MODULAR CURVES OF INFINITE LEVEL

JARED WEINSTEIN

1. Course Outline

The goal of this course is to investigate an object which might be called $X(p^\infty)$, and which appears as the inverse limit of the classical modular curves $X(p^m)$. Informally, $X(p^\infty)$ ought to classify elliptic curves E together with a $\mathbb{Z}_p$-basis for the Tate module $T_p(E)$. (A disclaimer is in order, lest I be accused of false advertising: We won't be studying all of $X(p^\infty)$, but rather a piece of it corresponding to those E with supersingular reduction.) A recurring theme is that moduli spaces at infinite level can actually be simpler than their finite counterparts, although one must be willing to work with rings which are non-Noetherian.

1.1. **Some motivation: local-global compatibility for GL_2.** Let p be prime. The modular curve $X(p^m)$ is acted upon by the finite group $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$. Passing to the limit, the projective system $\varprojlim X(p^m)$ is acted upon by the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$, but in fact this action can be promoted to an action of the locally compact group $\mathrm{GL}_2(\mathbb{Q}_p)$. (Or rather, a large subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, but we will gloss over this point for now.) This observation is important because it links the study of modular curves to the representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$.

To wit, let N be prime to p, and let X_m be the modular curve $X(\Gamma_1(N), \Gamma(p^n))$, considered over the base $\overline{\mathbb{Q}}_p$. The étale cohomology $H^1_{\text{ét}}(X_m, \overline{\mathbb{Q}}_\ell)$ admits an action of the product group $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. But then if we let

$$V = \varinjlim_m H^1_{\text{ét}}(X_m, \overline{\mathbb{Q}}_\ell),$$

then V admits an action of $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. It is a theorem of Deligne and Carayol that

$$V = \bigoplus_f \pi_{f,p} \otimes \rho_{f,p},$$

where

- f runs over cuspidal newforms of weight 2 and prime-to-p level dividing N,
- $\pi_{f,p}$ is the local component at p of the automorphic representation associated to f,
- $\rho_{f,p}$ is the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of the Galois representation associated to f.

(For precise statements, see [Car83].) Furthermore, $\pi_{f,p}$ and $\rho_{f,p}$ determine one another under (some normalization of) the local Langlands correspondence. This discussion suggests that whatever the object $X_\infty = \varprojlim X_m$ is, it should admit an action of $\mathrm{GL}_2(\mathbb{Q}_p)$, and its cohomology (the space V above) should contain interesting representations of that group.

1.2. **The Lubin-Tate tower.** Rather than consider the tower X_m of modular curves, we can consider a certain local analogue. Let G_0 be a formal group over $k = \overline{\mathbb{F}}_p$ of height n. In [Dri74], Drinfeld shows that the universal deformation ring of G_0 is the power series ring $A_0 = W(k)[\![u_1, \dots, u_{n-1}]\!]$. Drinfeld also defines a notion of level structure on a formal group, and shows that the functor of deformations of G_0 with p^m-level structure is representable by a regular local ring A_m, which gets an action of $\mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$.

In the case that $n = 1$, one recovers the beautiful theory of Lubin and Tate, who work out the "ramified part" of local class field theory using formal groups of height one, see [LT65].

In the case that $n = 2$, the rings A_m appear rather naturally in the context of modular curves. Let X_m be the modular curve as above, except this time let X_m be the Katz-Mazur model of $X(\Gamma_1(N)\cap\Gamma(p^m))$ ([KM85]). Then X_m is a regular scheme over $W(\overline{\mathbb{F}}_p)$. The special fiber $X_{m,s}$ has singularities exactly at the supersingular points. For each such supersingular point $x \in X_{m,s}$, the completed local ring $\hat{\mathcal{O}}_{X_m,x}$ is isomorphic to A_m.

