ARIZONA WINTER SCHOOL NOTES

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Dedicated to the memory of Torsten Ekedahl, who had a mind unlike any other.

The interplay among "arithmetic", "topology", and "geometry" has been a central theme in algebraic geometry since long before the Weil conjectures. This course is intended to give a taste of a number of related questions where ideas in one area directly imply ideas in another, or (more subtly) suggest through metaphor statements one should hope/believe/expect/prove to be true.

1. THE GROTHENDIECK RING OF VARIETIES

A central player in this series of lectures will be the Grothendieck ring of varieties, over a given field k. To set terms, let $\mathcal{V}_k$ be the set of finite type schemes over a field k, or (it will quickly not matter) varieties over k. You are welcome to imagine k as your favorite field, which depending on the person may be $\mathbb{Q}$ or $\mathbb{C}$ or $\mathbb{F}_p$, but we will want to be flexible about what k is.

Define the **Grothendieck ring of varieties** $R = R(\mathcal{V}_k)$ (nonstandard notation!) as follows.

- As an additive group, R is generated by symbols of the form $[X]$, where X is a variety (up to isomorphism).

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- The additive structure of R is generated by the following. If $Z \subset X$ is a closed embedding/subset, with complement U, then we say $[X] = [U] + [Z]$. (Exercise: show that $[X] = [X^{red}]$.)
- Multiplication in R is defined by the relation if W and X are varieties, then $[W][X] = [W \times X]$.

This turns R into a commutative ring.

1.1. *Examples.* $0 = [\varnothing]$, $1 = [pt]$. For convenience, define $\mathbb{L} = [\mathbb{A}^1]$ (an important definition for us!). The $[\mathbb{P}^2] = \mathbb{L}^2 + \mathbb{L} + 1$.

1.A. EXERCISE. if $W \to X$ is a $\mathbb{P}^n$-bundle (in the Zariski topology), show that $[W] = [X][\mathbb{P}^n]$.

For motivation: one of the central ideas of mathematics is turning general phenonena into linear phenomena, which we can calculate with. A first example is calculus, which turns differentiable things into linear things. Another example is "cohomology" which turns complicated things into groups or rings, from which much information can be extracted. This is some very blunt way of turning the category of varieties into a ring.

1.2. Example: "classical" geometry/topology.

Let's get a feel for this by applying this in an analogous case you might have some intuition for, that of "nice topological spaces" (say, finite CW-complexes).

Then $[\mathbb{R}] = [\mathbb{R}^{<0}] + [0] + [\mathbb{R}^{>0}]$, so $[\mathbb{R}] = 2[\mathbb{R}] + 1$, from which $[\mathbb{R}] = -1$ and $[\mathbb{R}^k] = (-1)^k$. Then by decomposing our spaces into such cells, we get that $[X]$ is some integer — and it turns out that $[X] = \chi_c(X)$, the Euler characteristic of cohomology with compact supports. (Important note: if X is smooth and compact, then this is just cohomology with compact supports agrees with usual cohomology: $h^i(X) = h^i_c(X)$, so $\chi(X) = \chi_c(X)$: $\sum(-1)^i h^i_c(X) = \sum(-1)^i h^i(X)$.)

You may think that there is not much going on here here. The Grothendieck ring is only $\mathbb{Z}$. Lots of information is lost. For example,

$$[S^1] = [S^1 \times S^1] = [S^1 \coprod S^1] = [\varnothing] = 0.$$

But in fact, even here there is something going on. How do we know that $1 \neq 0$?! How do we know that 1 is torsion-free? (Maybe $R = \mathbb{Z}/17$...) Making this work requires having some understanding of cohomology with compact supports.

And furthermore, once you have done this, you have a number of nontrivial results (such as, for example, that if you "cut up a genus g compact oriented surface into contractible pieces, with the "map" having v vertices, e edges, and f faces, then $v - e + f = 2 - 2g$ — Euler's formula.

This example drives home the fact that you should think of the map $V \to R$ from varieties to the Grothendieck ring as some sort of "universal Euler characteristic", at least for anything that you should think of as "cohomology with finite supports". For this reason, this should be seen as a baby version of a motive (encompassing only certain kinds of cohomology-like things), the ring R is sometimes called the ring of baby motives. But unlike the theory of motives, the definition is short.

So let's see what we get when we move into objects that are algebraic as well as geometric. It is hard to figure out what the ring is, but we can try to understand it through various quotients — in other words, maps from V to some ring A that descends to a map from R to that ring A. Such a map is called a *motivic measure*.

1.3. Example: $k = \mathbb{F}_q$**.**

First, suppose k is a finite field $\mathbb{F}_q$. Then each variety has a finite number of rational points — finitely many $\mathbb{F}_q$-valued points.

1.B. EXERCISE. Show that this respects the addition and multiplication relations.

Hence we have a "point-counting" map $\#: R_{\mathbb{F}_q} \to \mathbb{Z}$. And we have already shown that R is not the 0-ring. Furthermore, we have a clue that counting points should be thought of as some sort of Euler characteristic. We will return to this idea soon.

1.4. Example: $k = \mathbb{C}$**.**

Next suppore k is $\mathbb{C}$, a seemingly quite different situation. Certainly every complex variety is a finite type CW-complex, so we have a map $\chi_c(X, \mathbb{Z}) : R_{\mathbb{C}} \to \mathbb{Z}$. This is already useful. For example, as $\mathbb{CP}^2$ is smooth and complete (complete = proper over k; over $\mathbb{C}$, complete = compact), we can immediately see that $\chi(\mathbb{CP}^2) = 3$, as $\chi_c(\mathbb{L}) = 1$.

But there is more — the fact that a geometric object is *algebraic* imposes more structure than you might hope. If X is smooth and complete (a compact complex algebraic manifold), then *we recover each* $h^i_c(X, \mathbb{Q})$*, not just their alternating sum*. For example, for $\mathbb{CP}^2$, we can recover from $[\mathbb{CP}^2] = \mathbb{L}^2 + \mathbb{L} + 1$, we can recover that

$$h^0(\mathbb{CP}^2) = 1, \quad h^1(\mathbb{CP}^2) = 0, \quad h^2(\mathbb{CP}^2) = 1, \quad h^3(\mathbb{CP}^2) = 0, \quad h^4(\mathbb{CP}^2) = 1.$$

This is an incredibly deep, important, and amazing fact, and comes from Deligne's theory of weights. (We will see how to prove this shortly.)

If you haven't worked extensively with cohomology, I want to point out how striking this is. By considering h^0, we see that the number of components comes from this. In other words, if you take a bunch of smooth compact varieties, cut them into pieces, and rearrange them in a different way into a different bunch of smooth compact varieties, the number appearing will be the same. The dimension of the biggest component is also fixed. (The dimension of the smaller components are not determined, though — can you think of an example? One will come up soon — see (1).)

1.C. EXERCISE. find the cohomology groups of products of projective spaces, for example $\mathbb{P}^2 \times \mathbb{P}^1$. Find the cohomology groups of a Grassmannian, say the Grassmannian $G(2,4)$ of two-dimensional subspaces of $\mathbb{C}^4$.

Even without smoothness or compactness, you can still make predictions. Rather than making this precise, I will just give an examples.

