HARMONIC MAASS FORMS, MOCK MODULAR FORMS, AND QUANTUM MODULAR FORMS

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Abstract. This short course is an introduction to the theory of harmonic Maass forms, mock modular forms, and quantum modular forms. These objects have many applications: black holes, Donaldson invariants, partitions and q-series, modular forms, probability theory, singular moduli, Borcherds products, central values and derivatives of modular L-functions, generalized Gross-Zagier formulae, to name a few. Here we discuss the essential facts in the theory, and consider some applications in number theory. This mathematics has an unlikely beginning: the mystery of Ramanujan's enigmatic "last letter" to Hardy written three months before his untimely death. Section 15 gives examples of projects which arise naturally from the mathematics described here.

Modular forms are key objects in modern mathematics. Indeed, modular forms play crucial roles in algebraic number theory, algebraic topology, arithmetic geometry, combinatorics, number theory, representation theory, and mathematical physics. The recent history of the subject includes (to name a few) great successes on the Birch and Swinnerton-Dyer Conjecture, Mirror Symmetry, Monstrous Moonshine, and the proof of Fermat's Last Theorem. These celebrated works are dramatic examples of the evolution of mathematics.

Here we tell a different story, the mathematics of harmonic Maass forms, mock modular forms, and quantum modular forms. This story has an unlikely beginning, the mysterious last letter of the Ramanujan.

1. Ramanujan

The story begins with the legend of Ramanujan, the amateur Indian genius who discovered formulas and identities without any rigorous training[1] in mathematics. He recorded his findings in mysterious notebooks without providing any indication of proofs. Some say he did not see the need to provide proofs; indeed, it is said that his findings came to him as visions from Goddess Namagiri. As is well known, in 1913 Ramanujan wrote G. H. Hardy, a celebrated English analytic number theorist, a famous letter which included pages of formulas and claims (without proof). Some of his claims were well known, some were false, but many were so strange that Hardy (and his colleague Littlewood) were unable to evaluate them. Impressed by the creativity exhibited in this letter, Hardy invited Ramanujan to Cambridge. Ramanujan accepted the invitation, and he spent much of 1914-1919 in Cambridge where he wrote about 30 papers on an exceptionally wide variety of subjects in analytic number

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[1]Ramanujan flunked out of two different colleges.

theory and combinatorics. Despite his young age, heritage, and background, he was honored with his election as Fellow of the Royal Society.

This tale ends tragically. The enigmatic master of formulas returned to south India in the spring of 1919 seeking a return to good health and a forgiving climate. Unfortunately, his health steadily declined over the course of the ensuing year, and he died on April 26, 1920 in Madras. He was thirty two years old. Amazingly, despite his illness, he spent the last year of his life, again in mathematical isolation, conjuring a beautiful theory, one which was nearly lost. This is his "theory" of the *mock theta functions*.

1.1. **Ramanujan's motivation.** To properly understand the context of Ramanujan's last work, we must first recall his work on Euler's partition function $p(n)$. This research underscores the fact that the theory of partitions has historically served as a "testing ground" for some deep developments in the theory of modular forms. Indeed, Ramanujan's work with Hardy on asymptotics was the first triumph of the *circle method*, while the general theory of partition congruences exhibits the tight interplay between the Deligne-Serre theory of ℓ-adic Galois representations and Shimura's theory of half-integral weight modular forms.

To make this connection between partitions and modular forms, we simply require Euler's combinatorial generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots . \tag{1.1}$$

This formal power series is essentially a "weakly holomorphic modular form".

To make this a bit more precise, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup. An element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$, the upper-half of the complex plane, by the linear fractional transformation $\gamma z := \frac{az+b}{cz+d}$. Loosely speaking, a *weight k modular form on* Γ is a holomorphic function f on $\mathbb{H}$ which satisfies

$$f(\gamma z) = (cz+d)^k f(z) \tag{1.2}$$

for all $\gamma \in \Gamma$, which in addition is holomorphic "at the cusps". A *weakly holomorphic modular form* satisfies this definition but is permitted to have poles at cusps.

At the heart of the matter is Dedekind's weight $1/2$ modular form, the eta-function:

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q := e^{2\pi i z}$. For $z \in \mathbb{H}$, it turns out that

$$\eta(z+1) = e^{\frac{\pi i}{12}} \eta(z) \quad \text{and} \quad \eta(-1/z) = (-iz)^{\frac{1}{2}} \eta(z).$$

Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

it follows that $\eta(z)$ is a weight $1/2$ modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a "multiplier system". Since $\eta(z)$ is nonvanishing on $\mathbb{H}$, (1.1) then implies that

$$\frac{1}{\eta(z)} = \sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}} \tag{1.3}$$

is a weight $-1/2$ weakly holomorphic modular form.

1.2. **How did Ramanujan think of modular forms?** Since we view Ramanujan's results on the partition function, and the research it inspired, as a testing ground for the theory of modular forms, we are compelled to ask how Ramanujan thought of modular forms. A brief inspection of most of his works (for example, see [44, 203]) suggests that his view was consistent with the classical treatment found in most textbooks (for example, see [84, 100, 151, 160, 169, 182, 191, 205, 216, 224, 228]) on modular forms. Indeed, the bulk of his work on modular forms depends on the properties of the q-series

$$\begin{aligned} P(q) &:= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \\ Q(q) &:= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\ R(q) &:= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \end{aligned}$$

(where $\sigma_\nu(n) := \sum_{d|n} d^\nu$), which are the classical Eisenstein series $E_2(z), E_4(z)$ and $E_6(z)$ from the theory of modular forms. Therefore, much of his work followed classical lines.

This assessment, however, ignores Ramanujan's extensive work on combinatorial q-series, which at first glance has nothing to do with the theory of modular forms. Ramanujan, a master manipulator of power series, thought deeply about combinatorial power series such as infinite products, which are often modular like Dedekind's eta-function $\eta(z)$, and Jacobi's identity

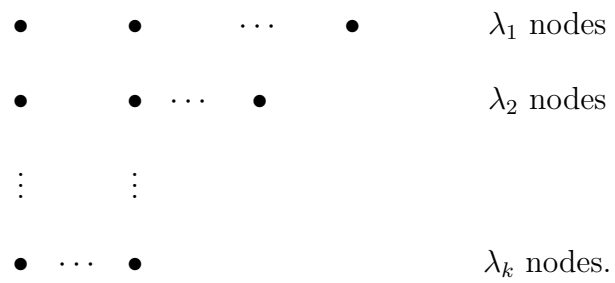
$$\prod_{n=1}^{\infty} (1-q^{2n})(1+t^2 q^{2n-1})(1+t^{-2} q^{2n-1}) = \sum_{n \in \mathbb{Z}} t^{2n} q^{n^2}.$$

He also thought deeply about power series he referred to as *Eulerian series*, such as

$$\Omega(t;q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-tq)^2(1-tq^2)^2 \cdots (1-tq^n)^2}, \tag{1.4}$$

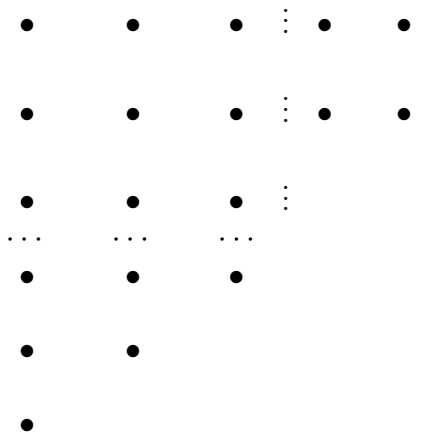
which seems to have nothing to do with modular forms. However, in some rare cases such series turn out to coincide with modular forms.

We now give one such coincidence, an identity which is particularly relevant for Ramanujan's work on partitions. We first recall some elementary combinatorial notions. One may visualize a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ as a *Ferrers diagram*, a left justified array of dots consisting of k rows in which there are λ_i dots in the ith row:



The *Durfee square* of such a Ferrers diagram is the largest square of nodes in the upper left hand corner of the diagram. The boundary of a Durfee square naturally then divides a partition into a perfect square and two partitions whose parts do not exceed the side length of the Durfee square.

Example. The Ferrers diagram of the partition $5+5+3+3+2+1$ is:



Therefore, this partition decomposes as the Durfee square of size 9, and the two partitions 2+2, and 3+2+1.

Armed with these notions, we have the following q-series identity for the generating function for $p(n)$.

Proposition 1.1. *The following combinatorial identity is true:*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2} = \Omega(1;q).$$

In particular, we have that

$$\frac{1}{\eta(z)} = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = q^{-\frac{1}{24}} + \sum_{m=1}^{\infty} \frac{q^{m^2-\frac{1}{24}}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2}.$$

Proof. For every positive integer m, the q-series

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = \sum_{n=0}^{\infty} a_m(n)q^n$$

is the generating function for $a_m(n)$, the number of partitions of n whose summands do not exceed m. Therefore by the discussion above, the q-series

$$\frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2} = \sum_{n=0}^{\infty} b_m(n)q^n$$

is the generating function for $b_m(n)$, the number of partitions of n whose Ferrers diagram has a Durfee square of size m^2. The theorem follows by summing in m. □

Although they are quite rare, there are further examples of such q-series identities in which an Eulerian series is essentially a modular form. Among them, perhaps the most famous are the Rogers-Ramanujan identities [199, 212, 213, 214]:

$$\begin{aligned} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}, \\ \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}. \end{aligned} \tag{1.5}$$

These infinite products are essentially the Fourier expansions of weight 0 weakly holomorphic modular forms of the type studied classically by Jacobi, Klein, and Siegel, and again more recently by Kubert and Lang in their work on *modular units* (for example, see [165, 217]).

Remark. The literature on such identities is extensive (for example, see works by Andrews, Gordon, Göllnitz, and Slater [9, 125, 124, 226] to name a few), and the pursuit of further identities and their interpretations remains an active area of research largely due to applications in combinatorics, Lie theory, number theory and physics (for example, see [14], [155] and [172] to name a few). In this direction, W. Nahm [186] has very interesting work related to the question of when a basic hypergeometric-type series is automorphic.

These identities stand out since there is no reason to believe, for function theoretic reasons, that an Eulerian series should ever be modular. Indeed, there is no general theory of transformation laws for Eulerian series.

It is not difficult to imagine Ramanujan's mindset. Understanding this quandary, it seems that Ramanujan, largely motivated by his work on partitions and the Rogers-Ramanujan identities, spent the last year of his life thinking deeply about the "near" modularity of Eulerian series. He understood the importance of developing a new theory, one which overlaps in spots with the classical theory of modular forms.

He discovered the *mock theta functions.*

1.3. **Ramanujan's last letter to Hardy.** The mathematics described here begins with Ramanujan's last letter to Hardy, dated January 12, 1920, just three months before his death. We quote (see pages 220-224 of [45]):

"I am extremely sorry for not writing you a single letter up to now.... I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

This letter contained 17 examples including:

$$\begin{aligned} f(q) &:= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},\\ \omega(q) &:= \sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}, \qquad (1.6)\\ \lambda(q) &:= \sum_{n=0}^{\infty}\frac{(-1)^n(1-q)(1-q^3)\cdots(1-q^{2n-1})q^n}{(1+q)(1+q^2)\cdots(1+q^{n-1})}. \end{aligned}$$

At first glance, these series indeed seem to resemble the Eulerian series in Proposition 1.1 and (1.5). For example, in terms of (1.4), we have that both

$$f(q)=\Omega(-1;q) \quad \text{and} \quad \frac{1}{\eta(z)}=q^{-\frac{1}{24}}\cdot\Omega(1;q).$$

However, as Ramanujan asserts, series such as $f(q),\omega(q)$ and $\lambda(q)$ are not modular. He conjectures that they are examples of functions he refers to as *mock theta functions.* His definition is quite vague, and he admits that he does not have a proof that his examples satisfy his definition.

To make this definition, he first recalls the asymptotic properties of certain modular forms which are given as *Eulerian series*. He refers to such properties as (A) and (B). These asymptotics are analogous to those he employed with Hardy in their work on the asymptotic properties of the partition function $p(n)$. Ramanujan then uses this discussion to give his definition. He writes:

"...Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q=e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: - is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is $O(1)$ at <u>all</u> the points $e^{2i\pi m/n}$? The answer is <u>it is not necessarily so.</u> When it is not so I call the function Mock ϑ-function. I have not proved rigorously that <u>it is not necessarily so</u>. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function."

Remark. By *ordinary theta function*, Ramanujan meant a *weakly holomorphic modular form* with weight $k\in\frac{1}{2}\mathbb{Z}$ on some $\Gamma_1(N)$.

Ramanujan does not have a proof that his examples, such as those in (1.6), satisfy his definition. However, after giving his vague definition of a mock theta function, he then gives

a surprising "near miss" which pertains to the Eulerian series

$$\Omega(-1;q) = f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots.$$

He claims that a certain weight $1/2$ modular form cuts out "half" of the exponential singularities, while its negative cuts out the other "half".

To be precise, we first note that $f(q)$ is convergent for $|q| < 1$ and those roots of unity q with odd order. For even order roots of unity, $f(q)$ has exponential singularities. For example, as $q \to -1$, we have

$$f(-0.994) \sim -1.08 \cdot 10^{31}, \quad f(-0.996) \sim -1.02 \cdot 10^{46}, \quad f(-0.998) \sim -6.41 \cdot 10^{90}.$$

To cancel the exponential singularity at $q = -1$, Ramanujan found the function $b(q)$, which is modular[2] up to multiplication by $q^{-\frac{1}{24}}$, defined by

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots\left(1 - 2q + 2q^4 - \cdots\right). \tag{1.7}$$

The exponential behavior illustrated above is canceled in the numerics below.

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\dots$	$3.969\dots$	$3.976\dots$	$3.984\dots$	$3.992\dots$

.

It appears that $\lim_{q\to -1}(f(q)+b(q)) = 4$. More generally, as q approaches an even order $2k$ root of unity, Ramanujan claimed that

$$f(q) - (-1)^k b(q) = O(1).$$

Ramanujan's claim is true. Indeed, recent work [118] by Folsom, Rhoades, and the author gives a simple closed formula for the suggested $O(1)$ constants as finite sums of q-hypergeometric-type series.

Theorem 1.2. *If ζ is a primitive even order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q\to\zeta}\left(f(q) - (-1)^k b(q)\right) = -4 \cdot \sum_{n=0}^{k-1} (1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

Remark. We shall relate such finite hypergeometric sums to *quantum modular forms* later in this paper. For now, the reader should simply view these formulas as strange finite expressions for Ramanujan's $O(1)$ numbers.

Example. Since empty products equal 1, Theorem 1.2 shows that

$$\lim_{q\to -1}\left(f(q) + b(q)\right) = 4.$$

Example. For $k = 2$, Theorem 1.2 gives $\lim_{q\to i}\left(f(q) - b(q)\right) = 4i$. The table below nicely illustrates this fact:

[2]Here $q^{-\frac{1}{24}}b(q)$ is modular with respect to z where $q := e^{2\pi i z}$.