There is a way of associating étale cohomology groups to the local rings A_m, although it is somewhat technically involved. One associates to A_m a certain nonarchimedean analytic space $\mathcal{M}_m$, and then applies Berkovich's theory to arrive at a compactly supported cohomology $H^i_c(\mathcal{M}_m \otimes \mathbb{C}_p, \mathbb{Q}_\ell)$. Then if we set

$$V = \varinjlim H_c^{n-1}(\mathcal{M}_m \otimes \mathbb{C}_p, \overline{\mathbb{Q}}_\ell),$$

then V admits an action of a triple product group $\mathrm{GL}_n(\mathbb{Q}_p) \times J \times W_{\mathbb{Q}_p}$, where $W_{\mathbb{Q}_p}$ is the Weil group and J is an inner twist of $\mathrm{GL}_n(\mathbb{Q}_p)$. Remarkably, V manifests the local Langlands and Jacquet-Langlands correspondences simultaneously ([HT01]). That is, the irreducible representations appearing in V take the form $\pi \otimes \pi' \otimes \sigma(\pi)$, where $\pi \mapsto \pi'$ is the Jacquet-Langlands correspondence and $\pi \mapsto \sigma(\pi)$ is (some normalization of) the local Langlands correspondence.

Let A be the I-adic completion of $\varinjlim A_m$, where I is the maximal ideal of A_0. In the final lecture, we will discuss a recent result which gives an explicit description of A.

2. Projects

2.1. **Exercises in equal characteristic.** Let $\mathcal{O}_L = \mathbb{F}_q[\![t^{1/q^\infty}]\!]$ be the t-adic completion of $\mathbb{F}_q[t^{1/q^\infty}]$. Let $L = \mathcal{O}_L[1/t]$ be the fraction field of $\mathcal{O}_L$. Let H be the multiplicative group $\mathbb{F}_q[\![\pi]\!]^\times$ of formal power series over $\mathbb{F}_q$ with nonzero constant coefficient. Have H operate on L as follows: the power series $a_0 + a_1\pi + a_2\pi^2 + \dots$ will act on L through the substitution $t \mapsto a_0 t + a_1 t^q + a_2 t^{q^2} + \dots$.

Give a nonconstant element of L which is fixed by all of H, or at least give an approximation theoreof (perhaps just for a particular value of q). Show that the fixed field $K = L^H$ is isomorphic to the field of formal Laurent series $\mathbb{F}_q(\!(\pi)\!)$. Now for $n \geq 1$, suppose K_n is the subfield of L fixed by the closed subgroup $1 + \pi^n \mathbb{F}_q[\![\pi]\!]$ of H. Show that K_n/K is a totally ramified extension with Galois group $(\mathcal{O}_K/\pi^n\mathcal{O}_K)^\times$, and that L is the completion of $K_\infty = \bigcup K_n$.

Conclude that there is a (very strange) isomorphism between the absolute Galois group $\mathrm{Gal}(K^s/K)$ and its subgroup $\mathrm{Gal}(K^s/K_\infty)$. Here K^s is a separable closure of K. (In fact, there is also an isomorphism between $\mathrm{Gal}(K^s/K)$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty}))$.)

2.2. **Formal linear algebra and determinants.** Let G be a one-dimensional formal group over $\mathbb{Z}_p$ of height 2. For instance, G could be the completion at the origin of an elliptic curve over $\mathbb{Z}_p$ with good supersingular reduction. Let $\hat{\mathbf{G}}_{\mathrm{m}}$ be the formal multiplicative group over $\mathbb{Z}_p$; this has height 1. (See [Sil09] for definitions of these concepts.)

Let $\mathcal{C}$ be the category of topological $\mathbb{Z}_p$-algebras whose topology is linear, meaning that there is an ideal $I \subset R$ for which $\{I^n\}$ is a system of neighborhoods around the origin. Then G determines a functor from $\mathcal{C}$ to $\mathbb{Z}_p$-vector spaces. Namely, for a topological $\mathbb{Z}_p$-algebra R, $G(R)$ is the set of topologically nilpotent elements of R, where the $\mathbb{Z}_p$-module structure is determined by G. It is not hard to see that $\tilde{G}$ is representable by the ring $\mathbb{Z}_p[\![T]\!]$.