From $[\mathbb{C}^*] = \mathbb{L} - 1$, we can interpret the $\mathbb{L}$ as $h^0(\mathbb{C}^*) = 1$ and $h^1(\mathbb{C}^*) = -1$. (There is a rough recipe, which we will not describe. But the $\mathbb{L}$ corresponds to the h^0, and the (-1) corresponds to the h^1, with the sign indicating that the cohomology is odd.)

You can get even more information on the cohomology groups. In particular, if X is smooth and complete, then you can recover the *Hodge structures* on the cohomology groups. If you don't know what that is, there is no need to worry; just be aware that it is an addition structure on the cohomology groups with $\mathbb{C}$-coefficients, taking into account the $\mathbb{Q}$-coefficients.

And even if X is not smooth and complete, you still get strong Hodge-theoretic information, by way of the notion of "mixed Hodge structures". (As with the case of $\mathbb{C}^*$, this allows you to make predicitons. This idea has been very fruitful — see for example [HN] and [HVR].)

1.5. How to get at this ring in general: Bittner's presentation in characteristic 0.

The fact that we can such different bits of information out of the ring, and in particular consequences of Deligne's theory of weights, forces us to ask: what other information is hiding in this ring? What other Euler characteristics are out there? What *is* this ring? We really have very little idea about this question, but what ideas we have are very interesting.

Note that $R = R(\mathcal{V}_k)$ is generated as an abelian group by classes of smooth affine varieties. This means that it is generated as an abelian group by classes of smooth projective varieties.

If $Z \hookrightarrow X$ is a closed embedding of a smooth projective variety in another smooth projective variety, then if $\mathrm{Bl}_Z X$ is the blow-up of X along Z with exceptional divisor $E_Z X$, then clearly

$$[Bl_Z X] - [E_Y X] = [X] - [Z] \tag{1}$$

1.6. *Theorem (Bittner,* [Bi]*).* — R *is isomorphic to the free abelian group on clsases of smooth irreducible projective varieties, modulo such relations.*

The proof is very short, and uses the (very useful) Weak Factorization Theorem of Abramovich, Karu, Matsuki, and Wlodarcyzk [AKMW], which states that any birational map between smooth varieties can be factored into a sequences of blow-ups and blow-downs along smooth center. The Weak Factorization Theorem is only proved in characteristic 0.

1.7. Applications of Bittner's presentation.

Because Bittner's presentation involves only smooth projective varieties, it allows us to determine new motivic measures, by defining them on smooth projective varieties, and showing that they satisfy this one single relation (1).

For example, there is a motivic measure $a_0 \colon V \to Z$, which is the "virtual number of components": write any variety X as a combination of smooth varieties. Write each of these smooth varieties in terms of smooth *projective* varieties. Then the number of such varieties (counted appropriately) depends only on X!

1.D. EXERCISE. Work out this invariant for a nodal cubic curve in $\mathbb{A}^2$.

1.E. EXERCISE. Show that there is motivic measure $a_i \colon V \to Z$ defined by $a_i(X) = h^i(X, \mathcal{O}_X)$ for $i > 0$ for smooth projective X.

1.F. EXERCISE. If you know a little about the properties of Hodge structures, and how they change by blowing up, show that the Hodge structure on h^i is also a motivic measure.

1.8. *Application: Bittner duality on* $R_{\mathbb{L}}$*.*

Here is an unexpected application.

1.9. *Theorem (Bittner duality). — There is a duality on* $R_{\mathbb{L}}$ *sending* X *to* $L^{-\dim X}X$ *for smooth irreducible projective varieties.*

In particular, $\mathbb{L} \mapsto 1/\mathbb{L}$, and $[\mathbb{P}^1] = \mathbb{L} + 1 \mapsto 1 + 1/\mathbb{L}$.

This is really a duality, and somehow has to do with Poincare duality! Notice what it does to $\mathbb{P}^1$.

I see this as one of the many ways in which nature is telling us to invert $\mathbb{L}$. But if you invert $\mathbb{L}$, you might lose information: Remember that $R_{\mathbb{L}} = R[1/\mathbb{L}] = R[x]/(x\mathbb{L} - 1)$. Localizations are not always injective; $R \to R_{\mathbb{L}}$ will kill any class α such that $\alpha\mathbb{L} = 0$.

For this reason there has long been a question/speculation/conjecture (Denef-Loeser [DeL] and many others):

1.10. *Conjecture. — The class* $\mathbb{L}$ *is not a* 0*-divisor. In other words,* $R \to R_{\mathbb{L}}$ *is an injection.*

This has recently been shown to be false in a dramatic work of Lev Borisov; more on this in §3.19.

1.11. *Proof of Bittner duality (Theorem 1.9).* Suppose Z is codimension c in X, and both are smooth. Then

$$\begin{aligned} [E_Z X] &= [\mathbb{P}^{c-1}][Z] \\ (\mathbb{L}-1)[E_Z X] &= (\mathbb{L}^c - 1)[Z] \\ \text{(subtracting (1))} \quad [Bl_Z X] - \mathbb{L}[E_Z X] &= [X] - \mathbb{L}^c[Z] \end{aligned}$$

Dividing by $\mathbb{L}^{\dim X}$, we get

$$[Bl_Z X]/\mathbb{L}^{\dim Bl_Z X} - [E_Z X]/\mathbb{L}^{\dim E_Z X} = [X]/\mathbb{L}^{\dim X} - [Z]/\mathbb{L}^{\dim Z}$$

as desired. □

1.12. *Application: connection to stable birational equivalence classes.*

Recall that two irreducible varieties X and Y are birational if they have "isomorphic open subsets". They are said to be *stably birational* if $X \times \mathbb{A}^m$ is birational to $Y \times \mathbb{A}^n$ for some m and n. Even taking into account the difference in dimension, stable birationality is a strictly weaker equivalence relation than birationality. (It is an amazing fact due to Beauville, Colliot-Thelene, Sansuc, and Swinnerton-Dyer [BCTSSD] that there are nonrational varieties that are nonethless stably rational.)

Stable birational equivalence classes of irreducible varieties form a commutative semigroup $\mathbb{Z}[SB]$.

1.13. *Theorem (Larsen-Lunts,* [LL]*). — There is a unique ring homomorphism* $R \to \mathbb{Z}[SB]$ *sending* X *to* X *for all smooth projective varieties* X. (But not for other varieties!)

Proof. Clearly there is at most one such map. And this map respects Bittner's relation (1). □

So we have a new motivic measure! What information have we lost? From the fact that $\mathbb{P}^1$ and a point map to the same thing (they are both rational), $\mathbb{L}$ maps to 0, so any multiple of $\mathbb{L}$ maps to 0. In fact we lose "nothing else".

1.14. *Theorem (Larsen-Lunts,* [LL]*). — The kernel is* $(\mathbb{L})$.

This is less easy.

1.G. EXERCISE. If you have some topological experience, try out some examples that are smooth but not projective, or projective but not smooth, or not smooth or projective.

2. SYMMETRIC POWERS, AND THE MOTIVIC ZETA FUNCTION

We now come to a central player in our discussions: the *motivic zeta function*. This is a notion which was formally defined remarkly late — in an unpublished manuscript of

Kapranov in 2000 — and should have been defined far earlier. (I think Grothendieck had discussed this idea in a letter to Serre, but I can't remember the reference.)