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$\sim 1.9 \cdot 10^6 - 4.6 \cdot 10^6 i$	$\sim 1.6 \cdot 10^8 - 3.9 \cdot 10^8 i$	$\sim 1.0 \cdot 10^{12} - 2.5 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.0577 + 3.855i$	$\sim 0.0443 + 3.889i$	$\sim 0.0303 + 3.924i$

Theorem 1.2 follows from a deeper understanding of Ramanujan's examples. As we shall explain, recent work by Zwegers shows that Ramanujan's examples are related to *harmonic Maass forms*, functions which will be defined in Section 4.1. Ramanujan's functions are the *holomorphic parts* of special weight 1/2 harmonic Maass forms. This realization makes it possible to now prove theorems such as Theorem 1.2. We shall give the proof in Section 13.

The connection between Ramanujan's mock theta functions and harmonic Maass forms does not immediately shed light on Ramanujan's vague definition. Moreover, Ramanujan's strange example concerning $f(q)$ and $b(q)$ shows that modular forms can almost exactly cut out the exponential singularities of his so-called mock theta functions. These facts prompted Berndt to remark [43] recently that

"*it has not been proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition.*"

Griffin, Rolen, and the author [131] have recently filled in this gap by proving the following theorem.

Theorem 1.3. *Suppose that $M(z)$ is one of Ramanujan's mock theta functions, and let γ and δ be integers for which $q^\gamma M(\delta z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q \to \zeta} (q^\gamma M(\delta z) - g(z)) = O(1).$$

Remark. The limits in Theorem 1.3 are *radial* limits taken from within the unit disk.

Remark. In a recent paper [209], Rhoades proves that Ramanujan's vague definition of a mock theta function is not equivalent to the definition of the holomorphic part of a weight 1/2 harmonic Maass form. He shows how to construct functions with modular-like asymptotics which are not holomorphic parts of weight 1/2 harmonic Maass forms up to a "trivial" function.

In Section 13 we shall prove Theorems 1.2 and 1.3. This shall be done after we develop some of the critical features of the theory of harmonic Maass forms. Before we give basic facts about the theory, we continue to recall the history which preceded the recent realization that Ramanujan's mock theta functions are related to harmonic Maass forms.

1.4. **Mock theta functions from 1920-1976.** Most of the surviving text of Ramanujan's last letter, which amounts to roughly 4 typewritten pages, consists of explicit formulas for his 17 strange formal power series. The theoretical content is rather obtuse, and he gives no indication of how he derived his 17 examples. He even divides these examples into groups based on their "order", a term he never defines. With such flimsy clues, how could one

rederive Ramanujan's theory? How could one proceed with his vague definition of a mock theta function?

Despite these formidable challenges, a few mathematicians such as G. E. Andrews, L. Dragonette[3], A. Selberg, and G. N. Watson [8, 101, 221, 239, 240] investigated Ramanujan's mock theta functions for what they were, a list of enigmatic power series. For example, Andrews and Dragonette [8, 101] investigated Ramanujan's claimed asymptotic formula for the mock theta function

$$f(q) = \Omega(-1; q) = \sum_{n=0}^{\infty} \alpha(n) q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

They proved, for positive n, that

$$\alpha(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n - \frac{1}{24}}} \cdot e^{\pi\sqrt{\frac{n}{6} - \frac{1}{144}}}.$$

Remark. Strictly speaking, Dragonette first proved the asymptotic [101], while Andrews refined her work to obtain [8] the asymptotic with an improved error term.

Despite the absence of a theory, or much less, just a simple useful definition of a mock theta function, these few early works bolstered the belief that Ramanujan had discovered something important. Watson, in his own words[4], proclaimed [239]:

"*Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine (a.k.a. Persephone)...*"

1.5. **The "lost notebook" and Dyson's "challenge for the future".** By the mid 1970s, little progress had been made on the mock theta functions. They remained a list of enigmatic power series, without any apparent connection to the theory of modular forms, or any other comprehensive theory for that matter.

Then in the spring of 1976, Andrews discovered Ramanujan's "lost notebook" in an old box of papers hidden away in the Trinity College Library at Cambridge University. This notebook, consisting of over 100 pages of mathematical scrawl [204], was archived among papers from Watson's estate. Miraculously, the "lost notebook" had somehow survived a circuitous journey from India in the early 1920s to lie forgotten in the Trinity College Library archives. The journey was indeed miraculous, for the contents of the box almost met a catastrophic end in 1968 when Rankin saved them just a few days before they were scheduled to be burned. Although the manuscript was never truly lost, it was long forgotten, and buried among Watson's random papers. Andrews proclaimed [10]:

[3]Leila Dragonette is better known under her married name, Leila Bram.

[4]This text is from Watson's 1936 Presidential Address to the London Mathematical Society

"... the fact that its existence was never mentioned by anyone for over 55 years leads me to call it "lost".

The discovery of the "lost notebook" was the catalyst which made it possible to begin chipping away at the puzzle of Ramanujan's mock theta functions. On top of giving 2 further mock theta functions, adding to the 17 from the last letter and 3 defined by Watson [239], the pages contained many clues: striking identities and relations. As usual, these were given without proof.

To illustrate the value of these clues, we consider some examples of Ramanujan's claimed identities, the famous "mock theta conjectures." To state them, we first fix notation. For non-negative integers n, let

$$(x)_n := (x;q)_n := \prod_{j=0}^{n-1}(1-xq^j), \tag{1.8}$$

and let

$$(x)_\infty := (x;q)_\infty := \prod_{j=0}^{\infty}(1-xq^j), \tag{1.9}$$

where an empty product equals 1. Let $f_0(q), f_1(q), \Phi(q)$, and $\Psi(q)$ be the mock theta functions

$$\begin{aligned} f_0(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q)_n},\\ \Phi(q) &:= -1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}\\ f_1(q) &:= \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q)_n},\\ \Psi(q) &:= -1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}. \end{aligned} \tag{1.10}$$

These q-series resemble (1.4), the Eulerian series in the Rogers-Ramanujan identities (1.5), and the series in (1.6). The mock theta conjectures are a list of ten identities involving these 4 functions. Thanks to work of Andrews and F. Garvan [29], these ten identities follow from the truth of the following pair of identities.

Conjecture (Mock Theta Conjectures)**.** *The following identities are true:*

$$\begin{aligned} \frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} &= f_0(q)+2\Phi(q^2),\\ \frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} &= f_1(q)+2q^{-1}\Psi(q^2). \end{aligned}$$

These surprising identities equate simple infinite products to linear combinations of mock theta functions. These infinite products, like those in Proposition 1.1 and (1.5), are well

known to essentially coincide with the Fourier expansions of certain weakly holomorphic modular forms. Therefore, the truth of these identities directly related mock theta functions to modular forms. These clues from the "lost notebook" finally placed Ramanujan's mock theta functions in the vicinity of the theory of modular forms.

Unfortunately, these clues would not prove to be enough to rederive Ramanujan's theory. Indeed, mathematicians even had to wait 10 years, until the work of D. Hickerson [144], just for a proof of these particularly vicious q-series identities.

Nonetheless, Andrews's discovery of the "lost notebook" in 1976 almost immediately sparked an explosion of research on the mock theta functions, largely spearheaded by Andrews. By the late 1990s, works by Andrews, Y.-S. Choi, H. Cohen, F. Dyson, Garvan, B. Gordon, Hickerson, R. McIntosh, M. Wakimoto [10, 12, 13, 15, 16, 23, 29, 30, 86, 87, 88, 89, 90, 95, 126, 127, 144, 145, 237], among numerous others, revealed many of the deeper properties of the mock theta functions. Thanks to these works, and too many others to list, Ramanujan's 22 mock theta functions had been related to a surprising collection of subjects:

- Artin L-functions in number theory
- Hypergeometric functions
- Partitions
- Lie theory
- Mordell integrals
- Modular forms
- Polymer chemistry
- ...

Despite this flurry of activity, the essence of Ramanujan's theory remained a mystery. The puzzle of his last letter to Hardy, thanks to the "lost notebook," had morphed into the enigmatic web of Ramanujan's 22 mock theta functions. The presence of this web strongly suggested the existence of a theory, and it also demanded a solution. In his plenary address at the Ramanujan Centenary Conference at the University of Illinois in 1987, Freeman Dyson beautifully summed up the situation [109]:

"*The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions...But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further.*"

2. Back to the future

By the late 1990s, the vast literature on Ramanujan's mock theta functions contained many important clues for Dyson's "challenge for the future". In addition to the identities comprising the mock theta conjectures, there were further clues such as q-series identities relating mock theta functions to Lambert-type series and indefinite theta series. We recall

two such identities involving the mock theta functions $f(q)$ and $f_0(q)$ (see (1.6) and (1.10)). In 1936, Watson [239] proved that

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} = \frac{2}{(q)_\infty} \cdot \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(3n^2+n)/2}}{1+q^n},$$

and in 1986 Andrews [13] proved that

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)^n} = \frac{1}{(q)_\infty} \cdot \left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} - \sum_{\substack{n+j<0\\ n-j<0}} \right) (-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}.$$

Such identities served as motivation for the 2002 Ph.D. thesis, written under the direction of D. Zagier, of S. Zwegers [252, 253]. Indeed, Zwegers researched the following two questions of Zagier (see page 2 of [253]):

(1) How do the mock ϑ-functions fit in the theory of modular forms?
(2) Is there a theory of indefinite theta functions?

His thesis brilliantly addressed these questions by combining and extending ideas from a number of sources such as works of Lerch [173, 174] on the functions

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi i n\nu}}{1-e^{2\pi i n\tau+2\pi i u}}, \tag{2.1}$$

where $\tau \in \mathbb{H}, \nu \in \mathbb{C}, u \in \mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$, work of L. Göttsche and Zagier on indefinite ϑ-functions [129], the theory of Jacobi forms [112], Mordell integrals [184, 185], and works of Andrews [11, 13, 15, 16].

Zwegers related Ramanujan's mock theta functions to real analytic vector-valued modular forms. Loosely speaking, he "completed" Ramanujan's mock theta functions by adding a non-holomorphic function, a so-called *period integral*, to obtain real analytic functions which obey desired modular transformation laws. We shall recount some of his work[5] in Section 3. The real analytic modular forms of Zwegers are examples of *harmonic Maass forms* (see Section 4.1 for the definition) which were defined about the same time by Bruinier and Funke [73], a coincidence which catalyzed much of the research described here.

These developments sparked an immediate explosion in a wide number of new directions. Indeed, recent works by the author, Andrews, Bringmann, Eguchi, Folsom, Griffin, Hikami, Kac, Lawrence, Malmendier, Mellit, Okada, Wakimoto, and Zagier [16, 19, 23, 29, 55, 56, 98, 110, 130, 146, 147, 148, 156, 157, 171, 179, 180, 244] apply this theory to:

- Black holes
- Donaldson invariants
- Gauge theory
- Representation theory of Lie superalgebras
- Moonshine

[5]Zagier delivered a Séminaire Bourbaki lecture on these recent works on Ramanujan's mock theta functions in 2007 [246].

- Knot theory
- Mathematical physics
- Probability theory
- Topology.

The author and his collaborators have aimed to employ this new perspective to answer deep questions about many of the number theoretic topics captured by the web of Ramanujan's mock theta functions. Here we describe the implications of this theory to:

- Partitions and q-series
- Modular forms
- Traces of singular moduli
- Borcherds products
- Modular L-functions à la Kohnen-Waldspurger and Kohnen-Zagier.
- Ramanujan's last letter
- Quantum modular forms

Remark. This paper is not a comprehensive treatise; instead, it is only meant to serve as an overview.

The remainder of the paper is organized as follows. In Section 3, we give one beautiful example of Zwegers's work, as it pertains to Ramanujan's mock theta functions $f(q)$ and $\omega(q)$, and we briefly describe his important construction of a weight 1/2 non-holomorphic Jacobi form. In Section 4 we give the formal definition of a harmonic Maass form, and we present their basic properties. In Section 5 we give various examples of such forms. Section 15 includes a list of sample projects which arise naturally from the work here. In the remaining sections we summarize our results on the following topics:

(Partition congruences aprés Dyson)

In an effort to provide a combinatorial explanation of Ramanujan's congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

Dyson introduced [108] the so-called *rank* of a partition, a delightfully simple statistic. The rank of a partition is defined to be its largest part minus the number of its parts.

For example, the table below includes the ranks of the partitions of 4.

Partition	Rank	Rank mod 5
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

One observes that each residue class modulo 5 occurs exactly once in the last column above. Based on further numerics, Dyson made the following conjecture whose truth provides a combinatorial explanation of Ramanujan's congruences modulo 5 and 7.

Conjecture (1944, Dyson)**.** *The partitions of* $5n+4$ *(resp.* $7n+5$*) form* 5 *(resp.* 7*) groups of equal size when sorted by their ranks modulo* 5 *(resp. 7).*

In 1954, Atkin and H. P. F. Swinnerton-Dyer proved [36] Dyson's conjecture[6]. In view of the more general theory of partition congruences, which includes examples such as

$$p(4063467631n + 30064597) \equiv 0 \pmod{31},$$

it is natural to investigate the role that Dyson's rank plays in the general theory of partition congruences. Using a new class of harmonic Maass forms, which will be described in Section 5.4, the author and K. Bringmann [53, 64] have obtained general results in this direction. These works will be described in in Section 6.

(Eulerian series as modular forms).

The Rogers-Ramanujan identities

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$
$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

and the mock theta conjectures provide examples of Eulerian series as modular forms. Thanks to the framework of the theory of harmonic Maass forms, we have a better understanding of the modular transformation properties of certain Eulerian series. In Section 7 we discuss work by the author, Bringmann, and R. Rhoades [66] on Eulerian series as modular forms. This work immediately gives many new identities such as

$$\sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} + \sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}^2} = 2\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^4}.$$

(Analytic Exact formulas)

As described earlier, Rademacher perfected the Hardy-Ramanujan asymptotic to obtain an exact formula for $p(n)$ (for example, see [196, 197]). To state his formula, let $I_s(x)$ be the usual I-Bessel function of order s, and let $e(x) := e^{2\pi i x}$. Furthermore, if $k \geq 1$ and n are integers, then let

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right), \tag{2.2}$$

[6]A short calculation reveals that the obvious generalization of the conjecture cannot hold for 11.

where the sum runs over the residue classes modulo $24k$, and where

$$\chi_{12}(x) := \left(\frac{12}{x}\right). \tag{2.3}$$

If n is a positive integer, then one version of Rademacher's formula reads [196]

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{2.4}$$

We address the following classical partition problem.

Problem. Determine exact formulas for $N_e(n)$ (resp. $N_o(n)$), the number of partitions of n with even (resp. odd) Dyson rank.

