ for the fiber of $t(\mathcal{X}) \to \operatorname{Spf} V$ at η, and call it the rigid generic fiber of $\mathcal{X}$. The composite map $\mathrm{sp}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{ad}}_\eta \hookrightarrow t(\mathcal{X}) \xrightarrow{\lambda} \mathcal{X} = \mathcal{X}_{\mathrm{red}}$ is called the specialization map.

(ii) Prove that $(\operatorname{Spf} V\langle T\rangle)^{\mathrm{ad}}_\eta = \operatorname{Spa}(F\langle T\rangle, V\langle T\rangle)$.

(iii) Observe that $(\operatorname{Spf} V[[T]])^{\mathrm{ad}}_\eta$ can be regarded as an open disk.
Hint: $(\operatorname{Spf} V[[T]])^{\mathrm{ad}}_\eta \subset t(\operatorname{Spf} V[[T]])$ is not a rational subset. Write it as an increasing union of rational subsets.

(iv) Let X be a scheme of finite type over V, and Y a closed subscheme of the special fiber of X. We write $\mathcal{X}$ (resp. $\mathcal{Y}$) for the formal completion of X along the special fiber (resp. Y). Prove that $\mathcal{Y}^{\mathrm{ad}}_\eta$ is isomorphic to the open adic subspace of $\mathcal{X}^{\mathrm{ad}}_\eta$ whose underlying space is the interior of $\mathrm{sp}_{\mathcal{X}}^{-1}(Y)$ in $\mathcal{X}^{\mathrm{ad}}_\eta$.

When $V = \mathcal{O}_{\mathbb{C}_p}$, I do not know whether $t(\mathcal{X})$ can be defined or not. Nevertheless, for a formal scheme $\mathcal{X}$ locally formally of finite type over $\mathcal{O}_{\mathbb{C}_p}$, one can define its rigid generic fiber $\mathcal{X}^{\mathrm{ad}}_\eta$, which is an adic space locally of finite type over $\operatorname{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$.

3.6 Let k be a non-archimedean field. We put $A = k\langle T\rangle$, and let A' be the integral closure of $k^\circ[A^{\circ\circ}]$ in A (recall that $A^{\circ\circ}$ denotes the set of topologically nilpotent elements in A).

(i) Show that A' equals $\{\sum_{n=0}^\infty a_n T^n \in k^\circ\langle T\rangle \mid a_n \in k^{\circ\circ}\ (n \geq 1)\}$.

(ii) Observe that $\operatorname{Spa}(A, A')$ is partially proper over $\operatorname{Spa}(k, k^\circ)$, and contains $\mathbb{D}^1 = \operatorname{Spa}(A, A^\circ)$ as an open subset.

(iii) Prove that $\overline{\mathbb{D}}^1 = \operatorname{Spa}(A, A')$ is the universal compactification of $\mathbb{D}^1$ in the following sense: for every partially proper adic space Y over $\operatorname{Spa}(k, k^\circ)$, a k-morphism $f\colon \mathbb{D}^1 \to Y$ extends uniquely to $\overline{f}\colon \overline{\mathbb{D}}^1 \to Y$.

(iv) Check that $\mathbb{A}^{1,\mathrm{ad}}$ is partially proper over $\operatorname{Spa}(k, k^\circ)$. Determine the image of the induced map $\overline{f}\colon \overline{\mathbb{D}}^1 \to \mathbb{A}^{1,\mathrm{ad}}$.

(v) Consider the questions (ii), (iii) for more general topologically finitely generated k-algebras.

3.7 Let (A, A^+) is a Tate Huber pair such that A is uniform (i.e., A° is bounded). We put $X = \operatorname{Spa}(A, A^+)$. Let $t \in A$, and consider rational subsets $U = \{v \in X \mid v(t) \leq 1\}$ and $V = \{v \in X \mid v(t) \geq 1\}$. We want to prove the exactness of $0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \to 0$.