2.1. Definition (Kapranov, [Ka]**).** The motivic zeta function of X is defined as

$$Z_X(t) := \sum [\mathrm{Sym}^n X] t^n \in R[[t]].$$

Let me try to motivate this definition — or more precisely, try to get across how incredibly well-motivated this is. First, note the following useful fact.

2.A. EXERCISE. Prove that if $X = U \coprod Z$ where Z is closed and U is open, then $Z_X(t) = Z_U(t) Z_Y(t)$.

Thus we can really define $Z_X(t)$ for any X in the Grothendieck ring R. We have a map of *groups*

$$Z\colon R \to (1 + tR[[t]]) \subset R[[t]]^\times,$$

turning $+$ into $\times$.

2.2. Example: "classical" geometry/topology.

To get a feel for this, let's start topologically, thinking about "usual geometric spaces" (say finite CW-complexes). For example, Sym^n of a point pt is just a point, so

$$Z_{pt}(t) = 1 + t + t^2 + \cdots = 1/(1-t).$$

We already know that the (topological analogue of the) Grothendieck ring contains no more information than the Euler characteristic with compact supports.

2.3. *Theorem (Macdonald, 1962,* [M]*). — We have*

$$Z_X(t) = \sum \chi_c(\mathrm{Sym}^n X) t^n = \frac{1}{(1-t)^{\chi_c(X)}}.$$

(We will prove this soon, as a consequence of Theorem 2.11.)

So the Euler characteristic of $\mathrm{Sym}^n X$ is

$$(-1)^n \binom{-\chi_c(X)}{n} = (-1)^n \frac{(-\chi_c(X))(-\chi_c(X)-1)\cdots(-\chi_c(X)-n+1)}{n!}$$

— it depends only on $\chi_c(X)$! If X is a compact manifold, then you can erase the c's — I won't say why.

This is a great theorem! Better yet, it has a one-line proof.

Proof. As $Z_X(pt) = 1/(1-t)$, and $[X] = \chi_c(X)[pt]$, and $R \to R[[t]]$ sends addition to multiplication, the result follows. □

2.4. Example: $k = \mathbb{F}_q$**.**

As before (§1.3), we next try this out in the case of a finite field $\mathbb{F}_q$. Here we have the point-counting function $\#\colon R \to \mathbb{Z}$. Applying this to the motivic zeta function, we get the Weil zeta function:

2.B. EXERCISE. Show that $\#(Z_X(t)) = \zeta_X(t)$.

Recall that the Weil zeta function is usually defined as follows. Take X, and count points over all extensions of $\mathbb{F}_q$. Put these in a generating function in an appropriate way, and massage the function a little.

Exercise 2.B tells us that we could instead just count $\mathbb{F}_q$-points of the symmetric powers — in a geometer's mind, a much more direct and short definition.

Weil's reason for introducing his zeta function was because of his thoughts around the Weil conjectures connecting number theory and topology and algebraic geometry, the famous "Rosetta stone" he described in a letter to his sister Simone Weil. In particular, the first part of the Weil conjectures is that the Weil zeta function is always *rational*. (This was first proved by Dwork [Dw] in 1960, well before the rest.)

So notice now that our two examples of motivic zeta functions (or at least zeta functions coming from motivic measures) are rational, for very different reasons (Macdonald, and Weil/Dwork). This leads to a natural question, that should have been asked in the 1960's, but was first asked (to my knowledge) in 2000.

2.5. *Question (Kapranov, unpublished* [Ka]*).* Is $Z_X(t)$ rational?

Here k can be any field. I want to convince you that this is an important question, and that you should really believe that the answer should be yes. Let me first start with evidence.

If X is a curve the answer is yes. (With a rational point: use Weil's proof of rationality of his zeta function. Without a rational point: needs more care; shown by Daniel Litt.)

If $k = \mathbb{C}$, then it is true for "Hodge structures" (Cheah's 1994 Chicago Ph.D. thesis [C1] — more soon, see Theorem 2.10).

If $k = \mathbb{F}_q$, then it is true for point-counting by the rationality of the Weil zeta function $\zeta_X(t)$ (Dwork, 1960).

So that is a lot of evidence.

But even better: if the conjecture is true, it would *imply* the rationality of the Weil zeta function, thereby by giving a direct and *very geometric* proof of that part of the Weil conjectures, and would give a serious attack on most of the rest of them! (Dwork's proof was very different, and was definitely a "number theorist's proof" rather than a "geometer's proof" — more on this later.)

And then this would even have strong consequences in characteristic 0. If I tell you that a power series is actual rational, say $f(t) = g(t)/h(t)$ (f a power series, g and t polynomials), then from $f(t)h(t) = g(t)$ will give you a recursive way to compute the coefficients of f.

So if we knew the first few $\mathrm{Sym}^n X$, we would know them all, by "cutting-and-pasting". For each X, there would be an explicit recipe!

Thus we have a statement which is short, beautiful, with lots of evidence over lots of fields, and which would have dramatic consequences. So you have to believe that this result is true. But:

2.6. *Theorem (Larsen-Lunts, 2003-4,* [LL, Theorem 1.6]*). — The answer is "no" if $k = \mathbb{C}$!! In fact, there is a motivic measure to a field, such that for all smooth projective surfaces X with $h^2(\mathcal{O}_X) = h^0(\omega_X) = h^{2,0}(X) \geq 2$, the motivic zeta function $Z_X(t)$ is not rational.*

This is a really amazing result. We do not know what the Grothendieck ring R is, and we understand it only through "motivic measures", which are sorts of Euler characteristics. In every case we knew, it looked polynomial. (More on this soon, see Theorem 2.10.) So Larsen and Lunts had to come up with a dramatically different kind of Euler characteristic. This Euler characteristic turns out to vanish on $\mathbb{L}$, so their argument *does not show that the zeta function is not rational in $R_{\mathbb{L}}$.*

2.7. *Big philosophical question.* Where in between R and $\mathbb{Z}$ does the zeta function start being rational? What makes the zeta function rational? Is the zeta function rational over $R_{\mathbb{L}}$? (Note: this is not ended to be an exercise for the evening sessions...)

So at this point we are disappointed and frustrated; we wanted to get at the Weil conjectures, and we did not succeed. But for now, we look for inspiration back over $\mathbb{C}$.

2.8. Example: $k = \mathbb{C}$.

Recall Macdonald's Theorem (Theorem 2.3, restated here in a slightly different way), which was the topological analogue of rationality.

2.9. *Theorem (Macdonald, 1962,* [M]*). —*

$$Z_X(t) = \sum \chi_c(\mathrm{Sym}^n X)t^n = \frac{(1-t)^{h^1}(1-t)^{h^3}\cdots}{(1-t)^{h^0}(1-t)^{h^2}\cdots}$$

When cohomology has additional structure, then we could hope to extend the theorem to this case. For example, Hodge structure is this kind of additional structure.

2.10. *Theorem (Cheah* [C1, C2]*). — If* X *is a complex variety, then*

$$\sum_{n,p,q,r\geq 0} (h^{p,q}(H_c^r(\mathrm{Sym}^n X))x^p y^q (-z)^r)\, t^n = \prod_{p,q,r\geq 0} \left(\frac{1}{1-x^p y^q z^r t}\right)^{(-1)^r h^{p,q}(H_c^r(X))}$$

(We will prove this below, as a consequence of Theorem 2.11.)

