

MODULAR CURVES AT INFINITE LEVEL: RECOMMENDED READING

JARED WEINSTEIN

For those AWS participants interested in my project on modular curves, I've put together a brief literature review.

This course is about modular curves. These are introduced in David Rohrlich's contribution to *Modular Forms and Fermat's Last Theorem*. That paper serves as a bridge between modular curves over the complex numbers (where they can safely be defined as quotients of the upper half-plane) and modular curves over number fields (which takes more work). However, the meat of this course involves studying modular curves over rings of integers, and for this there is no substitute for the book of Katz-Mazur, *Arithmetic Moduli of Elliptic Curves*.

As the course progresses we will begin to study formal groups. These are somewhat mysterious to the beginner, so I highly recommend reading about these in advance of AWS. An excellent introduction to the formal groups is given in Silverman's book *The Arithmetic of Elliptic Curves*, Ch. IV. The most spectacular application of formal groups to number theory is the paper *Formal complex multiplication in local fields*, by Lubin and Tate, where it is shown that abelian extensions of local fields are given explicitly by adjoining torsion points in formal groups of height 1. Lubin and Tate also investigate formal groups of larger height in their paper *Formal moduli for one-parameter formal Lie groups*, where it is shown that the deformation space of a one-dimensional formal group of height n is an open ball of dimension $n - 1$. This fact is also discussed in the introductory material of the (fascinating and very important) paper *Equivariant vector bundles on the Lubin-Tate moduli space*, by Gross and Hopkins.

This course doesn't use much about Dieudonné theory, but there is a very nice introduction to it in Katz's paper *Crystalline cohomology, Dieudonné modules, and Jacobi sums*, especially part V. Actually, I highly recommend this paper for understanding the "functorial" definition of a formal group, as well as the general yoga of divided powers. It also contains many cheerful facts about the p -adic Gamma function.

For an introduction to p -divisible groups, there is no better source than Tate's article *p -divisible groups*, which introduces the concept

and links it to the study of formal groups. The expository article by Brinon and Conrad has an excellent section on p -divisible groups, which places them in the broader context of p -adic Hodge theory. (In fact the material of our course is intimately related to p -adic Hodge theory, even though the lectures won't touch it.)

The goal of the course is to communicate the surprising fact that, even though the completed local ring of $X(p^n)$ around a supersingular point is very complicated, at infinite level it becomes much nicer (to the point of admitting an explicit description). This is a special case of my own paper Semistable models for modular curves of arbitrary level, which concerns deformations of 1-dimensional formal groups of any height (not just 2). That in turn is a special case of a much more general theorem about deformation spaces of arbitrary p -divisible groups. This is the subject of my joint paper with Peter Scholze, Moduli of p -divisible groups.

Project A concerns a funny phenomenon in characteristic p , which arises when you complete the maximal abelian extension of a field of Laurent series in one variable over a finite field. One of the exercises therein has to do with the Ax-Sen-Tate theorem, concerning Galois invariants in huge fields such as that one. For that, see Ax's paper Zeros of polynomials over local fields.

Project B concerns a strange-looking p -adic power series that doesn't seem like it should be integral but in fact is. Some methods for proving integrality of power series are discussed in Hazewinkel's book Formal Groups and Applications. The double-tailed power series appearing in this project were inspired by material from the recent preprint of Fargues-Fontaine concerning the "fundamental curve of p -adic Hodge theory"; see Example 7.27.