Take a ring of definition A_0 of A and a topologically nilpotent unit ϖ of A belonging to A_0. We put $B_0 = A_0[t]$ and write B for the ring A with the topology induced from the ϖ-adic topology on B_0. We put $C = A[1/t]$, $C_0 = A_0[1/t]$ and equip C with the topology induced from the ϖ-adic topology on C_0. Finally, we put $D = A[1/t]$, $D_0 = A_0[t, 1/t]$ and equip D with the topology induced from the ϖ-adic topology on D_0. Note that we have $\widehat{A} = \mathcal{O}_X(X)$, $\widehat{B} = \mathcal{O}_X(U)$, $\widehat{C} = \mathcal{O}_X(V)$, and $\widehat{D} = \mathcal{O}_X(U \cap V)$.

(i) We write $\phi\colon A \to A[1/t]$ for the natural map. Prove that $B_0 \cap \phi^{-1}(C_0) \subset A^\circ$. (In this step we do not need to assume that A is uniform.)

Hint: for $a \in B_0 \cap \phi^{-1}(C_0)$, find $f(T), g(T) \in A_0[T]$ and $c \geq \deg g$ such that $a = f(t)$ and $t^c a = g(t)$. Let $d = \deg f + c$, and $n \geq 0$ be an integer such that $\varpi^n t \in A_0$. Prove that $\varpi^{nd} t^i a^m \in A_0$ for every $m \geq 0$ and $0 \leq i \leq d$ by the induction on m.

(ii) By (i), there exists an integer $n \geq 0$ such that $\varpi^n(B_0 \cap \phi^{-1}(C_0)) \subset A_0$. By using this fact, prove that the exact sequence $0 \to A \to B \oplus C \to D \to 0$ remains exact after completion. This means that the sequence $0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \to 0$ is exact.

3.8 Let (A, A^+) be a stably uniform Tate Huber pair. Put $X = \operatorname{Spa}(A, A^+)$.

(i) Let $t_1, \ldots, t_n \in A$. For a subset $I \subset \{1, \ldots, n\}$, we put $U_I = \{v \in X \mid v(t_i) \leq 1\ (i \in I),\ v(t_i) \geq 1\ (i \notin I)\}$. They form an open covering $\{U_I\}_{I \subset \{1,\ldots,n\}}$ (such an open covering is called a Laurent covering). Prove that $\mathcal{O}_X$ satisfies the sheaf condition with respect to this covering.
Hint: use 3.7.

(ii) Let $a_1, \ldots, a_n \in A$ with $a_1 A + \cdots + a_n A = A$. For $1 \leq i \leq n$, we put $U_i = \{v \in X \mid v(a_i) \leq v(a_j) \neq 0\ (1 \leq j \leq n)\}$. They form an open covering $\{U_i\}_{1 \leq i \leq n}$ (such an open covering is called a rational covering). Assume moreover that $a_1, \ldots, a_n \in A^\times$. Prove that there exists a Laurent covering refining $\{U_i\}_{1 \leq i \leq n}$, and deduce from this fact that $\mathcal{O}_X$ satisfies the sheaf condition with respect to $\{U_i\}_{1 \leq i \leq n}$.

(iii) Let $\{U_i\}_{1 \leq i \leq n}$ be as in (ii), but we do not assume that $a_1, \ldots, a_n$ are units. Prove that there exists a Laurent covering $\mathcal{V} = \{V_J\}$ such that $\{U_i \cap V_J\}_{1 \leq i \leq n}$ is a rational covering of V_J of the type considered in (ii) for every J.

(iv) Prove that every open covering of X can be refined by a rational covering.

(v) Conclude that $\mathcal{O}_X$ is a sheaf.

3.9 Let k be a non-archimedean field, and $\varpi \in k$ a topologically nilpotent unit. We put $A = k[T, T^{-1}, Z]/(Z^2)$. Let A_0 be the k°-submodule of A generated by $\varpi^n T^{\pm n}$, $\varpi^{-n} T^{\pm n} Z$ with $n \geq 0$.

(i) Check that A_0 is a k°-subalgebra of A and $A = A_0[1/\varpi]$.