Thanks to Rademacher's formula, and the obvious fact that

$$p(n) = N_e(n) + N_o(n),$$

it turns out that this question is equivalent to the problem of deriving exact formulas for the coefficients

$$\alpha(n) = N_e(n) - N_o(n)$$

of the mock theta function $f(q)$. This deduction is a simple modification of the proof of Proposition 1.1.

As mentioned earlier, Andrews and Dragonette had already proved that

$$\alpha(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n-\frac{1}{24}}} \cdot e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}.$$

This result falls short of the problem of obtaining an exact formula, and as a consequence represents the obstruction to obtaining formulas for $N_e(n)$ and $N_o(n)$. In his plenary address "Partitions: At the interface of q-series and modular forms", delivered at the Millenial Number Theory Conference at the University of Illinois in 2000, Andrews highlighted this classical problem by promoting his conjecture[7] of 1966 (see page 456 of [8], and Section 5 of [18]) for the coefficients $\alpha(n)$.

Conjecture. (Andrews-Dragonette)
If n is a positive integer, then

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

The author and Bringmann [62] proved this conjecture using work of Zwegers and the theory of Maass-Poincaré series. Since $N_e(n) = (p(n)+\alpha(n))/2$ and $N_o(n) = (p(n)-\alpha(n))/2$, the proof of the conjecture, combined with Rademacher's exact formula (2.4), provides the desired formulas for $N_e(n)$ and $N_o(n)$.

The proof of the Andrews-Dragonette Conjecture is a special case of a more general problem.

[7]This conjecture is suggested as a speculation by Dragonette in [101].

Problem. Determine exact formulas for the coefficients of holomorphic parts of harmonic Maass forms.

We shall also give general theorems which provide such exact formulas for harmonic Maass forms with weight $\leq 1/2$. These new results include the classic results of Rademacher and Zuckerman [197, 198, 250, 251] which give exact formulas for the Fourier coefficients of negative weight weakly holomorphic modular forms. Indeed, these results follow since weakly holomorphic modular forms are harmonic Maass forms. They also recover some results of Bruinier, Hejhal, and Niebur [71, 143, 187, 188] for harmonic Maass forms of non-positive weight.

(Applications to classical modular forms)

Using the properties of various differential operators, it is possible to address classical problems about modular forms. For example, we address the classification of linear relations among cuspidal Poincaré series, and we obtain a theorem which detects the vanishing of Hecke eigenvalues for integer weight newforms. We also show that Lehmer's Conjecture, which asserts that none of the coefficients of the discriminant function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q + 252q^3 - \cdots$$

vanish, is implied by the irrationality of the coefficients of the "holomorphic part" of a certain Maass-Poincaré series.

The work on linear relations among cuspidal Poincaré series is contained in [113] and Rhoades's Ph.D. thesis [208] (also see [206]). The theorem which detects vanishing Fourier coefficients is in a recent paper by J. H. Bruinier, Rhoades, and the author [79]. These results are described in Section 9.

(Generating functions for singular moduli)

Let $j(z)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots .$$

The values of modular functions such as $j(z)$ at imaginary quadratic arguments in $\mathbb{H}$ are known as *singular moduli*. Singular moduli are algebraic integers which play many roles in number theory. For example, they generate class fields of imaginary quadratic fields, and they parameterize isomorphism classes of elliptic curves with complex multiplication.

In an important paper [245], Zagier gave a new proof of Borcherds's famous theorem on the infinite product expansions of integer weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with Heegner divisor. This proof, as well as all of the results of [245], are connected to his beautiful observation that the generating functions for traces of singular moduli are essentially weight 3/2 weakly holomorphic modular forms.

Zagier's paper has inspired an extraordinary number of research papers with generalizations in a variety of directions in works by the author, Bringmann, Bruinier, D. Choi, W. Duke, A. Folsom, J. Funke, O. Imamoḡlu, P. Jenkins, D. Jeon, S.-Y. Kang, C. Kim, R. Masri, A. Miller, A. Pixton, J. Rouse, and A. Toth [63, 67, 74, 75, 92, 93, 103, 104, 105, 106, 116, 181].

In Section 10 we describe two general theorems which gives typical examples of such generating functions. In addition to giving further generating functions for singular moduli, we shall also describe a beautiful theorem of Duke [103] related to the classical observation that

$$e^{\pi\sqrt{163}} = 262537412640768743.9999999999992\ldots.$$

is nearly an integer.

Remark. The recent work by Duke, O. Imamoḡlu and A. Toth [104, 105] is very interesting. Although we will not describe them here, we encourage the interested reader to read these papers. These nice papers consider "real quadratic" traces. Recent work by Ehlen [111], Duke and Li [107], and Viazovska [236] give deep arithmetic information in a another direction related to weight one harmonic weak Maass forms. In their work they consider weight one mock modular forms, and they relate them to singular moduli, and complex Galois representations.

The second general theorem pertains to traces of singular moduli for nonholomorphic modular functions obtained by applying differential operators to certain weight -2 weakly holomorphic modular (also see [78]). As a special case of this result, we obtain a finite "algebraic" formula for $p(n)$. Similar formulas have been obtained by Alfes in [7].

Here we recall this special result. We first recall the setting of Heegner points on modular curves (see [133]). Let $N > 1$, and let $D < 0$ be a quadratic discriminant coprime to N. The group $\Gamma_0(N)$ acts on the discriminant D positive definite integral binary quadratic forms

$$Q(X, Y) = [a, b, c] := aX^2 + bXY + cY^2$$

with $N \mid a$. This action preserves $b \pmod{2N}$. Therefore, if $\beta^2 \equiv D \pmod{4N}$, then it is natural to consider $\mathcal{Q}_{N,D,\beta}$, the set of those discriminant D forms $Q = [a, b, c]$ for which $0 < a \equiv 0 \pmod{N}$ and $b \equiv \beta \pmod{2N}$, and we may also consider the subset $\mathcal{Q}^{\text{prim}}_{N,D,\beta}$ obtained by restricting to primitive forms. The number of $\Gamma_0(N)$ equivalence classes in $\mathcal{Q}_{N,D,\beta}$ is the Hurwitz-Kronecker class number $H(D)$, and the natural map defines a bijection

$$\mathcal{Q}_{N,D,\beta}/\Gamma_0(N) \longrightarrow \mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z}),$$

where $\mathcal{Q}_D$ is the set of discriminant D positive definite integral binary quadratic forms (see the proposition on p. 505 of [133]). This bijection also holds when restricting to primitive forms, in which case the number of $\Gamma_0(N)$ equivalence classes in $\mathcal{Q}_{N,D,\beta}$, and the number of $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes in $\mathcal{Q}^{\text{prim}}_D$, is given by the class number $h(D)$.

As a special case of a general theorem, we shall study the "class polynomial" of the $\Gamma_0(6)$ nonholomorphic modular function

$$F_p(z) := -\partial_{-2}(P(z)) = \left(1 - \frac{1}{2\pi\mathrm{Im}(z)}\right) q^{-1} + \frac{5}{\pi\mathrm{Im}(z)} + \left(29 + \frac{29}{2\pi\mathrm{Im}(z)}\right) q + \ldots, \tag{2.5}$$

where $P(z)$ is the weight -2 weakly holomorphic modular form

$$P(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 10 - 29q + \ldots,$$

and where

$$\partial_h := \frac{1}{2\pi i}\cdot\frac{\partial}{\partial z} - \frac{h}{4\pi \mathrm{Im}(z)}. \tag{2.6}$$

These polynomials are defined as

$$H_n^{\mathrm{part}}(x) := \prod_{Q\in\mathcal{Q}_{6,1-24n,1}} (x - P(\alpha_Q)). \tag{2.7}$$

We then have the following finite algebraic formula for the partition numbers $p(n)$.

Theorem 2.1. *If n is a positive integer, then*

$$H_n^{\mathrm{part}}(x) = x^{h(1-24n)} - (24n-1)p(n)x^{h(1-24n)-1} + \dots,$$

where $p(n)$ is the partition function. In particular, we have that

$$p(n) = \frac{1}{24n-1}\sum_{Q\in\mathcal{Q}_{6,1-24n,1}} P(\alpha_Q),$$

and we have that each $(24n-1)P(\alpha_Q)$ is an algebraic integer.

We shall discuss the modular generating functions for traces of singular moduli in Section 10.

(Borcherds Products)

Recently, Borcherds [48, 49, 50] provided a striking description for the exponents in the naive infinite product expansion of many modular forms, those forms with a Heegner divisor. He proved that the exponents in these infinite product expansions are certain coefficients of modular forms of weight 1/2. For example, let $c(n)$ denote the integer exponents one obtains by expressing the classical Eisenstein series $E_4(z)$ as an infinite product:

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = (1-q)^{-240}(1-q^2)^{26760}\cdots = \prod_{n=1}^{\infty}(1-q^n)^{c(n)}.$$

Borcherds proved that there is a weight 1/2 meromorphic modular form

$$G(z) = \sum_{n\geq -3} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 + \cdots - 4096240q^9 + \dots$$

with the property that $c(n) = b(n^2)$ for every positive integer n.

These results may be generalized where the Borcherds exponents are given by certain Fourier coefficients of weight 1/2 harmonic Maass forms. As a nice example, consider Ramanujan's mock theta function (see (1.6))

$$\begin{aligned}\omega(q) &= \sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}\\ &= \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \cdots.\end{aligned}$$

Define integers $a(n)$ by the expression

$$\sum_{n\in\mathbb{Z}+1/3} a(n)q^n := -2q^{1/3}\big(\omega(q^{1/2})+\omega(-q^{1/2})\big) = -4\,q^{1/3} - 12\,q^{4/3} - 24\,q^{7/3} - 40\,q^{10/3} - \dots.$$

These new results on generalized Borcherds products imply that

$$\begin{aligned}(2.8)\quad &\prod_{n=1}^{\infty}\left(\frac{1+\sqrt{-2}q^n-q^{2n}}{1-\sqrt{-2}q^n-q^{2n}}\right)^{\left(\frac{n}{3}\right)a(n^2/3)}\\ &\quad = 1-8\sqrt{-2}q-(64-24\sqrt{-2})q^2+(384+168\sqrt{-2})q^3+(64-1768\sqrt{-2})q^4+\dots\end{aligned}$$

is a modular form on the congruence subgroup $\Gamma_0(6)$.

A brief indication of these results is given in Section 11.

(Derivatives and values of L-functions)

Once armed with a generalized Borcherds-type theorem, one is then able to construct modular forms with a prescribed divisor, a twisted Heegner divisor, with the additional property that one can determine the field of definition of the product Fourier expansion. For example, the modular form in (2.8) clearly has Fourier coefficients in $\mathbb{Z}[\sqrt{-2}]$, thanks to the fact that Ramanujan's mock theta function $\omega(q)$ has integer coefficients. This phenomenon is extremely useful in arithmetic geometry since it allows us to determine whether certain divisors vanish in the Jacobian of a modular curve, a condition which plays a central role in the work of Gross and Zagier on the Birch and Swinnerton-Dyer Conjecture. Here we explain how to make use of these generalized Borcherds products to extend deep theorems of Waldspurger and Kohnen.

In the 1980s, Waldspurger [238], and Kohnen and Zagier [161, 162, 163] established that certain half-integral weight modular forms serve as generating functions of a new type. Using the Shimura correspondence [223], they proved that certain coefficients of half-integral weight cusp forms essentially are square-roots of central values of quadratic twists of modular L-functions.

When the weight is $3/2$, these results appear prominently in works related to the Birch and Swinnerton-Dyer Conjecture. For example, Tunnell [234] made great use of explicit examples of these results in his work on the ancient "Congruent Number Problem": the determination of those positive integers which are areas of right triangles with rational sidelengths. More generally, these results of Kohnen, Waldspurger and Zagier play central roles in the deep works of Gross, Zagier and Kohnen [134, 133] on the Birch and Swinnerton-Dyer Conjecture.

The author and Bruinier [77] have generalized this theorem of Waldspurger and Kohnen to prove that the Fourier coefficients of weight $1/2$ harmonic Maass forms encode the vanishing and nonvanishing of both the central values and derivatives of quadratic twists of weight 2 modular L-functions.

Here we describe a special case of the main theorem. Suppose that

$$(2.9)\qquad G(z) = \sum_{n=1}^{\infty} B_G(n)q^n \in S_2(p)$$

is a weight 2 newform with prime level p. As usual, we let

$$L(G,s)=\sum_{n=1}^{\infty}\frac{B_G(n)}{n^s} \tag{2.10}$$

denote its Hecke L-function. If Δ is a fundamental discriminant of a quadratic field coprime to p, then we let $L(G,\chi_\Delta,s)$ be the *quadratic twist* Hecke L-function

$$L(G,\chi_\Delta,s)=\sum_{n=1}^{\infty}\frac{B_G(n)\chi_\Delta(n)}{n^s}, \tag{2.11}$$

where $\chi_\Delta(\bullet):=\left(\frac{\Delta}{\bullet}\right)$ denotes the Kronecker character for $\mathbb{Q}(\sqrt{\Delta})$. It is well known that $L(G,s)$ and $L(G,\chi_\Delta,s)$ have functional equations relating their values at s and $2-s$.

Here is a special case of the main result in [77].

Theorem 2.2. *Assume the hypotheses and notation above. In addition, suppose that the sign of the functional equation of $L(G,s)$ is $\epsilon(G)=-1$. Then there is a weight 1/2 harmonic Maass form $f(z)$ on $\Gamma_0(4p)$ with Fourier expansion*

$$f(z)=\sum_{n\gg-\infty}c_g^+(n)q^n+\sum_{n<0}c_g^-(n)\Gamma(1/2;4\pi|n|y)q^n,$$

where $y=\mathrm{Im}(z)$ and $\Gamma(\alpha;t)$ is the usual incomplete Gamma-function (see (4.5)), satisfying the following:

(1) *If $\Delta<0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then*

$$L(G,\chi_\Delta,1)=\alpha_G\cdot\sqrt{|\Delta|}\cdot c_g^-(\Delta)^2,$$

where α_G is an explicit non-zero constant.

(2) *If $\Delta>0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then $L'(G,\chi_\Delta,1)=0$ if and only if $c_g^+(\Delta)$ is algebraic.*

Remark. Theorem 2.2 is a special case of the general result which holds for all levels, and any arbitrary sign.

Remark. Bruinier [72] has recently obtained a nice formula for many of the coefficients $c_g^+(\Delta)$ in terms of periods of algebraic differentials of the third kind on modular and elliptic curves.

Example. Here we present an example which numerically illustrates the most general form of Theorem 2.2 for the weight 2 newform G which corresponds to the conductor 37 elliptic curve

$$E:\quad y^2=x^3+10x^2-20x+8.$$

The table below includes some of the coefficients of a suitable f which were numerically computed by F. Strömberg (also see [81]).

