ARIZONA WINTER SCHOOL 2014 COURSE OUTLINE AND PROJECT DESCRIPTION: ASYMPTOTICS FOR NUMBER FIELDS AND CLASS GROUPS

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1. Course Outline

This course will cover the questions of counting number fields and determining the distribution of class groups of number fields. The starting point is the very basic question: how many number fields are there? This is commonly interpreted as determining the aymptotics in X for

$$\#\{K \subset \bar{\mathbb{Q}} \mid \ |\mathrm{Disc}(K)| \leq X\}.$$

A typical refinement is to determine the asymptotics for number fields with a fixed Galois group G, i.e. for

$$\#\{K \subset \bar{\mathbb{Q}} \mid \ \mathrm{Gal}(K/\mathbb{Q}) \simeq G, |\mathrm{Disc}(K)| \leq X\}.$$

For class groups, what do random class groups look like? For instance, if A is a fixed odd finite abelian group and $Cl(K)_{\mathrm{odd}}$ denotes the odd part of the class group of K, what proportion of quadratic fields have $Cl(K)_{\mathrm{odd}} \simeq A$, i.e. what is

$$\lim_{X\to\infty} \frac{\#\{K \subset \bar{\mathbb{Q}} \mid \ [K:\mathbb{Q}]=2, |\mathrm{Disc}(K)| \leq X, Cl(K)_{\mathrm{odd}} \simeq A\}}{\#\{K \subset \bar{\mathbb{Q}} \mid [K:\mathbb{Q}]=2, |\mathrm{Disc}(K)| \leq X\}}?$$

We will discuss the various heuristics and conjectures that address such questions and their generalizations, as well as the relationship between the distribution of number fields and the distribution of class groups. For two specific questions, we will go into detail to show what can actually be proven and how. We will also provide an overview of what the most recent results are in the area.

Here is an outline of the course:

- Introduction
  - State the questions of counting number fields and class groups asymptotically
  - Connection between counting number fields and counting class groups
  - Moments of class group distributions
  - Changing the counting invariant
- Heuristics
  - Cohen–Lenstra heuristics for counting class groups
  - Malle–Bhargava heuristics for counting number fields
    - Formulation
    - Counterexamples
      - Order of Magnitude issues
      - Constant of the main term issues
- Counting abelian number fields

- Application of class field theory
- Multisection of the series
- Hardy–Littlewood Tauberian theorem
- Analytic continuation of Dirichlet series
- Results for counting by discriminant
- Results for counting by conductor

- Davenport–Heilbronn
  - Counting non-Galois cubic fields and 2-parts of class groups of quadratic fields
  - A modern understanding of the parametrization of cubic rings by binary cubic forms
  - Geometry of numbers and counting lattice points in a fundamental domain
  - Averaging fundamental domains
  - Cutting the cusp
  - Sieving for maximal orders, or maximal and non-totally ramified orders
  - Putting local conditions on the orders
  - Counting by discriminant of the Galois closure
- Statement of the state of the art
  - Number fields: Galois groups D_4, S_4, and S_5
  - Class groups: 2-part of class groups of cubic extensions

2. Project Description

The goal of this project will be to prove precise results relating the Malle–Bhargava heuristics for counting number fields to the Cohen–Lenstra heuristics for the distribution of class groups. In the case of the 2-part of class groups of cubic fields, we should be able to apply known results on counting quartic fields to then obtain new results about class groups.

(1) Using the relationship between unramified quadratic extensions K_6 of cubic fields K_3 and quartic fields K_4, as given in [Hei71], determine exactly what any local condition on the cubic field, e.g. split completely at 7, translates to for the quartic field.
(2) Using the relationship above, when a cubic field is split completely at some rational prime ℓ into $\ell_1\ell_2$, determine what the condition on the quartic field is for Frob_{ℓ_1} to be trivial in the Galois group $\mathrm{Gal}(K_6/K_3)$ of the unramified extension.
(3) Determine the prediction of (an extension of) the Cohen–Lenstra–Martinet heuristics (see [CM87, CM90, Mal10]) to the above scenario (i.e. determine the average number of order two quotients of the class group of K_3 in which ℓ_1 is trivial, and using Bhargava's results [Bha05] on counting quartic fields, prove (or disprove) this prediction).
(4) We now move on to other scenarios. Review or rediscover the fact that if $K/\mathbb{Q}$ is quadratic and L/K is abelian unramified, then $L/\mathbb{Q}$ is Galois with $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Gal}(L/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$. We will review the precise relationship between Malle–Bhargava and Cohen–Lenstra ([CL84]) in this case.
(5) Using [Hei71], for an abelian cubic $K_3/\mathbb{Q}$ and unramified quadratic K_6/K_3, determine the Galois group $\mathrm{Gal}(\tilde{K}_6/\mathbb{Q})$ as a permutation group (where $\tilde{K}_6$ is the Galois closure of K_3). Give a theorem that a certain asymptotic count of A_4 extensions

with restricted ramification would prove the prediction of the Cohen–Lenstra heuristics for the average size of the 2-part of Galois cubic fields. Include in this theorem possibilities for restricting the cubic fields with finitely many local conditions.

(6) For a prime p (we'll start with small p and then possibly do the general case), let $K_3/\mathbb{Q}$ be abelian cubic and K_{3p}/K_3 be an unramified degree p extension. Determine the possible Galois groups $\mathrm{Gal}(\tilde{K_{3p}}/\mathbb{Q})$ as permutation groups. It might be useful to do group computations in Sage or gap. There might be multiple possibilities, and we will determine which one is "generic". We will computationally see if there do appear to be fields giving the various possibilities we have narrowed things down to. For each of the possibilities, determine the prediction of the Malle–Bhargava heuristics for counting the fields $\tilde{K_{3p}}$ by $\mathrm{Disc}(K_3)$. Give a theorem that a certain asymptotic count of extensions with restricted ramification would prove an average size of the p-part of Galois cubic fields as predicted by the Cohen–Lenstra heuristics. Include in this theorem possibilities for restricting the cubic fields with finitely many local conditions. In particular, when do the Malle–Bhargava heuristics predict that this average is finite? Which Galois groups for $\tilde{K_{3p}}$ are even relevant to the asymptotics?

(7) Depending on our success so far, we can continue with the case of larger unramified extensions of Galois cubic fields, or the same problem for non-Galois cubic fields or abelian quartic fields. We may see patterns that will let us understand many cases simultaneously (I certainly hope so!). There are many possible avenues to pursue depending on what we find, including more closely understanding which primes are "bad" for the Cohen–Lenstra–Martinet heuristics (see [CM94, Bha05]), or extending our results to bad primes (see [Ger87a, Ger87b, FK07, FK06]).

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