(ii) We equip A with the topology such that $\{\varpi^n A_0\}_{n \geq 0}$ is a fundamental system of open neighborhoods of 0, and consider $X = \operatorname{Spa}(A, A^\circ)$. Let $U = \{v \in X \mid v(T) \leq 1\}$ and $V = \{v \in X \mid v(T) \geq 1\}$, which are rational subsets of X. Prove that $Z \in \mathcal{O}_X(X)$ is non-zero, and the image of Z under the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is zero. This means that the presheaf $\mathcal{O}_X$ on X is not a sheaf.
Hint: consider the intersection of kZ with A_0, $A_0[T]$ and $A_0[T^{-1}]$.

This problem is taken from [BV16, Proposition 12].

3.10 Let k and ϖ be as in 3.9. Let A_0 be a k°-submodule of $k[T, T^{-1}, Z]$ generated by $(\varpi T)^a (\varpi Z)^b$ with $b \geq 0$ and $a \geq -b^2$.

(i) Check that A_0 is a k°-subalgebra of $k[T, T^{-1}, Z]$.

(ii) We put $A = A_0[1/\varpi]$ and consider the topology on it such that $\{\varpi^n A_0\}_{n\geq 0}$ is a fundamental system of open neighborhoods of 0. Prove that the natural $\mathbb{Z}^2$-grading on $k[T, T^{-1}, Z]$ induces that on A°.
Hint: the crucial point is that the ring of definition A_0 is also graded.

(iii) By using (ii), show that $A^\circ = A_0$, hence A is uniform.

(iv) For a rational subset $U = \{v \in \mathrm{Spa}(A, A^\circ) \mid v(T) \leq 1\}$, prove that $\mathcal{O}(U)$ is not uniform. This means that (A, A°) is not stably uniform.
Hint: observe that $\varpi^{-1}Z \notin A_0[T]$ and $(\varpi^{-n}Z)^{n+1} \in A_0[T]$ for every $n \geq 1$.

This problem is taken from [BV16, Proposition 17]. By slight modification, one can also give a uniform Tate ring A such that $\mathrm{Spa}(A, A^\circ)$ is not sheafy. See [BV16, Proposition 18].

4 Perfectoid spaces

4.1 Let F be a non-archimedean local field. We fix a uniformizer ϖ of F. Let $\mathbb{X}$ be the Lubin-Tate formal group (= 1-dimensional formal $\mathcal{O}_F$-module of height 1) over $\mathcal{O}_F$ such that $[\varpi]_{\mathbb{X}}(T) = \varpi T + T^q$, where q is the cardinality of the residue field of F. We write F_m for the extension field of F obtained by adjoining all roots of $[\varpi^m]_{\mathbb{X}}(T) = 0$. Let $\widehat{F}_\infty$ be the completion of $\varinjlim_m F_m$. Prove that $\widehat{F}_\infty$ is a perfectoid field.

4.2 Let K be a perfectoid field. Prove that if $K^\flat$ is algebraically closed, so is K.

Hint: take an irreducible polynomial $P(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \in K^\circ[T]$. By changing the variable, we may assume that a_0 is a unit (why?). Take $Q(T) = T^d + b_{d-1}T^{d-1} + \cdots + b_0 \in K^{\flat\circ}[T]$ such that the image of $Q(T)$ in $(K^{\flat\circ}/\varpi^\flat)[T]$ is equal to that of $P(T)$ in $(K^\circ/\varpi)[T]$. Pick a root y of $Q(T)$ and approximate a root of $P(T)$ by $y^\sharp$.

4.3 Let A be a uniform complete Tate ring, and p a prime number.

(i) For a topologically nilpotent unit ϖ of A such that $p \in \varpi^p A^\circ$, prove that the pth power map $\Phi\colon A^\circ/\varpi A^\circ \to A^\circ/\varpi^p A^\circ$ is injective.

(ii) Show that the condition that $\Phi\colon A^\circ/\varpi A^\circ \to A^\circ/\varpi^p A^\circ$ is surjective is independent of the choice of a topologically nilpotent unit $\varpi \in A$ with $p \in \varpi^p A^\circ$.