Now let $\tilde{G}$ be the functor from $\mathcal{C}$ to $\mathbb{Q}_p$-vector spaces, defined by

$$\tilde{G}(R) = \varprojlim G(R).$$

Here the inverse limit is taken with respect to multiplication by p. Show that $\tilde{G}$ is representable by $\mathbb{Z}_p[\![T^{1/q^\infty}]\!]$, by which we mean the T-adic completion of $\mathbb{Z}_p[T^{1/q^\infty}]$.

The functor $\tilde{G}$ is a *formal vector space*, and the study of such objects might be called *formal linear algebra.* Show that there exists a nonzero natural transformation

$$\Delta \colon : \tilde{G} \times \tilde{G} \to \tilde{\hat{\mathbf{G}}}_{\mathrm{m}},$$

such that for each object R of $\mathcal{C}$, $\delta(R)$ is a $\mathbb{Q}_p$-alternating map $\tilde{G}(R) \times \tilde{G}(R) \to \tilde{\hat{\mathbf{G}}}_{\mathrm{m}}$. The idea is that the exterior square of $\tilde{G}$ is $\tilde{\hat{\mathbf{G}}}_{\mathrm{m}}$.

On the level of representing objects, Δ corresponds to a continuous homomorphism $\mathbb{Z}_p[\![T^{1/p^\infty}]\!] \to \mathbb{Z}_p[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]$. Let $\delta(X, Y)$ be the image of T under this homomorphism. Give a formula for $\delta(X, Y)$.

2.3. **The Lubin-Tate tower at infinite level, and CM points.** Let A be the completion of $\varinjlim A_m$ as described earlier, in the case that $n = 2$. A result in [Wei12] is that A represents the functor $\mathcal{M}$ from $\mathcal{C}$ to Sets which assigns to R the set of pairs $(x, y) \in \tilde{G}(R) \times \tilde{G}(R)$ such that

$$\Delta(x, y) = (1, \zeta_p, \zeta_{p^2}, \dots)$$

for a compatible system of primitive p^nth roots of unity $\zeta_{p^n} \in R$. (This characterization is independent of the choice of formal group G!)

Now suppose H is a deformation of G_0 to $\mathcal{O}_{\mathbb{C}_p}$ with endomorphisms by an order in a quadratic extension $L/\mathbb{Q}_p$. Suppose we are given a basis of the Tate module $T_p(H)$. These data define continuous homomorphisms $A_m \to \mathcal{O}_{\mathbb{C}_p}$ for all $m \geq 0$, and therefore a continuous homomorphism $A \to \mathcal{O}_{\mathbb{C}_p}$. By the above characterization of the functor $\mathcal{M}$, we get a pair $(x, y) \in \tilde{G}(\mathcal{O}_{\mathbb{C}_p}) \times \tilde{G}(\mathcal{O}_{\mathbb{C}_p})$. Such points might be called *CM points* (perhaps "local Heegner points" might be a better name). They are defined over the completion of the maximal abelian extension of L.

Given a point (x, y) of $\mathcal{M}(\mathcal{O}_{\mathbb{C}_p})$, give necessary and sufficient conditions for (x, y) to be a CM point.

References

[Car83] Henri Carayol, *Sur les représentations l-adiques attachées aux formes modulaires de Hilbert*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 15, 629–632.

[Dri74] V. G. Drinfel′d, *Elliptic modules*, Mat. Sb. (N.S.) **94(136)** (1974), 594–627, 656.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387.

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009.

[Wei12] Jared Weinstein, *Semistable models for modular curves of arbitrary level*, 2012, Available at `http://math.bu.edu/people/jsweinst/StableReduction/StabRed2012.pdf`.