

Arizona Winter School 2015

Lectures by J.-L. Colliot-Thélène, assisted by Y. Harpaz

Local-global principle for rational points and for zero-cycles

Outline

If a smooth projective algebraic variety over a number field has rational points over each completion (p -adic or real) of the number field, does it have a rational point over the number field? Or at least, is the index of the variety equal to one, that is, for any prime ℓ , does there exist a solution in an extension field of degree prime to ℓ ?

The affirmative answer for quadrics has been known for a long time (Hasse principle). That the answer is in general negative was shown by various examples. Analysing these examples led to the formulation of two types of obstructions : the Brauer-Manin obstruction together with its étale avatar, and the obstructions based on torsors under linear algebraic groups.

Two general conjectures have been proposed. Trying to disprove either of them would be a nice project.

Conjecture 1 (on rational points). For smooth, projective, rationally connected varieties over a number field, the Brauer-Manin obstruction to the Hasse principle and to weak approximation is the only obstruction.

Conjecture 2 (on zero-cycles). For an arbitrary smooth projective variety over a number field the Brauer-Manin obstruction is the only obstruction to the index one property.

Over the years, these conjectures were proved for some special classes of varieties. See the book by A. Skorobogatov, *Torsors and rational points*, Cambridge tracts in mathematics 144, as well as various papers, surveys, lectures, and slides of talks on my homepage <http://www.math.u-psud.fr/~colliot/>

For the study of Conjecture 1, two main tools were used : the descent method and the fibration method. Sometimes the circle method was called to the rescue. The Brauer group and descent were used on open varieties.

A conjectural approach for rational points involved Schinzel's hypothesis. If one allows field extensions, Schinzel's hypothesis has a nonconjectural analogue devised by P. Salberger, which leads to a proof of conjecture 2 for surfaces with a conic bundle structure over the projective line.

Over the last two years, there has been important progress.

Combination of the previous methods with powerful recent results in additive combinatorics (Green, Tao, Ziegler, Matthiesen), partially playing the rôle of a substitute for Schinzel's hypothesis, have led to striking results in the direction of conjecture 1 for rational points, for instance a proof of conjecture 1 for rational points of conic bundles over the projective line over the

rational points, when all the “bad fibres” are above rational points. This implies for instance that rational points are dense on surfaces given by an affine equation $y^2 - az^2 = \prod_{i=1}^n (x - e_i)$ with $a, e_i \in \mathbb{Q}$ and n an arbitrary integer.

The results appear in a series of papers by Browning, Harpaz, Matthiesen, Skorobogatov, Wittenberg.

As for Conjecture 2, crucial modifications in the earlier techniques have been introduced by Wittenberg and Harpaz, leading for instance to a proof of conjecture 2 for any variety fibred over projective space with generic fibre birational to a homogeneous space of a connected linear algebraic group (with connected geometric isotropy groups). As a simple example, let us quote smooth projective models of an affine hypersurface with equation $Norm_{K/k}(\Xi) = P(x)$, with K/k a finite field extension, $P(x)$ a polynomial in one variable over k and $\Xi \in K$ standing for $[K : k]$ variables over the ground field k . Even the case K/k biquadratic was not known.

In my lectures I shall try to explain the background and give the main ideas in these recent papers. Here are some references :

[CTSkSD] J.-L. Colliot-Thélène, A. N. Skorobogatov, Sir Peter Swinnerton-Dyer, Rational points and zero-cycles on fibred varieties : Schinzel’s hypothesis and Salberger’s device, *J. reine angew. Math.* 495 (1998), 1–28.

[W] O. Wittenberg, Zéro-cycles sur les fibrations au-dessus d’une courbe de genre quelconque. *Duke Math. J.* 161 (2012), no. 11, 2113–2166.

[BMS] Tim Browning, Lilian Matthiesen, Alexei Skorobogatov, Rational points on pencils of conics and quadrics with many degenerate fibers. *Ann. of Math.* (2) 180 (2014), no. 1, 381–402. <http://arxiv.org/abs/1209.0207>

[HSW] Yonatan Harpaz, Alexei N. Skorobogatov, Olivier Wittenberg, The Hardy–Littlewood conjecture and rational points, *Compositio math.* <http://arxiv.org/abs/1304.3333>

[HW] Yonatan Harpaz, Olivier Wittenberg, On the fibration method for zero-cycles and rational points, <http://arxiv.org/abs/1409.0993>

Rough outline for the student projects [Concrete examples will later be given.]

1) Computing the Brauer group of algebraic varieties over an arbitrary field. There are general formulas for computing the Brauer group of a smooth projective variety. For many interesting varieties, only an affine, possibly singular model is immediately available. It is then a challenge to compute the Brauer group of a smooth projective model.

2) Actually computing the Brauer-Manin obstruction to the existence of rational points. Here one encounters the problem of computing actual

elements in the Brauer group, rather than giving abstract formulas for that group.

3) Integral Brauer-Manin obstruction. This is an obstruction to the existence of integral points on an affine variety defined over the integers. Over the last few years, a number of papers have been devoted to this topic. Here again, a concrete problem is to compute the Brauer group of specific smooth affine varieties. A more ambitious project is to produce classes of varieties for which this obstruction is the only obstruction to the existence of integral points and to strong approximation.

4) Higher reciprocity laws and obstruction to existence of rational points over function fields. Work of Harbater, Hartmann, Krashen has led to the investigation of rational points of varieties over such fields as the function field of a curve over a complete discretely valued field. Counterexamples to a local-global principle have been devised, which build upon reciprocity laws on e.g. two-dimensional schemes. This raises the questions of computing these obstructions systematically, and of producing classes of varieties for which these are the only obstructions.