Δ	$c_g^+(-\Delta)$	$L'(E(\Delta),1) = L'(G,\chi_\Delta,1)$
-3	$1.0267149116\ldots$	$1.4792994920\ldots$
-4	$1.2205364009\ldots$	$1.8129978972\ldots$
-7	$1.6900297463\ldots$	$2.1107189801\ldots$
$\vdots$	$\vdots$	$\vdots$
-136	$-4.8392675993\ldots$	$5.7382407649\ldots$
-139	-6	0
-151	$-0.8313568817\ldots$	$6.6975085515\ldots$
$\vdots$	$\vdots$	$\vdots$
-815	$121.1944103120\ldots$	$4.7492583693\ldots$
-823	312	0

Strictly speaking, the cases where $\Delta = -139$ and -823 were not obtained numerically. We have that $L'(E(-139),1) = L'(E(-823),1) = 0$ by the Gross-Zagier formula. The evaluations $c_g^+(139) = -6$ and $c_g^+(823) = 312$ arise from explicit generalized Borcherds products (for example, see Example 8.3 of [77]). The rank 3 elliptic curve $E(-139)$ is quite famous, for it was used as input data for Goldfeld's celebrated effective solution to Gauss's "Class Number Problem". For the other Δ in the table, the derivatives are non-vanishing and the coefficients $c_g^+(-\Delta)$ are transcendental.

Theorem 2.2 relates the algebraicity of coefficients of harmonic Maass forms to the vanishing of derivatives of modular L-functions. It is then natural to ask whether these ideas can be used to exactly compute these derivatives. In other words, can the theory of harmonic Maass forms be used to obtain the deep formulas (as well as generalizations) of Gross and Zagier which relate heights of Heegner points to such derivatives? Bruinier and T. Yang have used these results, combined with their theory of Green's functions and theta lifts for harmonic Maass forms, to show that this is indeed the case. In addition to formulating a deep conjecture about derivatives of L-functions and heights (see Section 12.2), they have proved the following striking theorem (see [82]).

Theorem 2.3. *If $G \in S_2(N)$ is a weight 2 newform with the property that the sign of the functional equation of $L(G,s)$ is $\epsilon(G) = -1$, then there is a weight 1/2 harmonic Maass form f, a weight 3/2 cusp form g, and a Heegner divisor $Z(f)$ whose Neron-Tate height pairing is given by*

$$\langle Z(f), Z(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi \|g\|^2} L'\big(G,1\big).$$

Remark. To ease notation in Theorem 2.3, we did not describe the relationship between G, g, f and $Z(f)$. Loosely speaking, they are related as follows. We let g be a weight 3/2 cusp form whose image under the Shimura correspondence is G. The harmonic Maass form f in Theorems 2.2 and 2.3 is selected so that its principal part (see Section 4.1 for the definition)

is defined over the number field generated by the coefficients of G, and also satisfies

$$\xi_{\frac{1}{2}}(f) = \|g\|^{-2} g.$$

Here $\xi_{\frac{1}{2}}$ is the differential operator considered in Lemma 4.4. The Heegner divisor $Z(f)$ is then defined using the principal part of f.

These works shall be described in Section 12.

(Ramanujan's last letter revisited)

We return to the enigmatic last letter of Ramanujan. Ramanujan conjectured that his examples of the so-called mock theta functions have asymptotics which resemble those which arise in the theory of modular forms, but do not have the property that there are any weakly holomorphic modular forms which exactly cut out their exponential singularities. However, Ramanujan also found examples where modular forms very nearly cut out the singularities.

We recall Theorem 1.2 which gives formulas for Ramanujan's mysterious $O(1)$ numbers for his mock theta function $f(q)$.

Theorem 2.4. *If ζ is a primitive even order $2k$ root of unity, then, as q approaches ζ radially within the unit disk, we have that*

$$\lim_{q \to \zeta} \big(f(q) - (-1)^k b(q) \big) = -4 \cdot \sum_{n=0}^{k-1} (1+\zeta)^2 (1+\zeta^2)^2 \cdots (1+\zeta^n)^2 \zeta^{n+1}.$$

Remark. Since empty products equal 1, Theorem 1.2 shows that

$$\lim_{q \to -1} \left(f(q) + b(q) \right) = 4.$$

It turns out that this theorem is a special case of a much more general theorem, one which surprisingly relates two well known q-series in the theory of partitions. To make it precise, we require Dyson's rank function $R(w;q)$, the Andrews-Garvan crank function $C(w;q)$, and the recently studied q-hypergeometric series $U(w;q)$. The q-series $R(w;q)$ and $C(w;q)$ are among the most important generating functions in the theory of partitions. These famous series play a prominent role in the study of integer partition congruences (for example, see [28, 36, 64, 108, 178]).

The Andrews-Garvan crank function is defined by

$$C(w;q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} M(m,n) w^m q^n := \frac{(q;q)_\infty}{(wq;q)_\infty \cdot (w^{-1}q;q)_\infty}. \tag{2.12}$$

Here $M(m,n)$ is the number of partitions of n with crank m [28]. For roots of unity w, $C(w;q)$ is (up to a power of q) a modular form. The series $U(w;q)$ has recently been studied by several authors [21, 32, 83, 207] in work related to unimodal sequences. This q-hypergeometric series is defined by

$$U(w;q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} u(m,n) (-w)^m q^n := \sum_{n=0}^{\infty} (wq;q)_n \cdot (w^{-1}q;q)_n q^{n+1}. \tag{2.13}$$

Here $u(m,n)$ is the number of strongly unimodal sequences of size n with rank m [83].

Remark. In terms of the standard notation for q-hypergeometric series (for example, see p.4 of [122]), it turns out that $U(w;q)$ is a ${}_3\phi_2$ q-hypergeometric series. Namely, we have that

$$q^{-1}U(w;q) = {}_3\phi_2(wq, w^{-1}q, q; 0, 0; q, q).$$

Theorem 1.2 is a special case of the following general theorem which relates the asymptotic behavior of these three q-series. For roots of unity $w \neq 1$, we then have the mock theta function $R(w;q)$, the modular form $C(w;q)$, and the basic hypergeometric series $U(w;q)$. Throughout, we let $\zeta_n := e^{2\pi i/n}$.

Theorem 2.5. *Let $1 \leq a < b$ and $1 \leq h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b \mid k$. If h' is an integer satisfying $hh' \equiv -1 \pmod{k}$, then, as q approaches ζ_k^h radially within the unit disk, we have that*

$$\lim_{q \to \zeta_k^h} \left(R\left(\zeta_b^a; q\right) - \zeta_{b^2}^{-a^2h'k} C\left(\zeta_b^a; q\right) \right) = -(1-\zeta_b^a)(1-\zeta_b^{-a}) \cdot U(\zeta_b^a; \zeta_k^h).$$

Three remarks.
1) There is an integer $c(a,b,h,k)$ such that the limit in Theorem 2.5 reduces to the finite sum

$$-(1-\zeta_b^a)(1-\zeta_b^{-a}) \sum_{n=0}^{c(a,b,h,k)} (\zeta_b^a \zeta_k^h; \zeta_k^h)_n \cdot (\zeta_b^{-a} \zeta_k^h; \zeta_k^h)_n \cdot \zeta_k^{h(n+1)}.$$

2) Theorem 1.2 is the $a = 1$ and $b = 2$ case of Theorem 2.5 because $R(-1;q) = f(q)$, combined with the well known fact that $C(-1;q) = b(q)$.
3) A variant of Theorem 2.5 holds when $b \nmid k$. This is obtained by modifying the proof to guarantee that the two resulting asymptotic expressions match.

We now return to the original point of Ramanujan's last letter: the claim that his strange q-series are examples of his so-called *mock theta functions*. Earlier we presented the following fitting conclusion which confirms that his examples indeed satisfies his own vague definition of a mock theta function.

Theorem 2.6. *Suppose that $M(z)$ is one of Ramanujan's mock theta functions, and let γ and δ be integers for which $q^\gamma M(\delta z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q \to \zeta} (q^\gamma M(\delta z) - g(z)) = O(1).$$

This result is a special case of the following general theorem which we will prove.

Theorem 2.7. *Suppose that $f(z) = f^-(z) + f^+(z)$ is a harmonic weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_1(N)$, where $f^-(z)$ (resp. $f^+(z)$) is the nonholomorphic (resp. holomorphic) part of $f(z)$. If $f^-(z)$ is nonzero and $g(z)$ is a weight k weakly holomorphic modular form on any $\Gamma_1(N')$, then $f^+(z) - g(z)$ has exponential singularities at infinitely many roots of unity ζ.*

Remark. Harmonic weak Maass forms in this theorem are assumed to have principal parts at all cusps. Again, see Section 4.1 for definitions.

We shall prove Theorems 1.2,2.5, and 2.7 in Section 13.

(Quantum modular forms)

Recently, Bryson, Pitman, Rhoades and the author [83] investigated the q-series $U(w;q)$ defined in (2.13) in connection with the theory of quantum modular forms. Following Zagier[8] [247], a *weight k quantum modular form* is a complex-valued function f on $\mathbb{Q}$, or possibly $\mathbb{P}^1(\mathbb{Q}) \setminus S$ for some set S, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the function

$$h_\gamma(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a "suitable" property of continuity or analyticity. The $\epsilon(\gamma)$ are suitable complex numbers, such as those in the theory of half-integral weight modular forms when $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. In particular, Zagier offered a number of examples of such forms by making use of Dedekind sums, period polynomials, and a few curious q-series identities. Particularly interesting examples of such forms relate functions which are simultaneously defined on both $\mathbb{H} = \mathbb{H}^+$ and $\mathbb{H}^-$, the upper-half and lower-half of the complex plane respectively. The quantum form is the device which makes it possible to pass between the two half-planes.

We begin by recalling a particularly nice example related to the series $U(w;q)$ defined by (2.13). The q-series $U(1;q)$, the generating function for the number of size n sequences with even rank minus the number with odd rank, is intimately related to Kontsevich's strange function[9]

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n = 1 + (1-q) + (1-q)(1-q^2) + (1-q)(1-q^2)(1-q^3) + \dots. \tag{2.14}$$

This function is strange because it does not converge on any open subset of $\mathbb{C}$, but is well-defined at all roots of unity. Zagier [244] proved that this function satisfies the even "stranger" identity

$$F(q) = -\frac{1}{2} \sum_{n=1}^{\infty} n\chi_{12}(n) q^{\frac{n^2-1}{24}}, \tag{2.15}$$

where $\chi_{12}(\bullet) = \left(\frac{12}{\bullet}\right)$. Neither side of this identity makes sense simultaneously. Indeed, the right hand side[10] converges in the unit disk $|q| < 1$, but nowhere on the unit circle. The identity means that $F(q)$ at roots of unity agrees with the radial limit of the right hand side.

The series $U(1;q)$, which converges in $|q| < 1$, also gives $F(q^{-1})$ at roots of unity. The following theorem was proved in [83].

Theorem 2.8. *If q is a root of unity, then $F(q^{-1}) = U(1;q)$.*

Example. Here are two examples: $U(1;-1) = F(-1) = 3$ and $U(1;i) = F(-i) = 8 + 3i$.

[8]Zagier's definition is intentionally vague with the idea that sufficient flexibility is required to allow for interesting examples. Here we modify his defintion to include half-integral weights k and multiplier systems $\epsilon(\gamma)$.

[9]Zagier credits Kontsevich for relating $F(q)$ to Feynmann integrals in a lecture at Max Planck in 1997.

[10]As Zagier points out in Section 6 of [244], the right hand side of the identity is essentially the "half-derivative" of Dedekind's eta-function, which then suggests that the series may be related to a weight 3/2 modular object.

Remark. Theorem 2.8 is analogous to the result of Cohen [95, 247] that $\sigma(q) = -\sigma^*(q^{-1})$ for roots of unity q, for the well-known q-series $\sigma(q)$ and $\sigma^*(q)$ that Andrews, Dyson, and Hickerson [26] defined in their work on partition ranks.

Zagier [244] used (2.15) to obtain the following identity

$$e^{-\frac{t}{24}} \sum_{n=0}^{\infty} (1-e^{-t})(1-e^{-2t}) \dots (1-e^{-nt}) = \sum_{n=0}^{\infty} \frac{T_n}{n!} \cdot \left(\frac{t}{24}\right)^n, \tag{2.16}$$

where Glaisher's T_n numbers (see A002439 in [227]) are the algebraic parts of $L(\chi_{12}, 2n+2)$. As a companion to Theorem 2.8, we use $U(1;q)$ to give these same L-values.

Theorem 2.9. *As a power series in t, we have that*

$$e^{\frac{t}{24}} \cdot U(1; e^{-t}) = \sum_{n=0}^{\infty} \frac{T_n}{n!} \cdot \left(\frac{-t}{24}\right)^n = \frac{6\sqrt{3}}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{(2n+1)!}{n!} \cdot L(\chi_{12}, 2n+2) \cdot \left(\frac{-3t}{2\pi^2}\right)^n.$$

These results are related to the next theorem which gives a new quantum modular form. More precisely, we have that

$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(1; e^{2\pi i x}) \tag{2.17}$$

is a weight $\frac{3}{2}$ quantum modular form. Since $\mathrm{SL}_2(\mathbb{Z}) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ and

$$\phi(x) - e^{\frac{\pi i}{12}} \cdot \phi(x+1) = 0,$$

it suffices to consider $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The following theorem, proved in [83], establishes the desired relationship on the larger domain $\mathbb{Q} \cup \mathbb{H} - \{0\}$.

Theorem 2.10. *If $x \in \mathbb{Q} \cup \mathbb{H} - \{0\}$, then*

$$\phi(x) + (-ix)^{-\frac{3}{2}} \phi(-1/x) = h(x),$$

where $(ix)^{-\frac{3}{2}}$ is the principal branch and

$$h(x) := \frac{\sqrt{3}}{2\pi i} \int_0^{i\infty} \frac{\eta(\tau)}{(-i(x+\tau))^{\frac{3}{2}}} d\tau - \frac{i}{2} e^{\frac{\pi i x}{6}} (e^{2\pi i x}; e^{2\pi i x})_\infty^2 \cdot \int_0^{i\infty} \frac{\eta(\tau)^3}{(-i(x+\tau))^{\frac{1}{2}}} d\tau.$$

Here $\eta(\tau) := e^{\frac{\pi i \tau}{12}} (e^{2\pi i \tau}; e^{2\pi i \tau})_\infty$ is Dedekind's eta-function. Moreover, taking $\eta(x) = 0$ for $x \in \mathbb{R}$, $h : \mathbb{R} \to \mathbb{C}$ is a C^∞ function which is real analytic everywhere except at $x = 0$, and $h^{(n)}(0) = (-\pi i/12)^n \cdot T_n$, where T_n is the nth Glaisher number.

Remark. Zagier [244] proved that $e^{\frac{\pi i x}{12}} \cdot F(e^{2\pi i x})$ is a quantum modular form. Theorem 2.10 gives a dual quantum modular form, one whose domain naturally extends beyond $\mathbb{Q}$ to include $\mathbb{H}$. This is somewhat analogous to the situation for $\sigma(q)$ and $\sigma^*(q)$ discussed above. In a recent paper, Rolen and Schneider [215] obtained a similarly strange vector-valued quantum modular form.