4.4 Let R be a perfect $\mathbb{F}_p$-algebra, and $W(R)$ the ring of Witt vectors with coefficients in R. Let S be a p-adically complete ring. Let $t\colon R \to S$ be a multiplicative map such that the composite $R \xrightarrow{t} S \to S/pS$ is a ring homomorphism. Prove that the map $T\colon W(R) \to S$ defined by

$$T\Big(\sum_{n=0}^{\infty} p^n[a_n]\Big) = \sum_{n=0}^{\infty} p^n t(a_n) \quad (a_n \in R)$$

becomes a ring homomorphism. Check also that T is surjective if the composite $R \xrightarrow{t} S \to S/pS$ is.

4.5 Let R be a perfectoid $\mathbb{F}_p$-algebra, and $\xi = \sum_{n=0}^{\infty} p^n[a_n]$ ($a_n \in R^\circ$) be an element of $W(R^\circ)$. We say that ξ is primitive of degree 1 if a_0 is topologically nilpotent and a_1 is a unit of R°.

(i) Prove that a primitive element of degree 1 is a non-zero-divisor.

(ii) Prove that $\xi \in W(R^\circ)$ is primitive of degree 1 if and only if there exist $u \in W(R^\circ)^\times$, $\alpha \in W(R^\circ)$ and a topologically nilpotent unit $\varpi \in R^\circ$ such that $u\xi = p + \alpha[\varpi]$.
Hint: note that $(\xi - [a_0])/p$ is a unit of $W(R^\circ)$.

4.6 Let A be a perfectoid ring.

(i) Use 4.4 to construct a surjective ring homomorphism $\theta\colon W(A^{\flat\circ}) \to A^\circ$.

(ii) Check that $\theta([x]) = x^\#$ for $x \in A^{\flat\circ}$.

(iii) Take topologically nilpotent units $\varpi \in A$ and $\varpi^\flat \in A^\flat$ such that $p \in \varpi^p A^\circ$ and $(\varpi^\flat)^\sharp = \varpi$. Pick $\alpha \in W(A^{\flat\circ})$ such that $\theta(\alpha) = p/\varpi$ and put $\xi = p - \alpha[\varpi^\flat]$. Prove that ξ generates $\operatorname{Ker}\theta$.

4.7 (i) Let K be the completion of $\mathbb{Q}_p(\mu_{p^\infty})$, which is a perfectoid field of characteristic 0. Prove that $K^\flat$ is isomorphic to the completion of $\mathbb{F}_p((T^{p^{-\infty}}))$. Find a generator of $\operatorname{Ker}\theta$ (see 4.6) in this case.

(ii) Answer the same question for the completion of $\mathbb{Q}_p(p^{p^{-\infty}})$.

4.8 Let $\{X_i\}$ be a filtered projective system of adic spaces whose transition maps are quasi-compact and quasi-separated. For a perfectoid space X, we write $X \sim \varprojlim_i X_i$ if the following conditions are satisfied:

- A compatible family of morphisms $\phi_i\colon X \to X_i$ is given and the induced map $|X| \to \varprojlim_i |X_i|$ on the underlying spaces is a homeomorphism.
- For each $x \in X$, there exists an affinoid open neighborhood U of x such that the image of $\varinjlim_{(i,U_i\subset X_i)} \mathcal{O}_{X_i}(U_i) \to \mathcal{O}_X(U)$ is dense. Here U_i runs through affinoid open subsets of X_i which contain $\phi_i(U)$.

(i) For a perfectoid Huber pair (B, B^+), show that the map

$$\operatorname{Hom}(\operatorname{Spa}(B,B^+), X) \to \varprojlim_i \operatorname{Hom}(\operatorname{Spa}(B,B^+), X_i)$$

is bijective. Conclude that a perfectoid space X satisfying $X \sim \varprojlim_i X_i$ is unique up to isomorphism.

(ii) Let K be a perfectoid field of residue characteristic p. Let us consider the projective system $(\cdots \xrightarrow{\phi^{\mathrm{ad}}} \mathbb{A}^{n,\mathrm{ad}} \xrightarrow{\phi^{\mathrm{ad}}} \cdots \xrightarrow{\phi^{\mathrm{ad}}} \mathbb{A}^{n,\mathrm{ad}})$, where $\phi\colon \mathbb{A}^n \to \mathbb{A}^n$ is given by $(x_1,\ldots,x_n) \mapsto (x_1^p,\ldots,x_n^p)$. Check that there exists a perfectoid space X over K such that $X \sim \varprojlim_{\phi^{\mathrm{ad}}} \mathbb{A}^{n,\mathrm{ad}}$.