Here, $H_c^r(\mathrm{Sym}^n X) = H_c^r(\mathrm{Sym}^n X, \mathbb{C})$ is just rth cohomology with compact support, with coefficients in $\mathbb{C}$. This comes with a Hodge structure (technically, a "mixed Hodge structure", defined by Deligne). Although I haven't defined it, it comes with pieces of dmiension $h^{p,q}$ for various integers p and q. More preicsely, the vector space H^r is filtered by vector spaces of these dimensions. All you need to know is that this is some number, and $\sum h^{p,q}(H^r) = h^r$. So this is an enriched version of Macdonald's Theorem 2.9, which now takes into account the extra structure we have on the cohomology, thanks to the lucky fact that we are working in algebraic geometry, and not plain old topology.

We next try to replace numbers with vector spaces. I don't just want the size of the cohomology groups — I want the cohomology groups, along with whatever information they may have. (The modern call to arms: "Categorify!")

Let's try to make this work. I want to state a theorem, and then figure out how to define the words and symbols in it. We will find ourselves walking in Grothendieck's footsteps, and seeing where ideas come from.

"Hodge structures" are cohomology groups with some more information, so we want to remember more than vector spaces. So "cohomology" H^r takes as input varieties over k, the category $\mathcal{V}_k$. (In the case of Hodge structures, $k = \mathbb{C}$.) It maps to VS^+, the category finite-dimensional vector spaces (over some field) with some "additional structure", which respects the vector space structure (so we can have kernels, cokernels, etc. — VS^+ is an abelian category). There is no reason to have the vector space be over k, and we will see soon that we will want it to be characteristic 0, even if k is not. We want the additional structure to respect things like $\oplus$, $\otimes$, $\wedge$, etc.

Cohomology is functorial, so we want contravariant functors $H^i \colon \mathcal{V}_k \to VS^+$.

We want to talk about Euler characteristics $\sum(-1)^i H^i(X) \in K(VS^+)$, so to have this sum make sense, we require *Axiom 1 of 3:* For each $X \in \mathcal{V}_k$, $H^i(X) = 0$ for $i \gg 0$. Then we have an Euler characteristic

$$\chi(X) := \sum (-1)^i H^i(X) \in K(VS^+)$$

("virtual vector spaces with additional structure" — the Grothendieck group of vector spaces with additional structure).

2.11. *Theorem (??!!). —*

$$\sum_{i=0}^{\infty} \chi(\mathrm{Sym}^i X) t^i = \frac{1}{(1-t)^{\chi(X)}}$$

This is just a restatement of Macdonald's Theorem (in its incarnation Theorem 2.3), except that the Euler characteristic has grown up to be something extremely general. What does the statement of the theorem mean?! The left side lies in $K(VS^+)[[t]]$, so the right side should too. (And when we apply the "dimension" function $K(VS^+) \to \mathbb{Z}$, we should recover Macdonald's Theorem.)

For $V \in VS^+$, define

$$(1-t)^V := \sum_{i=1}^{\infty} (-1)^i (\wedge^i V) t^i.$$

Note that $(1-t)^V(1-t)^W = (1-t)^{V \oplus W}$ by properties of $\wedge$ and $\oplus$.

2.C. EXERCISE (IMPORTANT FUN FACT). Show that $1/(1-t)^V$ is the generating function for symmetric powers $\mathrm{Sym}^n V$. (Here, of course, we are taking symmetric powers of the vector space V in the usual way.)

So now Theorem 2.11 means something precise. We now prove Theorem 2.11, but we will require two additional axioms.

Axiom 2 of 3 (the Kunneth formula): We have a natural isomorphism

$$H^n(X \times Y) \cong \oplus_{i=0}^n H^i(X) \otimes H^{n-i}(Y).$$

If T is a variety and G is a finite group of automorphisms of T, then define $\phi \colon T \to T/G$. Then (as cohomology is a functor) we have $\phi^* \colon H^i(T/G) \to H^i(T)$. Also, by functoriality of cohomology, the G-action on T induces a G-action on $H^i(T)$, and ϕ^* maps $H^i(T/G)$ to $H^i(T)^G$. *Axiom 3 of 3:* ϕ^* identifies $H^i(T/G)$ with $H^i(T)^G$. (This is where we will need characteristic 0, as we will want G-representations to behave nicely.) This fact was proved in cohomology in Grothendieck's Tohoku paper [Gr]. So the Kunneth Axiom is reasonable to ask of any "cohomology-like" functor.

Given these axioms, we will now prove Theorem 2.11!

2.12. *Proof of Theorem 2.11.* We have $H^*(\mathrm{Sym}^n X) \cong H^*(X^n)^{S_n}$ by Axiom 3. But all of the cohomology groups of X^n can be formally written in terms of the cohomology groups of X, in some universal way, by Axiom 2. Then Axiom 3, (with the action of the symmetric group) is given to us (again, by functoriality of cohomology).

So now to check

$$\sum_{i=0}^{\infty} \chi(\mathrm{Sym}^i X) t^i = \frac{1}{(1-t)^{\chi(X)}}$$

we just have to check something completely formal, where we just expand both sides in terms of H^i for $i = 1$ through ∞. (In fancy-sounding language, we have a power series in "functors", which we can then apply to any X.) You should believe that this "must" be true, and can then go ahead and check this algebra fact.

2.D. EXERCISE (NOT SO EASY, BUT CAN BE DONE CLEANLY). Prove this fact.

□

Applying Theorem 2.11 to Hodge structures, we find:

2.13. *Corollary.* — Cheah's Theorem 2.10 is true.

(Why did it take until 1994 to know this?)

As another application: counting rational points on varieties over $\mathbb{F}_q$ can be interpreted in this way. It is not hard to prove that the number of $\mathbb{F}_q$ points is $\sum_i (-1)^i \operatorname{tr}(F|_{H^r(X,\mathbb{Q}_\ell)})$ where F is the action of Frobenius. So we take VS^+ to be the category of $\mathbb{Q}_\ell$ vector spaces, along with a representation of Frobenius. We have to check that etale cohomology satisfies our three axioms (which is true). Once we have done that, we have:

2.14. *Corollary.* — The Weil zeta function of any variety is rational!

3. MORE CONJECTURES/QUESTIONS/SPECULATIONS ABOUT THE GROTHENDIECK RING OF VARIETIES

The conjecture that the motivic zeta function was rational was false, but it took us to some very interesting places. This is why conjectures/questions/speculations about the Grothendieck ring of varieties R are so interesting — they can only be proved *or* disproved in an interesting way. Different people have different opinions about these conjectures, but my general opinion is that every such conjecture/question/speculation is wrong, but wrong for interesting reasons. I will now describe other examples.

3.1. The motivic stabilization of symmetric powers conjecture.

Given the Weil conjectures (or even our proof of rationality, modulo our three axioms), the Weil zeta function of X (proper geometrically irreducible of dimension d) looks like this:

$$\zeta_X = \frac{p_1(t)p_3(t)\cdots p_{2d-1}(t)}{p_0(t)p_2(t)\cdots p_{2d}(t)} = \sum a_i t^i$$

where $p_i(t)$ is a polynomial of degree $h^i(X)$, and $p_0(t) = 1 - t$ and $p_{2d} = (1 - q^d t^{2d})$.