Remark. Theorem 2.10 implies that $\Phi(z) := \eta(z)\phi(z)$ behaves analogously to a weight 2 modular form for $\mathrm{SL}_2(\mathbb{Z})$ for $z \in \mathbb{H}$ with a suitable error function. Namely, $\Phi(z+1) = \Phi(z)$ and $\Phi(z) - z^{-2}\Phi\left(-\frac{1}{z}\right) = \eta(z)h(z)$, see also Theorem 1.1 of [32].

In view of the roles that $R(w;q)$ (which is essentially a mock modular form for roots of unity $w \neq 1$) and $U(w;q)$ play in Theorem 2.5, it is natural to ask about the more general relationship between mock theta functions and quantum modular forms. To this end, we seek q-hypergeometric series related to mock theta functions which are defined on both $\mathbb{H}^+$ and $\mathbb{H}^-$. In doing so, we are led to the "False" ϑ-functions of Rogers and Fine, which Ramanujan claimed do not *"enter into mathematics as beautifully as the ordinary theta functions"*.

We recall these functions. In 1917 Rogers [213] defined the important q-hypergeometric series

$$F(\alpha, \beta, t; q) := \sum_{n=0}^{\infty} \frac{(\alpha q; q)_n t^n}{(\beta q; q)_n}. \tag{2.18}$$

This series does not typically specialize to modular forms, but instead often gives "halves" of modular theta functions. These include many of the primary examples of "false" and "partial" ϑ-functions. For example, we have the following special case of the work of Rogers and Fine [114]:

$$\frac{1}{1+w} \cdot F(wq^{-1}, -w, w; q) := \frac{1}{1+w} \cdot \sum_{n=0}^{\infty} \frac{(w;q)_n w^n}{(-wq;q)_n} = \sum_{n=0}^{\infty} (-1)^n w^{2n} q^{n^2}.$$

Here we consider the following specializations, where $q := e^{2\pi i z}$:

$$G(a,b;z) := \frac{q^{\frac{a^2}{b^2}}}{1-q^{\frac{a}{b}}} \cdot F\left(-q^{\frac{a}{b}-1}, q^{\frac{a}{b}}, -q^{\frac{a}{b}}; q\right), \tag{2.19}$$

$$H(a,b;z) := q^{\frac{1}{8}} \cdot F\left(\zeta_b^{-a} q^{-1}, \zeta_b^{-a}, \zeta_b^{-a} q; q^2\right). \tag{2.20}$$

We have the following false theta function identities, the second of which follows from equation (1) of [213], and the first of which is in [31].

$$G(a,b;z) = q^{\frac{a^2}{b^2}} \sum_{n=0}^{\infty} \frac{(-q^{\frac{a}{b}};q)_n}{(q^{\frac{a}{b}};q)_{n+1}} \cdot (-1)^n q^{n\frac{a}{b}} = \sum_{n=0}^{\infty} (-1)^n q^{\left(n+\frac{a}{b}\right)^2}, \tag{2.21}$$

$$H(a,b;z) = q^{\frac{1}{8}} \sum_{n=0}^{\infty} \frac{(\zeta_b^{-a} q; q^2)_n}{(\zeta_b^{-a} q^2; q^2)_n} \cdot (\zeta_b^{-a} q)^n = \sum_{n=0}^{\infty} \zeta_b^{-an} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}. \tag{2.22}$$

Remark. The second equalities in (2.21) and (2.22) are only valid for $|q| < 1$.

These specializations satisfy the following nice properties often associated to quantum modular forms: convergence in $\mathbb{H}^{\pm}$, a modular transformation law, and asymptotic expansions which are generating functions for values of L-functions. More precisely, we prove the following theorem.

Theorem 2.11. *Let $0 < a < b$ be coprime integers, with b even, and let*

$$Q_{a,b} := \left\{ \frac{h}{k} \in \mathbb{Q} : \gcd(h,k) = 1, h > 0, b \mid 2h, b \nmid h, k \equiv a \pmod{b}, k \geq a \right\}.$$

The following are true:
(1) The functions $G(a,b;z)$ and $H(a,b;z)$ converge for $z \in \mathbb{H}^+ \cup \mathbb{H}^-$.

(2) For $x \in Q_{a,b} \cup \mathbb{H}^+$, we have that

$$G(a,b;-x) + \frac{e^{-\frac{\pi i a}{b}}}{\sqrt{2ix}} \cdot H\left(a,b;\frac{1}{2x}\right) = -\int_0^{i\infty} \frac{(-iu)^{-\frac{3}{2}} T\left(a,b;-\frac{1}{u}\right) du}{\sqrt{-i(u+x)}},$$

where $T(a,b;z)$ is a weight $3/2$ modular form defined by

$$T(a,b;z) := i \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{4}\right) \cosh\left(2\pi i \left(n+\frac{1}{4}\right)\left(\frac{2a}{b}-1\right)\right) q^{\left(n+\frac{1}{4}\right)^2}.$$

That is, $G(a,b;x)$ and $H(a,b;x)$ are weight 1/2 quantum modular forms on $Q_{a,b} \cup \mathbb{H}^+$.
(3) Let $B_r(n)$ be the rth Bernoulli polynomial. For $\frac{h}{k} \in Q_{a,b}$, as $t \to 0^+$, we have

$$G\left(a,b;\frac{-h}{k}+\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r,c_G) \cdot \frac{(-t)^r}{r! \cdot b^{2r}},$$

$$H\left(a,b;\frac{k}{2h}+\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r,c_H) \cdot \frac{(-t)^r}{r! \cdot 8^r},$$

where

$$L(-r,c_G) = -\frac{(2kb^2)^r}{r+1} \sum_{n=1}^{2kb^2} c_G(n) B_{r+1}\left(\frac{n}{2kb^2}\right),$$

$$L(-r,c_H) = -\frac{(16h)^r}{r+1} \sum_{n=1}^{16h} c_H(n) B_{r+1}\left(\frac{n}{16h}\right),$$

$$c_G(n) := \begin{cases} \zeta_{kb^2}^{-hn^2}, & \text{if } n \equiv a \pmod{2b}, \\ -\zeta_{kb^2}^{-hn^2}, & \text{if } n \equiv a+b \pmod{2b}, \\ 0, & \text{otherwise}, \end{cases}$$

$$c_H(n) := \begin{cases} \zeta_b^{-a\left(\frac{n-1}{2}\right)} \zeta_{16h}^{kn^2}, & \text{if } n \equiv 1 \pmod{2}, \\ 0, & \text{otherwise}. \end{cases}$$

Two remarks.
1) For $x = \frac{h}{k} \in Q_{a,b}$, we have that $G(a,b;-x)$ and $H\left(a,b;\frac{1}{2x}\right)$ converge. Moreover, they are explicitly given by the finite sums

$$G\left(a,b;-\frac{h}{k}\right) = \zeta_{b^2k}^{-a^2h} \sum_{n=0}^{m} \frac{(-\zeta_{bk}^{-ah};\zeta_k^{-h})_n (-\zeta_{bk}^{-ah})^n}{(\zeta_{bk}^{-ah};\zeta_k^{-h})_{n+1}}, \tag{2.23}$$

$$H\left(a,b;\frac{k}{2h}\right) = \zeta_{16h}^{k} \sum_{n=0}^{\ell} \frac{(\zeta_b^{-a}\zeta_{2h}^{k};\zeta_h^{k})_n (\zeta_b^{-a}\zeta_{2h}^{k})^n}{(\zeta_b^{-a}\zeta_h^{k};\zeta_h^{k})_n}, \tag{2.24}$$

where the non-negative integers ℓ and m satisfy $b(2\ell+1) = 2h$ and $a+bm = k$, respectively.
2) In Theorem 2.11 (2), we are using the vector-valued notion of a quantum modular form.

Example. Here we will illustrate how the different parts of Theorem 2.11 may be used to understand the Rogers-Fine functions (2.21) and (2.22), and relations between them, at rational numbers z by way of an example. If $a = 1$ and $b = 2$, then (2.21) and (2.22) give the identities

$$G(1,2;z) := q^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}};q)_n (-1)^n q^{\frac{n}{2}}}{(q^{\frac{1}{2}};q)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{(2n+1)^2}{4}},$$

$$H(1,2;z) := q^{\frac{1}{8}} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n (-q)^n}{(-q^2;q^2)_n} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{(2n+1)^2}{8}},$$

so that $G(1,2;z) = H(1,2;2z)$. For simplicity, we consider the rational number $\frac{h}{k} = \frac{1}{1} \in Q_{1,2}$.

On one hand, from Theorem 2.11 (3), we find that asymptotically, as $t \to 0^+$, we have

$$G\left(1,2;-1+\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r, c_G) \cdot \frac{(-t)^r}{r! \cdot b^{2r}}.$$

We compute (using that $B_1(x) = x - 1/2$) that $L(0, c_G) = -\frac{i}{2}$, and so $G(1,2;-1) \sim -\frac{i}{2}$. On the other hand, Thereom 2.11 gives $G(1,2;-1)$ exactly, as a finite sum, using (2.23). In particular, we have $m = 0$ for $(a,b,h,k) = (1,2,1,1)$, and hence we compute that at the root of unity -1, the function $G(1,2;-1)$ is exactly equal to $G(1,2;-1) = \frac{\zeta_4^{-1}}{1-\zeta_2^{-1}} = -\frac{i}{2}$.

Similarly, using Theorem 2.11 (3), we find that asymptotically, $H\left(1,2;\frac{1}{2}\right) \sim \zeta_{16}$. On the other hand, we may evaluate $H\left(1,2;\frac{1}{2}\right)$ exactly as a finite sum using (2.24). Indeed, we find that $H\left(1,2;\frac{1}{2}\right) = \zeta_{16}$.

We may combine these calculations with Theorem 2.11 (2), to find an exact value for the integral expression appearing in Theorem 2.11 (2). Namely, we have that

$$\frac{i}{4} \int_0^{i\infty} \frac{(-iu)^{-\frac{3}{2}} \Theta_{1,4}\left(-\frac{1}{u}\right) du}{\sqrt{-i(u+1)}} = \frac{i}{2} - \sqrt{\frac{i}{2}} \cdot \zeta_{16},$$

where $\Theta_{1,4}(z) := \sum_{n \equiv 1 \pmod 4} n q^{\frac{n^2}{16}}$.

In Section 14 we shall sketch the proof of Theorem 2.11.

3. The mock theta functions of Ramanujan d'après Zwegers

Now we begin to describe recent results beginning with the work of Zwegers. We first give a typical example of Zwegers's results on Ramanujan's mock theta functions, and we briefly describe his construction of a weight 1/2 non-holomorphic Jacobi form.

3.1. **Ramanujan's mock theta functions $f(q)$ and $\omega(q)$.** We first consider Ramanujan's third order mock theta function $f(q)$. Important results concerning its modularity properties were first obtained by G. N. Watson in [239]. Although $f(q)$ is not the Fourier expansion of a usual meromorphic modular form, in this classic paper Watson determined its complicated modular transformation properties. Watson's modular transformation formulas are difficult to grasp at first glance; they involve another third order mock theta function, as well as terms arising from Mordell integrals.

Zwegers [252] nicely packaged Watson's results in terms of real analytic vector-valued modular forms. We now describe this result. As before, let $\omega(q)$ be the third order mock theta function defined by (1.6). Define the vector-valued function $F(z)$ by

$$F(z) = (F_0(z),\ F_1(z),\ F_2(z))^T := (q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T. \tag{3.1}$$

Similarly, let $G(z)$ be the vector-valued non-holomorphic function defined by

$$G(z) = (G_0(z),\ G_1(z),\ G_2(z))^T := 2i\sqrt{3}\int_{-\overline{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau, \tag{3.2}$$

where the $g_i(\tau)$ are the cuspidal weight $3/2$ theta functions

$$\begin{aligned} g_0(\tau) &:= \sum_{n=-\infty}^{\infty} (-1)^n \left(n+\frac{1}{3}\right) e^{3\pi i\left(n+\frac{1}{3}\right)^2\tau}, \\ g_1(\tau) &:= -\sum_{n=-\infty}^{\infty} \left(n+\frac{1}{6}\right) e^{3\pi i\left(n+\frac{1}{6}\right)^2\tau}, \\ g_2(\tau) &:= \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{3}\right) e^{3\pi i\left(n+\frac{1}{3}\right)^2\tau}. \end{aligned} \tag{3.3}$$

Using these vector-valued functions, Zwegers defined the vector-valued function $H(z)$ by

$$H(z) := F(z) - G(z). \tag{3.4}$$

The following description of $H(z)$ is the main result of [252].

Theorem 3.1. (Zwegers)
The function $H(z)$ is a vector-valued real analytic modular form of weight $1/2$ satisfying

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z),$$

$$H(-1/z) = \sqrt{-iz}\cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $\zeta_n := e^{2\pi i/n}$. Furthermore, $H(z)$ is an eigenfunction of the Casimir operator $\Omega_{\frac{1}{2}} := -4y^2\frac{\partial^2}{\partial z\partial\overline{z}} + iy\frac{\partial}{\partial\overline{z}} + \frac{3}{16}$ with eigenvalue $\frac{3}{16}$, where $z = x+iy$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, and $\frac{\partial}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Remark. This Casimir operator is essentially the weight $1/2$ hyperbolic Laplacian operator (see (4.1)) which plays an important role in the definition of harmonic Maass forms.

This beautiful theorem nicely describes the modular transformations of both $f(q)$ and $\omega(q)$. In particular, they are essentially the "holomorphic parts" of components of a 3-dimensional vector-valued weight $1/2$ real analytic modular form. The "non-holomorphic"

parts of these components are "period integrals" of classical weight 3/2 functions which turn out to be single variable theta functions. The functions in Theorem 3.1 can be thought of as prototypes for the theory of harmonic Maass forms.

3.2. **Zwegers's weight 1/2 non-holomorphic Jacobi form.** In his thesis, Zwegers constructed weight 1/2 harmonic Maass forms by making use of the transformation properties of Lerch sums. Here we briefly recall some of these important results which address the difficult problem of constructing weight 1/2 harmonic Maass forms.

For $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, Zwegers defined the function

$$\mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v; \tau)} \cdot \sum_{n \in \mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n}, \tag{3.5}$$

where $z := e^{2\pi i u}$, $w := e^{2\pi i v}$, $q := e^{2\pi i \tau}$ and

$$\vartheta(v; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} e^{\pi i \nu} w^{\nu} q^{\nu^2/2}. \tag{3.6}$$

Remark. We stress that $q := e^{2\pi i \tau}$ in this subsection, which is a brief departure from our convention that $q := e^{2\pi i z}$.

Zwegers (see Section 1.3 of [253]) proves that $\mu(u, v; \tau)$ satisfies the following important properties.