4.9 Let $X = \operatorname{Spa}(A, A^+)$ be an affinoid perfectoid space, and Z a closed subset of X defined by $f_1 = \cdots = f_n = 0$ for $f_1,\ldots,f_n \in A$.

(i) Fix a topologically nilpotent unit $\varpi \in A$. For $m \geq 0$, let U_m be the open neighborhood of Z defined by $\{v \in X \mid v(f_i) \leq v(\varpi^m)\}$. Prove that there exists a perfectoid space $\widetilde{Z}$ such that $\widetilde{Z} \sim \varprojlim_m U_m$.

(ii) Let Y be a perfectoid space and $\phi\colon Y \to X$ a morphism whose set-theoretic image is contained in Z. Prove that ϕ uniquely factors through $Y \to \widetilde{Z}$. In particular, $\widetilde{Z}$ is independent of the choice of $f_1, \dots, f_n$ and ϖ.

4.10 Let K be a perfectoid field of characteristic 0, and p the residue characteristic of K.

(i) Let A be a complete Tate K-algebra satisfying the following conditions:

(a) Every element of $1 + A^{\circ\circ}$ has a pth root in A.

(b) A is uniform.

Prove that A is a perfectoid K-algebra.
Hint: first observe that a pth root of $a \in 1 + A^{\circ\circ}$ can be taken from $1 + A^{\circ\circ}$.

(ii) Let A be a Tate K-algebra satisfying the condition (a) in (i). Take a topologically nilpotent unit ϖ of K and equip A with the new topology such that $\{\varpi^m A^\circ\}$ is a fundamental system of open neighborhoods of 0. Let $\widehat{A}$ denote the completion of A with respect to this topology. Prove that $\widehat{A}$ satisfies the conditions (a), (b) in (i), hence is a perfectoid K-algebra.

(iii) Let $X = \operatorname{Spa}(B, B^\circ)$ be an affinoid adic space of finite type over $\operatorname{Spa}(K, K^\circ)$. Prove that there exist a filtered projective system $\{X_i\}$ of finite étale covers of X and a perfectoid space X_∞ over K such that $X_\infty \sim \varprojlim_i X_i$.

This problem is taken from [Col02, §2.8] and [Sch13, Proposition 4.8].

4.11 Let K be a perfectoid field of characteristic 0, and G a finite group acting on K. Let us prove that K^G is a perfectoid field. Note that the surjection $\theta\colon W(K^{\flat\circ}) \to K^\circ$ in 4.6 is G-equivariant.

(i) Prove that for every integer $m \geq 0$ there exists a topologically nilpotent unit ϖ in K^G such that $p \in \varpi^{p^{m+1}} K^\circ$.
Hint: find ϖ of the form $\theta([u])$ with $u \in K^{\flat\circ}$.

(ii) Assume first that $|G| = p^m$. Take ϖ as in (i). For $x \in K^{G\circ}$, pick $y \in K^{\flat\circ}$ such that $\theta([y]) \equiv x \pmod{pK^\circ}$ and put $z = \prod_{g\in G} g(y)^{1/p^{m+1}}$. Check that $\theta([z]) \in K^{G\circ}$ and $x \equiv \theta([z])^p \pmod{\varpi^p K^{G\circ}}$. This shows that K^G is a perfectoid field.
Hint: use 4.3.

(iii) Prove that K^G is a perfectoid field for general G.

(iv) Repeat the argument above to prove the following claim: for a perfectoid K-algebra A and a finite group G acting on A, A^G is a perfectoid K^G-algebra.

This problem is taken from [KL16, Theorem 3.3.25].

4.12 Let K be a perfectoid field of characteristic $p > 0$. Modify 3.9 to construct a Huber K-algebra A satisfying the following condition: $X = \operatorname{Spa}(A, A^\circ)$ is covered by affinoid perfectoid spaces, but $\mathcal{O}_X$ is not a sheaf.

This problem is taken from [BV16, Proposition 13].

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