How do these coefficients grow? As motivation, we consider a question from when we learned about sequences and series. Consider a rational function whose coefficients are in $\mathbb{C}$, expanded as a power series (around 0). How do its coefficients grow? Answer: You take the root in the denominator of smallest size — assume for convenience that there is only 1 — then the coefficients will grow like powers of the inverse of that root.

3.A. EXERCISE. Prove this, perhaps using partial fractions.

The Weil conjectures include implications about the sizes of the roots. The biggest inverse root is q^d, coming from the top cohomology class. This implies that the t^i coefficient of the Weil zeta function $\zeta_X(t)$ grows like $(q^d)^i$ (as i gets large, and X is fixed). This can be interpreted as the Lang-Weil estimates, which are much easier than the Weil conjectures. In other words, we (as a species) knew this rate of growth (and it is very very useful) without knowing rationality of the Weil zeta function. In the case of the motivic zeta function, perhaps the analogue of this might be true, despite the lack of rationality.

3.2. *Motivic Stabilization of Symmetric Powers Conjecture/Question/Speculation* [VW]. — *Suppose* X *is geometrically irreducible. Then* $\mathrm{Sym}^n X/\mathbb{L}^{n \dim X} \in \hat{R}_{\mathbb{L}}$ *converges.*

(If X is also smooth and projective, then the left side should be interpreted as essentially the Bittner dual of $\mathrm{Sym}^n X$.)

We need to make sense of this completion on the right. Things on the left are in $R_{\mathbb{L}}$, but the denominators will get bigger and bigger. So we need to complete in some sense. We do this by filtering R (and hence $R_{\mathbb{L}}$) by dimension; I will skip the details. (This completion was first introduced by Kontsevich in a quite different context, in his theory of motivic integration [Kon].)

So motivated by the Lang-Weil estimates, we wonder if the Conjecture 3.2 is true.

There are other reasons to hope that it is true.

(i) We know it is true when you specialize to Hodge structures, because we have rationality of the zeta function with that motivic measure.
(ii) We know the analogue is true when you specialize to point-counting, because we know the rationality of the Weil zeta function.
(iii) I had mentioned that the motivic zeta function is rational when X is a curve (and the case when X has a point you can do as an exercise).
(iv) It is true for stably rational varieties, even though these motivic zeta functions may not be rational [VW]. (But as discussed above, maybe this motivic zeta function with $\mathbb{L}$ inverted *is* rational!)

All of these feel number theoretic. But there are very topological reasons for this.

If X is a reasonable topological space, with a point p, then we have maps $\mathrm{Sym}^n X \to \mathrm{Sym}^{n+1} X$ (by "adding a copy of p"). Long before we arithmetic and algebraic geometers got into this, the topologists knew that as n gets big this "stabilizes". To first approximation, and most relevant for us: for any i, once you have a big enough n, then, this map becomes an isomorphism on H^i. But better yet: the homotopy stabilizes up to "codimension i". (Arnav Tripathy can explain exactly how high you have to go for each i. Ask him — he has a very pretty proof.) So we have some "limiting homotopy type" $\mathrm{Sym}^\infty X$. And we even know what it is — this is the Dold-Thom theorem, which gives the answer in terms of the cohomology groups of X — it doesn't depend on the homotopy type of X, just the cohomology groups.

This is an example of a stabilization result in topology, and such results long predate our analogous understanding in arithmetic and algebraic geometry. Harer stability is another example.

Another remark: we had shown, modulo an axiom or two, that the cohomology groups of Sym^n of a topological space depend only on the cohomology groups of the topological space, and the analogue was arithmetically powerful, taking us to the Weil conjectures. In topology, more is true: Sym^n works "up to homotopy" — Sym^n is a functor from homotopy types to homotopy types. This is actually a surprising statement, and leads to interesting facts on the arithmetic/algebraic side.

I want to mention two results in this vein.

Algebraically, we would want X and $X \times \mathbb{A}^1$ to be "homotopic" to each other. But $\mathrm{Sym}^n(X)$ and $\mathrm{Sym}^n(X \times \mathbb{A}^1)$ can be very different. For example, $\mathrm{Sym}^n(\mathbb{A}^m)$ is singular.

3.3. *Theorem (Totaro's observation, about 2001,* [Gö]*). — In the Grothendieck ring of varieties,* $[\mathrm{Sym}^n(X \times \mathbb{A}^1)] = [\mathrm{Sym}^n X] \times [\mathbb{A}^n]$.

(Totaro did the case where X is a point, but his argument works in general.) He had a very clever one-line argument using Hilbert 90 — can you find it? To give you a more precise question:

3.B. EXERCISE. Show that you can cut-and-paste $\mathrm{Sym}^n \mathbb{A}^m$ into $\mathbb{A}^{nm}$. Example: $\mathrm{Sym}^n \mathbb{A}^1 = \mathbb{A}^n$, even without cut-and-pasting. (This is a priori stronger than just being equal in the Grothendieck ring — you can cut $\mathrm{Sym}^n \mathbb{A}^m$ into locally closed pieces, and rearrange them to form $\mathbb{A}^{nm}$. We discuss this more in §3.6 below.)

Before Totaro's idea, it had been a big deal to show that $\mathrm{Sym}^n \mathbb{A}^m$ was even rational! And his proof is nonconstructive (just as Hilbert 90 can be) — and after his argument, an actual cut-and-pasting was found. This should really have been known in the 19th century.

Much harder, and related to this winter school:

3.4. *Theorem (Tripathy, 2015* [Tr]*). —* "Sym" and "étale realization" commute.

(If you know something about étale homotopy theory, you will realize that this opens the door to a lot of great things. As just one example, it leads to an "étale Dold-Thom Theorem".)

3.5. *Back to the Motivic Stabilization of Symmetric Powers Conjecture 3.2.*

So this conjecture is simultaneously motivated by the Weil conjectures in arithmetic geometry, and the Dold-Thom theorem in topology. This combination would potentially lead to something very useful. Recall that when we proved the rationality of various zeta functions by a "universal method", we needed cohomology functors. It isn't clear that we have such things in the Grothendieck ring R. More generally, one of Grothendieck's standard conjectures (which is clearly motivated by this point of view) is that there exist such functors for motives (not baby motives — the real thing).

But in Dold-Thom, the functor $X \mapsto \mathrm{Sym}^n X$ takes something and breaks it up into something that depends only on its cohomology groups — so if the Motivic stabilization conjecture 3.2 is true, it might give us a place in which we could break up a baby motive into cohomology groups, and thus do many many things. (Again, I should repeat that I do not believe the conjecture is true... But at least it makes clear that the conjecture is interesting...)

3.6. The Piecewise Isomorphism Conjecture.

If X and Y are "piecewise isomorphic" — if they can both be cut up into the same locally closed pieces — then clearly $[X] = [Y]$.

For example, $[Bl_p\mathbb{P}^2] = [\mathbb{P}^1 \times \mathbb{P}^1]$.

3.7. *Piecewise Isomorphism Conjecture (see for example* [LL, Question 1.2]*). — If* X *and* Y *are varieties with* $[X] = [Y]$, *then* X *is piecewise isomorphic to* Y.