Lemma 3.2. *Assuming the notation above, we have that*

$$\begin{aligned}
\mu(u, v; \tau) &= \mu(v, u; \tau), \\
\mu(u + 1, v; \tau) &= -\mu(u, v; \tau), \\
z^{-1} w q^{-\frac{1}{2}} \mu(u + \tau, v; \tau) &= -\mu(u, v; \tau) - i z^{-\frac{1}{2}} w^{\frac{1}{2}} q^{-\frac{1}{8}}, \\
\mu(u, v; \tau + 1) &= \zeta_8^{-1} \mu(u, v; \tau) \qquad (\zeta_N := e^{2\pi i/N}) \\
(\tau/i)^{-\frac{1}{2}} e^{\pi i (u-v)^2/\tau} \mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) &= -\mu(u, v; \tau) + \frac{1}{2i} h(u - v; \tau),
\end{aligned}$$

where

$$h(z; \tau) := \int_{-\infty}^{\infty} \frac{e^{\pi i x^2 \tau - 2\pi i x z} dx}{\cosh \pi x}.$$

Remark. The integral $h(z; \tau)$ is known as a *Mordell integral*.

Lemma 3.2 shows that $\mu(u, v; \tau)$ is nearly a weight 1/2 Jacobi form, where τ is the modular variable. Zwegers then uses μ to construct weight 1/2 harmonic Maass forms. He achieves this by modifying μ to obtain a function $\widehat{\mu}$ which he then uses as building blocks for such Maass forms. To make this precise, for $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$, let $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$, and let

$$R(u; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\nu - \frac{1}{2}} \left\{ \mathrm{sgn}(\nu) - E\left((\nu + c) \sqrt{2\mathrm{Im}(\tau)} \right) \right\} e^{-2\pi i \nu u} q^{-\nu^2/2}, \tag{3.7}$$

where $E(x)$ is the odd function

$$E(x) := 2\int_0^x e^{-\pi u^2} du = \mathrm{sgn}(x)(1-\beta(x^2)), \tag{3.8}$$

where for positive real x we let $\beta(x) := \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u} du$.

Using μ and R, Zwegers defines the real analytic function

$$\widehat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2} R(u-v;\tau). \tag{3.9}$$

Zwegers construction of weight 1/2 harmonic Maass forms depends on the following theorem (see Section 1.4 of [253]).

Theorem 3.3. *Assuming the notation and hypotheses above, we have that*

$$\begin{aligned}
\widehat{\mu}(u,v;\tau) &= \widehat{\mu}(v,u;\tau),\\
\widehat{\mu}(u+1,v;\tau) &= z^{-1} w q^{-\frac{1}{2}} \widehat{\mu}(u+\tau,v;\tau) = -\widehat{\mu}(u,v;\tau),\\
\zeta_8 \widehat{\mu}(u,v;\tau+1) &= -(\tau/i)^{-\frac{1}{2}} e^{\pi i (u-v)^2/\tau} \widehat{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = \widehat{\mu}(u,v;\tau).
\end{aligned}$$

Moreover, if $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *then*

$$\widehat{\mu}\left(\frac{u}{\gamma\tau+\delta}, \frac{v}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \chi(A)^{-3} (\gamma\tau+\delta)^{\frac{1}{2}} e^{-\pi i \gamma (u-v)^2/(\gamma\tau+\delta)} \cdot \widehat{\mu}(u,v;\tau),$$

where $\chi(A) := \eta(A\tau) / \left((\gamma\tau+\delta)^{\frac{1}{2}} \eta(\tau)\right)$.

Theorem 3.3 shows that $\widehat{\mu}(u,v;\tau)$ is essentially a weight 1/2 non-holomorphic Jacobi form. In analogy with the classical theory of Jacobi forms, one may then obtain harmonic Maass forms by making suitable specializations for u and v by elements in $\mathbb{Q}\tau + \mathbb{Q}$, and by multiplying by appropriate powers of q. Without this result, it would be very difficult to explicitly construct examples of weight 1/2 harmonic Maass forms.

Harmonic Maass forms of weight k are mapped to classical modular forms (see Lemma 4.4), their so-called *shadows*, by the differential operator

$$\xi_k := 2iy^k \cdot \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

The following lemma makes it clear that the shadows of the real analytic forms arising from $\widehat{\mu}$ can be described in terms of weight 3/2 theta functions.

Lemma 3.4. [Lemma 1.8 of [253]] *The function R is real analytic and satisfies*

$$\frac{\partial R}{\partial \overline{u}}(u;\tau) = \sqrt{2} y^{-\frac{1}{2}} e^{-2\pi c^2 y} \vartheta(\overline{u}; -\overline{\tau}),$$

where $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$. *Moreover, we have that*

$$\frac{\partial}{\partial \overline{\tau}} R(a\tau - b;\tau) = -\frac{i}{\sqrt{2y}} e^{-2\pi a^2 y} \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} (-1)^{\nu-\frac{1}{2}} (\nu+a) e^{-\pi i \nu^2 \overline{\tau} - 2\pi i \nu (a\overline{\tau} - b)}.$$

4. Harmonic Maass forms

For the remainder of the paper, we shall assume that the reader is familiar with the classical theory of elliptic modular forms (for example, see [84, 100, 151, 160, 169, 182, 191, 205, 216, 224, 228]).

D. Niebur [187, 188] and D. Hejhal [143] constructed certain non-holomorphic Poincaré series which turn out to be examples of *harmonic Maass forms*. Bruinier [70] made great use of these Poincaré series in his early work on Borcherds lifts and Green's functions. He then realized the importance of developing a "theory of harmonic Maass forms" in its own right. Later in joint work with Funke [73], he developed the fundamental results of this theory, some of which we describe here. After making the necessary definitions, we shall discuss Hecke operators and various differential operators. The interplay between harmonic Maass forms and classical modular forms shall play an important role throughout this paper.

4.1. **Definitions.** In 1949, H. Maass introduced the notion of a *Maass form*[11] (see [176, 177]). He constructed these non-holomorphic automorphic forms using Hecke characters of real quadratic fields, in analogy with Hecke's theory [141] of *modular forms with complex multiplication* (see [210] for a modern treatment).

To define these functions, let $\Delta = \Delta_0$ be the *hyperbolic Laplacian*

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

where $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. It is a second-order differential operator which acts on functions on $\mathbb{H}$, and it is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

A *Maass form* for a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have
$$f\left(\frac{az+b}{cz+d}\right) = f(z).$$
(2) We have that f is an eigenfunction of Δ.
(3) There is some $N > 0$ such that
$$f(x+iy) = O(y^N)$$
as $y \to +\infty$.

Furthermore, we call f a *Maass cusp form* if

$$\int_0^1 f(z+x)dx = 0.$$

There is now a vast literature on Maass forms thanks to the works of many authors such as Hejhal, Iwaniec, Maass, Roelcke, Selberg, Terras, Venkov, among many others (for example, see [142, 143, 150, 152, 176, 177, 211, 222, 231, 232, 235]).

This paper concerns a generalization of this notion of Maass form. Following Bruinier and Funke [73], we define the notion of a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ as follows. As

[11] In analogy with the eigenvalue problem for the vibrating membrane, Maass referred to these automorphic forms as *Wellenformen*, or *waveforms*.

before, we let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. We define the weight k hyperbolic Laplacian Δ_k by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{4.1}$$

For odd integers d, define ϵ_d by

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases} \tag{4.2}$$

Definition 4.1. If N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then a *weight k harmonic Maass form on* $\Gamma \in \{\Gamma_1(N), \Gamma_0(N)\}$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$M\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k M(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k \ M(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and $\sqrt{z}$ is the principal branch of the holomorphic square root.

(2) We have that $\Delta_k M = 0$.

(3) There is a polynomial $P_M = \sum_{n \leq 0} c^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that

$$M(z) - P_M(z) = O(e^{-\epsilon y})$$

as $y \to +\infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps.

Remark. Maass forms and classical modular forms are required to satisfy moderate growth conditions at cusps, and it is for this reason that harmonic Maass forms are often referred to as "harmonic weak Maass forms". The term "weak" refers to the relaxed condition Definition 4.1 (3) which gives rise to a rich theory. For convenience, we use the terminology "harmonic Maass form" instead of "harmonic weak Maass form".

Remark. We refer to the polynomial P_M as the *principal part* of $M(z)$ at ∞. Obviously, if P_M is non-constant, then $M(z)$ has exponential growth at ∞. Similar remarks apply at all cusps.

Remark. We do not have time here to discuss the interesting recent work of Candelori [85] which interpretes some essential features of the theory of harmonic Maass forms in a p-adic context.

Remark. Bruinier and Funke [73] define two types of harmonic Maass forms based on varying the growth conditions at cusps. For a group Γ, they refer to these spaces as $H_k(\Gamma)$ and $H_k^+(\Gamma)$. Definition 4.1 (3) corresponds to their $H_k^+(\Gamma)$ definition.

Remark. Since holomorphic functions on $\mathbb{H}$ are harmonic, it follows that weakly holomorphic modular forms are harmonic Maass forms.

Remark. Here we recall the congruence subgroups. If N is a positive integer, then define the level N *congruence subgroups* $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \pmod{N}, \text{ and } c \equiv 0 \pmod{N} \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\}.$$

Remark. For $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, the transformation law in Definition 4.1 (1) coincides with those in Shimura's theory of half-integral weight modular forms [223].

Remark. Later we shall require the classical "slash" operator. For convenience, we recall its definition here. Suppose that $k \in \frac{1}{2}\mathbb{Z}$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ ($\Gamma_0(4)$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), define $j(A, z)$ by

$$j(A, z) := \begin{cases} \sqrt{cz+d} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \tag{4.3}$$

where ε_d is defined by (4.2), and where $\sqrt{z}$ is the principal branch of the holomorphic square root as before. For functions $f : \mathbb{H} \to \mathbb{C}$, we define the action of the "slash" operator by

$$(f \mid_k A)(z) := j(A, z)^{-2k} f(Az) = j(A, z)^{-2k} f\left(\frac{az+b}{cz+d}\right). \tag{4.4}$$

Notice that Definition 4.1 (1) may be rephrased as

$$(M \mid_k A)(z) = M(z).$$

Remark. We shall also consider *level N weight $k \in \frac{1}{2}\mathbb{Z}$ forms with Nebentypus χ*. To define such forms, suppose that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), and let χ be a Dirichlet character modulo N. To define these forms, one merely requires[12], for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, that

$$M\left(\frac{az+b}{cz+d}\right) = \begin{cases} \chi(d)(cz+d)^k M(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \chi(d)(cz+d)^k \ M(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Throughout, we shall adopt the following notation. If $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup, then we let

$$\begin{aligned} S_k(\Gamma) &:= \text{weight } k \text{ cusp forms on } \Gamma, \\ M_k(\Gamma) &:= \text{weight } k \text{ holomorphic modular forms on } \Gamma, \\ M_k^!(\Gamma) &:= \text{weight } k \text{ weakly holomorphic modular forms on } \Gamma, \\ H_k(\Gamma) &:= \text{weight } k \text{ harmonic Maass forms on } \Gamma. \end{aligned}$$

[12]This replaces (1) in Definition 4.1.

Furthermore, if χ is a Dirichlet character modulo N, then we let

$$\begin{aligned} S_k(N,\chi) &:= \text{level } N \text{ weight } k \text{ cusp forms with Nebentypus } \chi, \\ M_k(N,\chi) &:= \text{level } N \text{ weight } k \text{ holomorphic modular forms with Nebentypus } \chi, \\ M_k^!(N,\chi) &:= \text{level } N \text{ weight } k \text{ weakly holomorphic modular forms with Nebentypus } \chi, \\ H_k(N,\chi) &:= \text{level } N \text{ weight } k \text{ harmonic Maass forms with Nebentypus } \chi. \end{aligned}$$

When the Nebentypus character is trivial, we shall suppress χ from the notation.

The real analytic forms in Theorem 3.1 provide non-trivial examples of weight $1/2$ harmonic Maass forms. More generally, the work of Zwegers [252, 253], shows how to complete all of Ramanujan's mock theta functions to obtain weight $1/2$ harmonic Maass forms. In Section 5, we shall present further examples of harmonic Maass forms.

4.2. **Fourier expansions.** In this paper we consider harmonic Maass forms with weight $2-k \in \frac{1}{2}\mathbb{Z}$ with $k > 1$. Therefore, throughout we assume that $1 < k \in \frac{1}{2}\mathbb{Z}$.

Harmonic Maass forms have distinguished Fourier expansions which are described in terms of the incomplete Gamma-function $\Gamma(\alpha; x)$

$$\Gamma(\alpha; x) := \int_x^\infty e^{-t} t^{\alpha-1}\, dt, \tag{4.5}$$

and the usual parameter $q := e^{2\pi i z}$. The following characterization is straightforward (for example, see Section 3 of [73]).

Lemma 4.2. *Assume the notation and hypotheses above, and suppose that N is a positive integer. If $f(z) \in H_{2-k}(\Gamma_1(N))$, then its Fourier expansion is of the form*

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n, \tag{4.6}$$

where $z = x + iy \in \mathbb{H}$, with $x, y \in \mathbb{R}$.

As Lemma 4.2 reveals, $f(z)$ naturally decomposes into two summands. In view of this fact, we make the following definition.

Definition 4.3. Assuming the notation and hypotheses in Lemma 4.2, we refer to

$$f^+(z) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

as the *holomorphic part of $f(z)$*, and we refer to

$$f^-(z) := \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n$$

as the *non-holomorphic part of $f(z)$*.

Remark. A harmonic Maass form with trivial non-holomorphic part is a weakly holomorphic modular form. We shall make use of this fact as follows. If $f_1, f_2 \in H_{2-k}(\Gamma)$ are two harmonic Maass forms with equal non-holomorphic parts, then $f_1 - f_2 \in M_{2-k}^!(\Gamma)$.

4.3. **The ξ-operator and period integrals of cusp forms.** Harmonic Maass forms are related to classical modular forms thanks to the properties of differential operators. The first nontrivial relationship depends on the differential operator

$$\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \overline{z}}}. \tag{4.7}$$

The following lemma[13], which is a straightforward refinement of a proposition of Bruinier and Funke (see Proposition 3.2 of [73]), shall play a central role throughout this paper.

Lemma 4.4. *If $f \in H_{2-k}(N, \chi)$, then*

$$\xi_{2-k} : H_{2-k}(N, \chi) \longrightarrow S_k(N, \overline{\chi})$$

is a surjective map. Moreover, assuming the notation in Definition 4.3, we have that

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n.$$

Thanks to Lemma 4.4, we are in a position to relate the non-holomorphic parts of harmonic Maass forms, the expansions

$$f^-(z) := \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n,$$

with "period integrals" of modular forms. This observation was critical in Zwegers's work on Ramanujan's mock theta functions.

