p-ADIC HODGE THEORY

BHARGAV BHATT

p-adic Hodge theory is a p-adic counterpart of classical Hodge theory: it studies the natural structures found on the cohomology of algebraic varieties over a p-adic field. Some of these structures (such as a Galois action) arise from the arithmetic of the base field, while others (such as a Frobenius action) arise from the geometry of integral models. The aim of this lecture series is to discuss some of these structures, and paint a picture of the rich web that connects them; we will use the language of perfectoid spaces [Sc1] to develop this story.

1. Lecture series

Goals. The lecture series will be loosely structured around various aspects of the Hodge-Tate decomposition, which is the p-adic counterpart of the Hodge decomposition over the complex numbers.

Rational aspects. Our main goal is to build up the techniques necessary to establish the Hodge-Tate decomposition, due to Tate [Ta] (for abelian varieties) and Faltings [Fa1] (in general).

Theorem 1.1. *Let C be the completion of an algebraic closure of $\mathbf{Q}_p$. Let X/C be a smooth projective variety. Then there exists a "Hodge-Tate" filtration on $H^n(X_{et}, C)$ whose graded pieces are given by the Hodge cohomology groups of X. Further, if X is defined over a finite extension $K/\mathbf{Q}_p$ contained in C, then this filtration can be upgraded to a Galois equivariant decomposition*

$$H^n(X_{et}, C) \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/C})(-j), \tag{1}$$

where the twist on the right is the Tate twist.

We will approach Theorem 1.1 using perfectoid geometry, as in [Sc2], emphasizing examples such as curves and abelian varieties (to make contact with the lectures of Caraiani). Our primary objective is to clearly explain why differential forms arise naturally in relating X to perfectoid spaces lying over X.

Integral aspects. Towards the end of the lecture series, we will discuss the integral analog of the Hodge-Tate decomposition. In fact, a decomposition cannot exist integrally (as examples show). Nevertheless, there is a close relation between Hodge and étale cohomology integrally, which has the following concrete consequence from [BMS]:

Theorem 1.2. *Fix X and C as in Theorem 1.1; let $k := \overline{\mathbf{F}_p}$ be the residue field of the ring of integers $\mathcal{O}_C \subset C$. Assume that X has good reduction, i.e., there exists a smooth proper $\mathcal{O}_C$-scheme $\mathfrak{X}$ such that $X := \mathfrak{X} \otimes_{\mathcal{O}_C} C$ is the generic fibre of $\mathfrak{X}$. Let $\mathfrak{X}_k := \mathfrak{X} \otimes_{\mathcal{O}_C} k$. Then we have an inequality*

$$\dim_{\mathbf{F}_p} H^n(X_{et}, \mathbf{F}_p) \leq \sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}). \tag{2}$$

The primary objective of this part of the series is to explain a picture that realizes Hodge cohomology as a "specialization" of étale cohomology over a base that one might call "$\mathbf{Z}_p \otimes_{\mathbf{F}_1} \mathbf{Z}_p$" (cf, the lectures of Kedlaya and Weinstein).

2. PROJECTS

(1) **Understanding torsion discrepancies.** There are (at least) 4 natural integral cohomology theories attached to a scheme $\mathfrak{X}$ as in Theorem 1.2: étale, Hodge, de Rham and crystalline. Each of these is a finitely presented module over a p-adic valuation ring, and all 4 have the same rank by fundamental results of p-adic Hodge theory. The torsion subgroups, however, may be quite different; for example, [BMS, §2] shows that its possible for crystalline cohomology to have torsion even when the étale cohomology does not.

The goal of this project is to find examples where the torsion subgroups in all 4 theories are distinct. A natural starting point, as in [BMS, §2], is to construct "interesting" finite flat group schemes over $\mathcal{O}_C$, and to consider cohomology of quotients of smooth projective schemes by free actions of such groups.

More ambitiously, one might try to bound the discrepancy in the torsion orders in terms of the geometry of X, or even prove that the torsion in one theory is always "$\leq$" the torsion in another theory. For example, one of the main results of [BMS] asserts that the torsion in étale cohomology is a lower bound for that in either de Rham or crystalline cohomology. However, as far as I know, no other analogous implications relating the other 3 theories are known.

(2) **Extensions to non-compact varieties.** In practical applications, the properness assumptions on X and $\mathfrak{X}$ in Theorem 1.2 are somewhat restrictive: it is much more natural to allow that both X and $\mathfrak{X}$ be non-compact provided they admit compactifications $\overline{X}$ and $\overline{\mathfrak{X}}$ with a normal crossings boundary. The goal of this project is to investigate the extent to which the inequality (2) is valid in this context. As a first step, one might consider the so-called "vertical" case, i.e., consider compact X equipped with a semistable model $\mathfrak{X}$, and compare the étale cohomology of X with the *logarithmic* Hodge cohomology of $\mathfrak{X}_k$.

REFERENCES

[BMS] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*, available at https://arxiv.org/abs/1602.03148

[Fa1] G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. 1, 1 (1988), 255-299.

[Sc1] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245-313.

[Sc2] P. Scholze, *p-adic Hodge theory for rigid analytic varieties*, Forum of Mathematics, Pi (2013), Vol. 1, e1, 77 pages.

[Ta] J. Tate, *p-divisible groups*, in Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, pages 158-183, 1967.