For example, you can cut $\mathrm{Sym}^n \mathbb{A}^m$ into pieces, and rearrange them to make $\mathbb{A}^{mn}$ (Exercise 3.B). This is a priori a stronger statement than $[\mathrm{Sym}^n \mathbb{A}^m] = [A^{mn}]$ in R.

3.8. *Theorem (Liu-Sebag* [LS]*). — Suppose* k *is algebraically closed of characteristic* 0, *and* X *and* Y *are varieties over* k, *with* $[X] = [Y]$ *(in* R*). If one of the following holds:*

(i) $\dim X = 1$
(ii) X *is a smooth projective surface*
(iii) X *contains a finite number of rational curves*

Then X *and* Y *are piecewise isomorphic. In other words, the Piecewise Isomorphism Conjecture 3.7 holds in these cases.*

This is yet another theorem I find remarkable. (How would you even start to go about proving such a thing?) Notice in particular the "geometric" case (iii) — somehow the geometry of rational curves has a lot to do with cutting and pasting. (This will come up again in the discussion of Litt's work below, see Theorem 3.15.)

3.9. *Question.* Is the Piecewise Isomorphism Conjecture 3.7 true in positive characteristic?

3.10. Another false hope: is R an integal domain?.

Recall that there are many clues that we want to invert $\mathbb{L}$ in R. (Some additional ones: Kontsevich's theory of motivic integration; the idea that X and $X \times \mathbb{A}^1$ should be "homotopic" in some sense; ...) Thus it is important to ask if $\mathbb{L}$ is zero-divisor (or equivalently, that $R \to R_{\mathbb{L}}$ is an injection).

But there is a prior question: is R an integral domain? The answer is "no" in characteristic 0 in general. Kollár gave an example where the field is not algebraically closed, when there is a smooth conic not isomorphic to $\mathbb{P}^1$ [Kol]. And Poonen gave an example where the field *is* algebraically closed, [P1]. The key ideas for both are very simple (and thus clever).

Here is a sketch of Poonen's argument. We work over $\mathbb{C}$. It is not too hard to come up with nonisomorphic abelian varieties A and B such that $A \times A \cong B \times B$, by thinking in terms of lattices.

Thus $([A]+[B])([A]-[B]) = 0$. Now we know that $[A]+[B] \neq 0 = [\varnothing]$, as we know that the class in the Grothendieck ring of a smooth projective variety determines the number of components.

So the key to show that $[A] \neq [B]$. To do this, Poonen finds a useful motivic measure. To each smooth projective variety X, we associate its Albanese variety $\operatorname{Alb} X$, which is an abelian variety that X maps to. The formation of the Albanese commutes with products ($\operatorname{Alb}(X \times Y) = \operatorname{Alb} X \times \operatorname{Alb} Y$). Even better, it depends only the stable biraionality type of the variety (because it is easily seen to be a birational invariant, and $\operatorname{Alb} X \cong \operatorname{Alb} X \times \mathbb{P}^1$. Thus we have a map from the Grothendieck ring to the semiring generated by abelian varieties

$$R \to \mathbb{Z}[AV].$$

by composing Larsen and Lunts' map $R \to \mathbb{Z}[SB]$ with the Albanese map $\mathbb{Z}[SB] \to \mathbb{Z}[AV]$.

In fact, R isn't even reduced — Ekedahl gives an example on Mathoverflow, see [E2]. Ekedahl's example is, like Poonen's, constructed using clever number theory. (He starts with a maximal order in a definite quaternion algebra over $\mathbb{Q}$.)

Poonen's example even works over $\mathbb{Q}$. Ekedahl's works over some number field.

3.11. *Question.* Can anyone produce examples over finite fields?

3.12. Demolishing conjectures with geometry.

Recall the importance of $\hat{R}_{\mathbb{L}} = \lim R[1/\mathbb{L}]$. An analogous construction, that is "Bittner-dual", is the limit $R' = \lim R/\mathbb{L}^n$.

3.13. *Question.* Is $\cap(\mathbb{L}^n) = 0$? In other words, is $R \to R'$ is an injection? Or is information lost?

3.14. *Properties of* R'.

(i) In some sense, this is a better completion than $\hat{R}_{\mathbb{L}}$. More measures on R extend to R', such as point-counting — if $k = \mathbb{F}_q$, then we get a map $R_{\mathbb{L}}$ to the p-adics.

(ii) Because (L) is in the kernel of the map $R \to Z[SB]$, we have $R' \to Z[SB]$.

(iii) $R'[\mathbb{L}^{-1}] \to \hat{R}_{\mathbb{L}}$ is surjection. If $\mathbb{L}$ is not a zero-divisor, then this is an isomorphism (of topological rings).

We begin with a geometric fact.

3.15. *Theorem (Litt* [Li1]*). — Suppose X is smooth irreducible projective surface with $h^0(X, \omega_X) \neq 0$. Suppose $m > 0$ is fixed. Then for $n \gg 0$, $\mathrm{Sym}^n(X)$ is not stably birational to $\mathrm{Sym}^m(X)$.*

In other words, in the list of symmetric powers of $\mathbb{C}$, each stable birational class appears only finitely many times! This is in stark contrast to the case of rational varieties (where $\mathrm{Sym}^n X$ is always stably birational to a point, using Totaro's trick, Theorem 3.3), and the case of irreducible curves with a rational point, (where $Sym^n X$ is stably birational to the Jacobian of the associated smooth projective curve, for $n \gg 0$).

The proof fundamentally uses the geometry of rational curves on X.

This has some alarming implications for the Grothendieck ring (all due to Litt).

3.16. *Corollary. — For such an X, the limit $\lim \mathrm{Sym}^n X$ does not exist in R', beause it doesn't exist even in $R/(\mathbb{L})$.*

The definition of this limit R' looks "Bittner-dual" to the limit in the Motivic Stabilization of Symmetric Powers Conjecture 3.2.

3.17. *Corollary. — The MSSP Conjecture 3.2 and the "$\mathbb{L}$ not a 0-divisor" conjecture cannot both be true.*

Proof. Recall §3.14(iii): if $\mathbb{L}$ is not a 0-divisor, then $R'_{\mathbb{L}} \to \hat{R}_{\mathbb{L}}$ is an isomorphism. So if the limit doesn't exist on the left, then it doesn't exist on the right.

3.18. *Corollary. — The MSSP Conjecture 3.2 contradicts the Piecewise isomorphism Conjecute* **??**.

Proof: If piecewise-isomorphism holds, then $[X]/\mathbb{L}^{\dim X} = [Y]/\mathbb{L}^{\dim Y}$ modulo negative dimensional pieces if and only if X is stably birational to Y. □

Thus these conjectures are so strong that they seem to contradict each other!

3.19. Borisov's contruction.

A recent preprint of Lev Borisov [Bo] is beautiful and stunning.

3.20. *Theorem (Borisov, [Bo]). — The piecewise polynomiality conjecture fails. And* $\mathbb{L}$ **is** *a zero-divisor.*

The construction is elegant and classical, and the proof is elegant and modern. His argument is very short (the preprint is 6 pages long and readable).