To make this connection, we must relate the Fourier expansion of the cusp form $\xi_{2-k}(f)$ with $f^-(z)$. This connection is made by applying the simple integral identity

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau+z))^{2-k}} \, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y) q^{-n}. \tag{4.8}$$

This identity follows by the direct calculation

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau+z))^{2-k}} \, d\tau = \int_{2iy}^{i\infty} \frac{e^{2\pi i n(\tau - z)}}{(-i\tau)^{2-k}} \, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y) \, q^{-n}.$$

In this way, we may think of the non-holomorphic parts of weight $2-k$ harmonic Maass forms as period integrals of weight k cusp forms, where one applies (4.8) to

$$\int_{-\overline{z}}^{i\infty} \frac{\sum_{n=1}^{\infty} a(n) e^{2\pi i n \tau}}{(-i(\tau+z))^{2-k}} \, d\tau,$$

where $\sum_{n=1}^{\infty} a(n) q^n$ is a weight k cusp form. In short, $f^-(z)$ is the period integral of the cusp form $\xi_{2-k}(f)$.

This connection is nicely illustrated in the case of certain integer weight harmonic Maass forms, where one can use holomorphic parts of these forms to compute the classical "period polynomials". As is well known, these polynomials are generating functions for the critical values of modular L-functions. Here we make this precise in one nice setting (see [59]).

[13]The formula for $\xi_{2-k}(f)$ corrects a typographical error in [73].

Suppose that $\mathcal{F}(z) := \mathcal{F}^-(z) + \mathcal{F}^+(z) \in H_{2-k}(1)$, where k is a positive even integer. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define the *$\gamma$-mock modular period function* for $\mathcal{F}^+$ by

$$\mathbb{P}\left(\mathcal{F}^+, \gamma; z\right) := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot \left(\mathcal{F}^+ - \mathcal{F}^+|_{2-k}\gamma\right)(z), \tag{4.9}$$

where for any function g, we let $\left(g|_w\gamma\right)(z) := (cz+d)^{-w} g\left(\frac{az+b}{cz+d}\right)$. The map

$$\gamma \mapsto \mathbb{P}\left(\mathcal{F}^+, \gamma; z\right)$$

gives an element in the first cohomology group of $\mathrm{SL}_2(\mathbb{Z})$ with polynomial coefficients, and we shall see that they are intimately related to classical "period polynomials".

For positive c, let $\zeta_c := e^{2\pi i/c}$, and for $0 \le d < c$, let $\gamma_{c,d} \in \mathrm{SL}_2(\mathbb{Z})$ be any matrix satisfying $\gamma_{c,d} := \left(\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right)$. Here the integers $0 \le d' < c'$ are chosen so that $\frac{d}{c} = \frac{d'}{c'}$ in lowest terms. The following result is motivation for the paper [59] by Bringmann, Guerzhoy, Kent, and the author.

Theorem 4.5. *Suppose that $4 \le k \in 2\mathbb{Z}$, and suppose that $\mathcal{F} \in H_{2-k}(1)$ and $f = \xi_{2-k}(\mathcal{F})$. Then we have that*

$$\overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{1,0}; \overline{z}\right)} = \sum_{n=0}^{k-2} \frac{L(f, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}.$$

Moreover, if $\chi \pmod c$ is a Dirichlet character, then

$$\frac{1}{c} \sum_{m \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(m) \sum_{d=0}^{c-1} \zeta_c^{md} \cdot \overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; \overline{z} - \frac{d}{c}\right)} = \sum_{n=0}^{k-2} \frac{L(f, \chi, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}.$$

Here $L(f,s)$ (resp. $L(f,\chi,s)$) is the usual L-function (resp. twisted by χ) for f.

Sketch of the proof. The proof follows from the fact that the non-holomorphic part $\mathcal{F}^-$ can be described in terms of a "period integral" of f as described above. In particular, it then suffices to consider the integral

$$\int_{-\overline{z}}^{i\infty} f^c(\tau)(z+\tau)^{k-2} d\tau.$$

Here we let $f^c(z) := \overline{f(-\overline{z})}$. The theorem then follows from the standard fact, for $0 \le n \le k-2$, that

$$L(f, n+1) = \frac{(2\pi)^{n+1}}{n!} \cdot \int_0^\infty f(it) t^n dt.$$

We leave the details to the reader. □

In addition to these important observations, we require the following fact concerning the nontriviality of certain principal parts of harmonic Maass forms.

Lemma 4.6. *If $f \in H_{2-k}(\Gamma)$ has the property that $\xi_{2-k}(f) \neq 0$, then the principal part of f is nonconstant for at least one cusp.*

Sketch of the proof. This lemma follows from the work of Bruinier and Funke [73]. Using their pairing $\{\bullet, \bullet\}$, one finds that $\{\xi_{2-k} f, f\} \neq 0$ thanks to its interpretation in terms of Petersson norms. On the other hand, Proposition 3.5 of [73] expresses this quantity in terms of the principal part of f and the coefficients of the cusp form $\xi_{2-k}(f)$. An inspection of this formula reveals that at least one principal part of f must be nonconstant. □

4.4. **The D-operator.** In addition to the differential operator ξ_{2-k}, which defines the surjective map

$$\xi_{2-k} : H_{2-k}(N, \chi) \longrightarrow S_k(N, \overline{\chi}),$$

we consider the differential operator

$$D := \frac{1}{2\pi i} \cdot \frac{d}{dz}. \tag{4.10}$$

We have the following theorem for integer weights.

Theorem 4.7. *Suppose that $2 \leq k \in \mathbb{Z}$ and $f \in H_{2-k}(N)$, then*

$$D^{k-1}(f) \in M_k^!(N).$$

Moreover, assuming the notation in (4.6), we have

$$D^{k-1} f = D^{k-1} f^+ = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n.$$

To prove this theorem, we must first recall some further differential operators, the Maass raising and lowering operators (for example, see [73, 84]) R_k and L_k. They are defined by

$$R_k = 2i\frac{\partial}{\partial z} + ky^{-1} = i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + ky^{-1},$$
$$L_k = -2iy^2\frac{\partial}{\partial \bar{z}} = -iy^2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With respect to the Petersson slash operator (4.4), these operators satisfy the intertwining properties

$$R_k(f \mid_k \gamma) = (R_k f) \mid_{k+2} \gamma,$$
$$L_k(f \mid_k \gamma) = (L_k f) \mid_{k-2} \gamma,$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$. The Laplacian Δ_k can be expressed in terms of R_k and L_k by

$$-\Delta_k = L_{k+2} R_k + k = R_{k-2} L_k. \tag{4.11}$$

If f is an eigenfunction of Δ_k satisfying $\Delta_k f = \lambda f$, then

$$\Delta_{k+2} R_k f = (\lambda + k) R_k f, \tag{4.12}$$
$$\Delta_{k-2} L_k f = (\lambda - k + 2) L_k f. \tag{4.13}$$

For any positive integer n we put

$$R_k^n := R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k.$$

We also let R_k^0 be the identity. The differential operator $D := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$ satisfies the following relation

$$R_k = -4\pi D + k/y.$$

The next well known lemma (for example, see (4.15) of [175]) is often referred to as Bol's identity.

Lemma 4.8. *Assuming the notation and hypotheses above, we have*

$$D^{k-1} = \frac{1}{(-4\pi)^{k-1}} R_{2-k}^{k-1}.$$

By Lemma 4.8, we see that D^{k-1} defines a linear map

$$D^{k-1} : M_{2-k}^!(N) \longrightarrow M_k^!(N).$$

Theorem 4.7 asserts that this map extends to harmonic Maass forms. Moreover, the theorem provides a simple description of the images.

Proof of Theorem 4.7. It is clear that $D^{k-1}f$ has the transformation behavior of a modular form of weight k.

We now show that $L_k D^{k-1} f = 0$. This implies that $D^{k-1}f$ is holomorphic on $\mathbb{H}$. By Lemma 4.8, it suffices to show that $L_k R_{2-k}^{k-1} f = 0$. Since $\Delta_{2-k} f = 0$, it follows from (4.12) by induction that

$$\Delta_{k-2} R_{2-k}^{k-2} f = (2-k) R_{2-k}^{k-2} f.$$

Using (4.11), we obtain

$$L_k R_{2-k}^{k-1} f = (L_k R_{k-2}) R_{2-k}^{k-2} f = (-\Delta_{k-2} - (k-2)) R_{2-k}^{k-2} f = 0.$$

Finally, the growth behavior of f at the cusps implies that $D^{k-1}f$ is meromorphic at the cusps. Therefore, D^{k-1} indeed extends to $H_{2-k}(N)$.

To complete the proof, we compute the Fourier expansion of $D^{k-1}f$. Assuming the notation in (4.6), a straightforward calculation gives

$$R_{2-k}^{k-2} f(z) = \sum_{n \gg -\infty} c_f^+(n) \Gamma(k-1, 4\pi n y)(-y)^{2-k} e^{2\pi i n \bar{z}} + (k-2)!^2 \sum_{n<0} c_f^-(n)(-y)^{2-k} e^{2\pi i n \bar{z}}.$$

Moreover, $R_{2-k}^{k-1} f$ has the Fourier expansion

$$R_{2-k}^{k-1} f(z) = \sum_{n \gg -\infty} c_f^+(n)(-4\pi n)^{k-1} q^n.$$

In particular, we have

$$D^{k-1} f = D^{k-1} f^+ = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n.$$

The first two formulas follow from the Fourier expansion of f and the differential equations $\Delta_{k-2} R_{2-k}^{k-2} f = (2-k) R_{2-k}^{k-2} f$ and $\Delta_k R_{2-k}^{k-1} f = 0$. The third formula is a consequence of the second and Lemma 4.8. □

Remark. Note that $g := y^{k-2}\overline{R_{2-k}^{k-2}f}$ is a harmonic Maass form of weight $2-k$ in the (slightly more general) sense of Section 3 of [73]. Moreover, $\xi_{2-k}g = y^{-k}\overline{L_{2-k}g} = R_{2-k}^{k-1}f$. This can also be used to compute the Fourier expansions in the proof of Theorem 4.7.

Theorem 4.7 implies that the coefficients $c_f^+(n)$, for non-zero n, are obtained by dividing the nth coefficient of some fixed weakly holomorphic modular form by n^{k-1}. Therefore we are compelled to determine the image of the map

$$D^{k-1} : H_{2-k}(N) \longrightarrow M_k^!(N).$$

It is not difficult to see that this map is not generally surjective. Our next result determines the image of D^{k-1} in terms of regularized inner products. The following result is obtained by Bruinier, Rhoades and the author in [79].

Theorem 4.9. *If $2 \leq k \in \mathbb{Z}$, then the image of the map*

$$D^{k-1} : H_{2-k}(N) \longrightarrow M_k^!(N)$$

consists of those $h \in M_k^!(N)$ which are orthogonal to cusp forms with respect to the regularized inner product, which also have constant term 0 at all cusps of $\Gamma_0(N)$.

To make Theorem 4.9 precise, we must define what it means for a weakly holomorphic modular form to be orthogonal to cusp forms. To this end, we first recall the *regularized inner product*.

We stress again that $k \geq 2$ is an integer. Obviously, $\Gamma_0(N)$ has finite index in $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. We define a regularized inner product of $g \in M_k(N)$ and $h \in M_k^!(N)$ as follows. For $T > 0$ we denote by $\mathcal{F}_T(\Gamma(1))$ the truncated fundamental domain

$$\mathcal{F}_T(\Gamma(1)) = \{z \in \mathbb{H} \ : \ |x| \leq 1/2, \, |z| \geq 1, \text{ and } y \leq T\}$$

for $\Gamma(1)$. Moreover, we define the truncated fundamental domain for $\Gamma_0(N)$ by

$$\mathcal{F}_T(N) = \bigcup_{\gamma \in \Gamma_0(N)\backslash\Gamma(1)} \gamma \mathcal{F}_T(\Gamma(1)).$$

Following Borcherds [50], we define the regularized inner product $(g,h)^{reg}$ as the constant term in the Laurent expansion at $s = 0$ of the meromorphic continuation in s of the function

$$\frac{1}{[\Gamma(1) : \Gamma_0(N)]} \lim_{T \to \infty} \int_{\mathcal{F}_T(N)} g(z)\overline{h(z)} y^{k-s} \, \frac{dx\, dy}{y^2}.$$

Arguing as in Section 6 of [50], it can be shown that $(g,h)^{reg}$ exists for any $g \in M_k(N)$ and $h \in M_k^!(N)$. (It also exists for $g \in M_k(N)$ and $h \in H_k(N)$. But note that it does *not* exist in general if g and h are both weakly holomorphic with poles at the cusps.) For cusp forms g and h, the regularized inner product reduces to the classical Petersson inner product.

Remark. If $h \in M_k^!(N)$ has vanishing constant term at every cusp of $\Gamma_0(N)$, then

$$(g,h)^{reg} = \frac{1}{[\Gamma(1) : \Gamma_0(N)]} \lim_{T \to \infty} \int_{\mathcal{F}_T(N)} g(z)\overline{h(z)} y^{k} \, \frac{dx\, dy}{y^2}.$$

The following theorem provides a formula for the regularized inner product in terms of Fourier coefficients.

Theorem 4.10. *If $g \in M_k(N)$ and $f \in H_{2-k}(N)$, then*

$$(g, R_{2-k}^{k-1} f)^{reg} = \frac{(-1)^k}{[\Gamma(1) : \Gamma_0(N)]} \sum_{\kappa \in \Gamma_0(N) \backslash P^1(\mathbb{Q})} w_\kappa \cdot c_g(0, \kappa) \overline{c_f^+(0, \kappa)},$$

where $c_g(0, \kappa)$ (resp. $c_f^+(0, \kappa)$) denotes the constant term of the Fourier expansion of g (resp. f) at the cusp $\kappa \in P^1(\mathbb{Q})$, and w_κ is the width of the cusp κ.

Proof. For simplicity, we carry out the proof only in the special case $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. We put $H := y^{k-2} \overline{R_{2-k}^{k-2} f}$, and let $h := R_{2-k}^{k-1} f = y^{-k} \overline{L_{2-k} H}$. Since the constant terms at all cusps of h vanish, we have

$$\begin{aligned}
(g, R_{2-k}^{k-1} f)^{reg} &= \lim_{T \to \infty} \int_{\mathcal{F}_T(1)} g(z) \overline{h(z)} y^k \, \frac{dx\, dy}{y^2} \\
&= \lim_{T \to \infty} \int_{\mathcal{F}_T(1)} g(z) (L_{2-k} H) \, \frac{dx\, dy}{y^2} \\
&= \lim_{T \to \infty} \int_{\mathcal{F}_T(1)} g(z) (\frac{\partial}{\partial \bar{z}} H) \, dz\, d\bar{z} \\
&= - \lim_{T \to \infty} \int_{\mathcal{F}_T(1)} (\bar{\partial} H) \wedge g(z) \, dz.
\end{aligned}$$

Using the holomorphy of g, we obtain, by Stokes' theorem, the expression

$$\begin{aligned}
(g, R_{2-k}^{k-1} f)^{reg} &= - \lim_{T \to \infty} \int_{\mathcal{F}_T(1)} d(H(z) g(z) \, dz) \\
&= - \lim_{T \to \infty} \int_{\partial \mathcal{F}_T(1)} H(z) g(z) \, dz \\
&= \lim_{T \to \infty} \int_{x=-1/2}^{1/2} H(x + iT) g(x + iT) \, dx.
\end{aligned}$$

The integral over x gives the constant term in the expansion of $H(x + iT) g(x + iT)$, and it can be computed using the Fourier expansion

$$H(z) = (-1)^k \sum_{n \gg -\infty} \overline{c_f^+(n)} \Gamma(k - 1, 4\pi n y) e^{-2\pi i n z} + (-1)^k (k-2)!^2 \sum_{n<0} \overline{c_f^-(n)} e^{-2\pi i n z}$$

of H (see the proof of Theorem 4.7) and the expansion of g. To complete the proof, notice that in the limit $T \to \infty$ the only contribution is $(-1)^k c_g(0) \overline{c_f^+(0)}$. □

Proof of Theorem 4.9. By Theorem 4.10, it follows that if $g \in S_k(N)$, then

$$(g, R_{2-k}^{k-1} f)^{reg} = 0.$$

Therefore, if $f \in H_{2-k}(N)$, it follows from Lemma 4.8 and Theorem 4.10 that $D^{k-1} f$ is orthogonal to cusp forms.