The counterexample involves the difference of two Calabi-Yau threefolds X and Y — he shows that $([X] - [Y])(\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = 0$. The idea is motivated by thoughts on mirror symmetry, and the Pfaffian-Grassmannian double mirror correspondence. He then shows that X and Y are not stably birational. Notice that his counterexample, and Poonen's counterexample, both involved the difference of two "Calabi-Yau" manifolds (where the canonical bundle is trivial) — is this a coincidence?

3.21. *Question.* Is there a counterexample to these conjectures in positive characteristic (assuming, for example, resolution of singularities)?

4. Stabilization in the Grothendieck ring of varieties

After giving some rather negative results, we continue on a happier note, with positive results, on discriminants. With regards to discriminants, there are three perspectives on the world: topological, arithmetic, and algebro-geometric. The theme we will see: discriminant (complements) stabilize as "the problem goes to ∞", and the limit can be explicitly described in terms of the zeta functions.

4.1. Baby Example: Points on a line (no topology).

Space of n unordered points on a line $\mathbb{A}^1$ (over a given field k) is precisely $\mathbb{A}^n$, because n unordered numbers are determined by the polynomial with those numbers as roots:

$$\{x^n + a_{n-1}x^n + \cdots + a_0 = 0\} = \mathbb{A}^n_k.$$

For each partition ν of n, define the ν*-discriminant locus*

$$\Delta_\nu \subset \{x^n + a_{n-1}x^n + \cdots + a_0 = 0\}$$

corresponding to the locus where the "root pattern" is ν, or some degeneration thereof. For conveience, let Δ°_ν be the locally closed subset where the "root pattern" is precisely ν, so $\Delta_\nu = \overline{\Delta^\circ_\nu}$.

For example, the traditional "discriminant locus" is $\Delta_{1^{n-2}2} \subset \mathbb{A}^n$. The case of $\nu = m1^{n-m}$, of polynomials with a root of multiplicity m (or worse) will be our jumping-off point. We consider this problem from the perspective of three different fields of mathematics.

4.2. *Arithmetic:* $k = \mathbb{F}_q$. There are finitely many polynomials of degree n (q^n in fact). The probability of having m-fold root (or worse) is $\#\Delta_{1^{n-m}m}/q^n = 1/q^{m-1}$ if $n \geq m$.

4.A. EXERCISE. Prove this! (How you prove this will show whether you are a number theorist or an algebraic geometer.)

4.3. *Topology:* $k = \mathbb{C}$.

4.4. *Theorem (Arnol'd, plus a bit more). — We have* $h^i(\mathbb{A}^n_{\mathbb{C}} \setminus \Delta_m) = 1$ *if* $i = 0, 2m-1$, *and* 0 *otherwise.*

Thus Δ_m has only one nonzero compactly supported cohomology group.

4.5. *Algebraic geometry.* Δ_m cut-and-pastes to $\mathbb{A}^{n-m+1}$.

4.B. EXERCISE. Prove this. (You may have already done this, depending on how you solved Exercise 4.A.)

This exercise is not too hard — certainly easier than Arnol'd's Theorem 4.4. Thus we should have *predicted* Arnol'd result. But we didn't.

4.6. *Stabilization.*

Notice that each of these answers "stabilizes" as $n \to \infty$. In arithmetic, this stabilization is stupid — the answer is constant (once $n \geq m$).

In topology, it initially looks like this stabilization is stupid, but in fact, we have something better than an isomorphism — this isomorphism comes from a map.

4.7. *Theorem (Arnol'd/Galatius). — The space of complex polynomials with no* m*-fold root has precisely two nonzero cohomology groups:* $h^0 = h^{2m-1} = 1$. *The map from the space of monic complex polynomials of degree* n *with no* m*-fold root to this space is a homotopy equivalence once* $n \geq m$.

This argument does not generalize to more complicated discriminants — it is "local" in nature.

However, we *do* have stabilization (in the algebro-geometric and arithmetic settings) for more complicated discriminants. For example:

4.8. *Algebraic geometry.*

$$[\Delta_{1^{n-a-b}ab}]/[\mathbb{A}^n] = 1/\mathbb{L}^{a+b-2}.$$

This immediately implies the following.

4.9. *Arithmetic.* The probability of a polynomial having an a-fold root and a b-fold root, or a degeneration thereof, is $1/q^{a+b-2}$. (We do not know how to show this in any other way!)

4.10. *Topology.* Just as we should have predicted Arnol'd's Theorem 4.4, this gives a explicit prediction on cohomology. But this prediction is not proved!

We get even more interesting discriminants once we have more multiple points. As the simplest example: $\Delta_{1^n 223}$ stabilizes, but is not eventually constant. It is instead a power series in $1/\mathbb{L}$ (which yet again suggests the importance of $\hat{R}_{\mathbb{L}}$).

4.11. Configuration spaces of points on an arbitrary variety X.

We can replace $\mathbb{A}^1$ ("the line") in this question by an arbitrary variety X. Roughly, $\mathbb{A}^1$ (or more generally $\mathbb{A}^m$) is the case with "no topology", and now we are adding topology.

The stabilization story starts in topology. The topology of the space Δ^0_ν was recently shown to stabilize by Church [Ch] and Randal-Williams [RW].

Then in algebraic geometry, assuming that X satisfies the Motivic Stabilization of Symmetric Powers Conjecture 3.2 (for example, if X is stably rational), or working in Hodge strucutres (if $k = \mathbb{C}$), or working in point-counting (if $k = \mathbb{F}_q$), then $[\Delta^0_{1^n\nu}]$ stabilizes. The result can be stated in terms of motivic zeta functions. (See [VW] for all discussion of the stabilization in the Grothendieck ring.)

It turns out that $[\Delta_{1^n\nu}]$ stabilizes, and the result is much cleaner. Thus somehow tells us that the closure is the (a?) "right" object to look at. This suggests in turn topological conjectures (stated in the introduction to [VW]), without any suggestion of how to prove it. Many of these conjectures are proved (in improved and better forms) by Kupers, Miller, and Tran [KMT], and Tommasi [To1, To2].

The algebro-geometric results also directly imply the corresponding statements in arithmetic (not relying on the Motivic Stabilization of Symmetric Powers Conjecture 3.2). There is no other (e.g. directly arithmetic) proof of these facts.

The case of the affine line $\mathbb{A}^1$ discussed above generalizes essentially without change to affine space $\mathbb{A}^d$ in general. This can be seen as the case where there is "no topology".

The probability of n points being distinct is $1/Z_X(1/\mathbb{L}^{2d})$; this it not hard. Better: the probability of there being exactly m multiple points (where "n points being distinct" is the case $m = 0$) is:

$$\lim_{n\to\infty} = \frac{?}{Z_X(1/\mathbb{L}^{2d})} = \frac{\cdots + (\mathrm{Sym}^{N \text{ into } m \text{ points (with mult.)}} X)(1/\mathbb{L}^{2d})^N + \cdots}{1 + \cdots + (\mathrm{Sym}^N X)(1/\mathbb{L}^{2d})^N + \cdots} \tag{2}$$

(This can be written in closed form in terms of motivic zeta values, and symmetirc powers of X. For example, for $m = 0$, we get $1/Z_X(t)$, and for $m = 1$ get $X\mathbb{L}(1 - 1/\mathbb{L}^{2d})/Z$.)

These result lead to specific predictions/conjectures/speculations in arithmetic geometry and topology. For example, $h^i_c(\Delta_{1^n\gamma}(X))$ should stabilize as $n \to \infty$, and if $X = \mathbb{A}^d$, then precisely two of these groups should be non-zero (and should be one-dimensional). For more statements of this sort (and also serious progress by a number of different authors), see the introduction to [VW].

4.12. Hypersurfaces.

The "baby example" of points on a line can be generalized in another way too: it is the first case of hypersurfaces on an arbitrary variety. Here are some clues that we should expect to see some structure.

(i) The first motivation, of course, is the structure of the space of points on a line.

(ii) A second motivation is the classical problem: what is the probability that an integer is square-free? The answer is $6/\pi^2$, which should be understood of as a (Riemann) zeta-value: $1/\zeta(2)$. This should be further understood as $1/\zeta(\dim \mathbb{Z} + 1)$, as we will soon see.

(iii) A third motivation, much harder, is a theorem of Poonen [P2]. Suppose X is a smooth projective variety over $\mathbb{F}_q$. An important question (unresolved before [P2]) is: does there exist a smooth very ample divisor on X? (This would allow the use of various inductive arguments, for example.) Poonen answers this question as follows. Suppose $\mathcal{L}$ is an ample line bundle on X. Let $p(n)$ be the probability that a section of $\mathcal{L}^{\otimes n}$ is smooth. We wish to show that $p(n) \neq 0$ for some $n > 0$.

4.13. *Theorem (Poonen).* — $\lim_{n\to\infty} p(n) = 1/\zeta_X(1/q^{\dim X+1})$.

Here ζ is the Weil zeta function.

(iv) A fourth motivation is from a theorem of Vassiliev in topology.

4.14. *Theorem (Vassiliev).* — *The space of smooth divisors on $\mathbb{C}^n$ has two nonzero cohomology groups.*

(One is obviously h^0. Can you guess the other? Possible hint: Theorem 4.7.) For a wonderful discussion on this and related results, see [Va].

This should leave you believing that related results should hold in the Grothendieck ring, somehow philosophically connecting all of the above. That is indeed the case.

4.15. *Theorem* [VW]. — *Suppose X is a smooth projective variety of dimension n over some field k, and that $\mathcal{L}$ is an ample line bundle on X. Let $p(n)$ be the "motivic probability" of a section of*

$\mathcal{L}^{\otimes n}$ *to have* exactly m *singularities. Then*

$$\begin{aligned}\lim_{n\to\infty} p(n) &= \frac{\cdots + (\mathrm{Sym}^{N \textit{ into } m \textit{ parts}} X)(1/\mathbb{L}^{d+1})^N + \cdots}{Z_X(1/\mathbb{L}^{d+1})} \\ &= \frac{\cdots + (\mathrm{Sym}^{N \textit{ into } m \textit{ parts}} X)(1/\mathbb{L}^{d+1})^N + \cdots}{\cdots + (\mathrm{Sym}^N X)(1/\mathbb{L}^{d+1})^N + \cdots}\end{aligned}$$

4.16. *Remarks.*

(i) Notice the similarity to (2). But the zeta function is evaluated at a different value. It is not clear why the answers to two diferent questions are the same.

(ii) Theorem 4.15 does not depend on the choice of ample line bundle $\mathcal{L}$.

(iii) Even if you are only interested in working over the complex numbers (unlikely at the Arizona Winter School!), the proof uses finite fields.

(iv) If $m = 0$, we get $1/Z_X(1/\mathbb{L}^{d+1})$. This is parallel to Poonen's Theorem 4.13, but the statement logically independent, and the proof is completely different.

(v) The answer can be cleanly written in a closed form in terms of zeta-values and symmetric powers of X.

(vi) Theorem 4.15 suggests enhanced versions of results in arithmetic. For example, it suggests how the statement of Poonen's Theorem 4.13 should be extended to count hypersurfaces with any given number of singularities. This was proposed as one of the problems for the Arizona Winter School, but before the school even began, Joseph Gunther proved it [Gu]!

(vii) Theorem 4.15 suggests enhanced versions of results in topology, extending Vassiliev's Theorems. For example, it suggests that the cohomology of space of smooth divisors in $|\mathcal{L}^{\otimes n}|$ stabilizes. For example, in the case of smooth plane curves was shown to stabilize by Tommasi [To1]. Another example in this vein is Kupers-Miller-Tran [KMT].

(viii) Vague speculative question: when we see a motivic zeta value in a limit, what should we expect to see topologically?

5. BHARGAVOLOGY

I now want to take this point of view somewhere different. As above, any new ideas here are joint with Melanie Wood.

Here is a motivating question. Fix a positive integer $d > 1$. How many number fields of degree d are there? In other words, how many degree d extensions of $\mathbb{Q}$ are there? Taken at face value, the answer is obvious: there are an infinite number of extensions. There is only one way to make sense of this: by turning it into a limit, each of whose entries is appropriatelly normalized. So instead we count how many degree d number fields there are with discrminant at most N, figure out how that might grow, normalize, and take the limit as $N \to \infty$, suitably normalized. It turns out that the right growth rate (provable for small d, conjectural in general) is linear. So we divide the number of degree d extensions of discriminant at most N, and take the limit as $N \to \infty$.

(If you squint, this is exactly what is going on in the Motivic Stabilization of Symmetric Powers Conjecture 3.2.)

For $N = 2$ and 3, this was known "classically" (by which I mean "pre-Bhargava"). Bhargava then dealt with the cases $N = 4$ and $N = 5$ in his totemic papers [Bh1, Bh2]. His arguments including serious work in the geometry of numbers, and so seem far removed from arithmetic geometric methods. And his answers turn out, magically, to be written in terms of *Riemann zeta values* (and other similar gadgets).

But under the surface, until the last part of his argument, his ideas use classical and ancient geometry, which has been used in the past on analogous problems, and (I argue) should be used in the future on other analogous problems. And we should look for analogous answers involving zeta functions!

(In the final lecture, some ongoing work, and speculative prospects for new work, were described. I may update these notes further to include some of that discussion.)

6. Conclusion

The Grothendieck ring of varieties R is a very blunt tool. In some sense we know very little about it, but we should hope that any "blunt" facts in geometry/topology/arithmetic should be understandable in this "baby motivic" language. Bittner's presentation allows us to get a handle on it, giving (for example) a way to prove things about weights, Hodge theory, etc. Unfortunately, it is known only in characteristic 0.

Rationality questions lead us to the right way to think about the Weil conjectures, even though it does not lead to a new proof.

All conjectures about the Grothendieck ring seem to be wrong, but all for interesting reasons, to do (simultaneously) with geometry and arithmetic.

The completed localized ring $\hat{R}_{\mathbb{L}}$ seems to be the "right" gadget for many things. Possibly the fact that $\mathbb{L}$ is not a 0-divisor is good news, not bad news — $\mathbb{L}$ may "kill off" things making R behave badly.

Motivated by decades of results about stabilization in topology, there is a recent burst of activity about stabilization in the Grothendieck ring, as well as in arithmetic settings. As examples, we have seen results about configutations of points on varieties, hypersurfaces on varieties, and a host of facts paralleling Bhargava's theorem.

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