Conversely, assume that $h \in M_k^!(N)$ is orthogonal to cusp forms and has vanishing constant term at every cusp of $\Gamma_0(N)$. By Lemma 3.11 of [73], we may chose $f \in H_{2-k}(N)$ such that

the principal parts of $D^{k-1}f$ and h at the cusps agree up to the constant terms. Since the constant terms of h and $D^{k-1}f$ vanish, they trivially agree as well. Consequently,

$$h - D^{k-1}f \in S_k(N).$$

In view of Theorem 4.10 and the hypothesis on h, we find that $h - D^{k-1}f$ is orthogonal to cusp forms, and so it is zero. □

4.5. **Hecke operators.**

Remark. There is an Eichler cohomology theory for harmonic Maass forms in the context of Theorem 4.5 . The interested reader should see [59] for details.

It is natural to investigate the action of the Hecke operators on harmonic Maass forms. Here we illustrate how to combine facts about Hecke operators with Lemma 4.4 to obtain weakly holomorphic modular forms from harmonic Maass forms.

First we recall the classical Hecke operators. Suppose that p is prime, and that $k \in \frac{1}{2}\mathbb{Z}$. Suppose that

$$F(z) = \sum_{n \gg -\infty} a_F(n) q^n$$

is a weight k weakly holomorphic modular form on $\Gamma_0(N)$ with Nebentypus χ. If $k \in \mathbb{Z}$, then the Hecke operator $T_k(p)$ is defined by

$$F \mid T_k(p) := \sum_{n \gg -\infty} \left(a_F(pn) + \chi(p) p^{k-1} a_F(n/p) \right) q^n. \tag{4.14}$$

If $k = \lambda + \frac{1}{2} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then we have that

$$F \mid T_k(p) := \sum_{n \gg -\infty} \left(a_F(p^2 n) + \chi(p) \left(\frac{(-1)^\lambda n}{p} \right) p^{\lambda - 1} a_F(n) + \chi(p^2) p^{2\lambda - 1} a_F(n/p^2) \right) q^n. \tag{4.15}$$

Remark. For the sake of uniformity, we abuse notation in the case of the half-integral weight Hecke operators. Our operator $T_k(p)$ is traditionally referred to as $T(p^2)$ or $T_k(p^2)$.

These operators map modular forms to modular forms, and they define linear transformations on $M_k(N, \chi)$ and $S_k(N, \chi)$. Using Lemma 4.2, it is not difficult to see that the definitions of these Hecke operators extend to harmonic Maass forms and their spaces in the obvious way. Using Lemmas 4.2 and 4.4, it is then not difficult to obtain the following useful theorem. As before, here we suppose that $2 - k \in \frac{1}{2}\mathbb{Z}$ with $k > 1$.

Theorem 4.11. *Suppose that $f(z) \in H_{2-k}(N, \chi)$, and that $p \nmid N$ is a prime for which $\xi_{2-k}(f) \in S_k(N, \overline{\chi})$ is an eigenform of $T_k(p)$ with eigenvalue $\lambda(p)$.*

(1) *If $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then*
$$f \mid T_{2-k}(p) - p^{2-2k}\lambda(p) f$$
is a weakly holomorphic modular form in $M^!_{2-k}(N, \chi)$.

(2) *If $k \in \mathbb{Z}$, then*
$$f \mid T_{2-k}(p) - p^{1-k}\lambda(p) f$$
is a weakly holomorphic modular form in $M^!_{2-k}(N, \chi)$.

5. Examples of harmonic Maass forms

In this section we present further examples of harmonic Maass forms. The interested reader should consult Zwegers's works [252, 253] to see those weight 1/2 harmonic Maass forms whose holomorphic parts are given in terms of Lambert-type series and generating functions for indefinite binary quadratic forms. Recent work of Bringmann, Folsom, and the author [57] concerns further properties of these forms, and it also gives related weight 3/2 harmonic Maass forms. All of these harmonic Maass forms f have the property that $\xi_w(f)$ is a linear combination of single variable theta functions.

Here we consider further examples of harmonic Maass forms. We recall the classical Eisenstein series $E_2(z)$, Zagier's weight 3/2 Eisenstein series, certain Maass-Poincaré series, and a combinatorial power series which plays a central role in the Dyson-Ramanujan theory of partition congruences.

5.1. **The Eisenstein series $E_2(z)$.** The differential operator $D := \frac{1}{2\pi i}\frac{d}{dz} = q \cdot \frac{d}{dq}$ plays an important role in the theory of modular forms and q-series. Indeed, as an operator on formal power series we have that

$$D\left(\sum_{n=h}^{\infty} a(n)q^n\right) := \sum_{n=h}^{\infty} na(n)q^n.$$

Ramanujan proved [200] that

$$D(E_4) = (E_4E_2 - E_6)/3 \quad \text{and} \quad D(E_6) = (E_6E_2 - E_8)/2,$$

where E_k, for every even integer $k \geq 2$, is the standard Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Here B_k denotes the usual kth Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$.

As is well known, for even $k \geq 4$ we have that $E_k(z)$ is a weight k modular form on $\mathrm{SL}_2(\mathbb{Z})$. However, the Eisenstein series $E_2(z)$

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

is not modular, and so the derivative of a modular form is not (in general) a modular form.

It is not difficult to pin down the obstruction which prevents $E_2(z)$ from being a modular form. Indeed, it is simple to show (for example, see page 113 of [160]), for $z \in \mathbb{H}$, that

$$z^{-2}E_2(-1/z) = E_2(z) + \frac{6}{\pi i z}. \tag{5.1}$$

Using this fact, it is not difficult to modify $E_2(z)$ to obtain a weight 2 harmonic Maass form. Using (5.1), and the fact that

$$\mathrm{SL}_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$

one finds that

$$E_2^*(z) := -\frac{3}{\pi y} + E_2(z), \tag{5.2}$$

where $y = \mathrm{Im}(z)$, is a weight 2 harmonic Maass form on $\mathrm{SL}_2(\mathbb{Z})$ in the more general sense of Bruinier and Funke. It fails to satisfy Definition 4.1 (3). Obviously, its holomorphic part is $E_2(z)$, and its non-holomorphic part is the simple function $-\frac{3}{\pi y}$.

Remark. Although $E_2^*(z)$ is not a harmonic Maass form in the strict sense of Definition 4.1, it still behaves well under the differential operator ξ_2. One easily checks that $\xi_2(E_2^*(z))$ is constant, and so it is a modular form of weight 0. This is in line with Lemma 4.4.

5.2. **Zagier's Eisenstein series.** In their work on the intersection theory of Hilbert modular surfaces [149, 243], Hirzebruch and Zagier required a non-holomorphic weight 3/2 Eisenstein series $G(z)$ on $\Gamma_0(4)$. The non-holomorphic part of this Eisenstein series is essentially the "period integral" of the classical Jacobi theta function

$$\Theta_0(z) := 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots .$$

This is analogous with the work of Zwegers on Ramanujan's mock theta functions (see Theorem 3.1 and the discussion after Lemma 4.4) in that the non-holomorphic parts of his harmonic Maass forms are also period integrals of single variable theta functions. Unlike Zwegers's harmonic Maass forms, where the holomorphic parts turned out to be combinatorial q-series such as the mock theta functions of Ramanujan, the holomorphic part of the Zagier form is the generating function for Hurwitz class numbers. More precisely, we have (in Zagier's notation) the following theorem.

Theorem 5.1. *If $H(0) = \zeta(-1) = -\frac{1}{12}$, and if $H(n)$ denotes the usual Hurwitz class numbers for discriminants $-n$, then the function*

$$G(z) = \sum_{n=0}^{\infty} H(n)q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y)q^{-n^2}$$

is a weight 3/2 harmonic Maass form on $\Gamma_0(4)$ in the sense of Bruinier and Funke. Here we have that $\beta(s) := \int_1^{\infty} t^{-3/2}e^{-st}dt$.

Remark. Zagier's β-function is essentially the incomplete Gamma-function.

As in the case of $E_2^*(z)$, we have that $G(z)$ is a weight 3/2 harmonic Maass form on $\Gamma_0(4)$ in the more general sense of Bruinier and Funke. Again, the culprit is Definition 4.1 (3). Nevertheless, one sees that $\xi_{\frac{3}{2}}(G)$ is a multiple of $\Theta_0(z)$, which is in line with the phenomenon in Lemma 4.4. The only difference is that $\Theta_0(z)$ is not a cusp form.

Remark. In recent work, Bringmann and Lovejoy [60] have found a striking identity relating a certain rank generating function for overpartitions to $G(z)$.

5.3. **Poincaré series.** Here we describe certain Maass-Poincaré series. Throughout, we rely on classical special functions whose properties and definitions may be found in [1, 22]. Such Poincaré series, which were first given by D. Hejhal, are discussed in [65, 71, 75, 143].

Suppose that $k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). Let m be an integer, and let $\varphi_m : \mathbb{R}^+ \to \mathbb{C}$ be a function which satisfies $\varphi_m(y) = O(y^\alpha)$, as $y \to 0$, for some $\alpha \in \mathbb{R}$. If $e(\alpha) := e^{2\pi i \alpha}$ as usual, then let

$$\varphi_m^*(z) := \varphi_m(y)e(mx). \tag{5.3}$$

This function is fixed by the group of translations

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\}.$$

Using the slash operator (4.4), we define the Poincaré series $\mathbb{P}(m, k, N, \varphi_m; z)$ by

$$\mathbb{P}(m, k, N, \varphi_m; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^* \mid_k A)(z). \tag{5.4}$$

The Fourier expansions of these series are given in terms of the Kloosterman sums

$$K_k(m, n, c) := \begin{cases} \sum_{v (c)^\times} e\left(\frac{m\overline{v}+nv}{c}\right) & \text{if } k \in \mathbb{Z}, \\ \sum_{v (c)^\times} \left(\frac{c}{v}\right)^{2k} \varepsilon_v^{2k} e\left(\frac{m\overline{v}+nv}{c}\right) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases} \tag{5.5}$$

In the sums above, v runs through the primitive residue classes modulo c, and $\overline{v}$ denotes the multiplicative inverse of v modulo c. The following lemma gives the fundamental properties of such Poincaré series (for example, see Proposition 3.1 of [75] where $N = 4$).

Lemma 5.2. *If $k > 2 - 2\alpha$, then the following are true.*

1. *Each Poincaré series $\mathbb{P}(m, k, N, \varphi_m; z)$ is a weight k $\Gamma_0(N)$-invariant function.*
2. *Near the cusp at ∞, the function $\mathbb{P}(m, k, N, \varphi_m; z) - \varphi_m^*(z)$ has moderate growth. Near the other cusps, $\mathbb{P}(m, k, N, \varphi_m; z)$ has moderate growth.*
3. *If $\mathbb{P}(m, k, N, \varphi_m; z)$ is twice continuously differentiable, then it has the locally uniformly absolutely convergent Fourier expansion*

$$\mathbb{P}(m, k, N, \varphi_m; z) = \varphi_m^*(z) + \sum_{n \in \mathbb{Z}} a(n, y)e(nx),$$

where

$$a(n, y) := \sum_{\substack{c > 0 \\ c \equiv 0 \pmod N}}^{\infty} c^{-k} K_k(m, n, c) \int_{-\infty}^{\infty} z^{-k} \varphi_m\left(\frac{y}{c^2|z|^2}\right) e\left(-\frac{mx}{c^2|z|^2} - nx\right) dx.$$

We now recall the classical family of Poincaré series (for example, see [151, 195]) which arises from Lemma 5.2. If $2 \leq k \in \frac{1}{2}\mathbb{Z}$, N is positive, and if m is an integer, then let

$$P(m, k, N; z) = q^m + \sum_{n=1}^{\infty} a(m, k, N; n)q^n =: \mathbb{P}(m, k, N, e(imy); z). \tag{5.6}$$

These series are modular, and their Fourier expansions are given in terms of the I-Bessel and J-Bessel functions, and the Kloosterman sums above.

Theorem 5.3. *If $2 \leq k \in \frac{1}{2}\mathbb{Z}$, and $m, N \geq 1$ (with $4 \mid N$ if $k \in \frac{1}{2}Z \setminus \mathbb{Z}$), then the following are true.*

1) We have that $P(m, k, N; z) \in S_k(N)$, and for positive integers n we have

$$a(m, k, N; n) = 2\pi(-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \cdot \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K_k(m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

2) We have that $P(-m, k, N; z) \in M_k^!(N)$, and for positive integers n we have

$$a(-m, k, N; n) = 2\pi(-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \cdot \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K_k(-m, n, c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Now we recall one family of Maass-Poincaré series which have appeared in earlier works [71, 75, 143, 187]. To define them, again suppose that $2 < k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). To employ Lemma 5.2, we first select an appropriate function φ.

Let $M_{\nu,\mu}(z)$ be the usual M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0.$$

For complex s, let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|),$$

and for positive m let $\varphi_{-m}(z) := \mathcal{M}_{1-\frac{k}{2}}(-4\pi m y)$. We now let

$$Q(-m, k, N; z) := \mathbb{P}(-m, 2-k, N, \varphi_{-m}; z). \tag{5.7}$$

Lemma 5.2 leads to the following theorem (see [63, 65, 71, 143]).

Theorem 5.4. *If $2 < k \in \frac{1}{2}\mathbb{Z}$ and $m, N \geq 1$ (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then $Q(-m, k, N; z)$ is in $H_{2-k}(N)$. Moreover, if I_{k-1} is the usual I-Bessel function, then*

$$Q(-m, k, N; z) = (1-k)\left(\Gamma(k-1, 4\pi m y) - \Gamma(k-1)\right) q^{-m} + \sum_{n\in\mathbb{Z}} c_m(n, y)\, q^n.$$

1) If $n < 0$, then

$$c_m(n, y) = 2\pi i^k (1-k)\,\Gamma(k-1, 4\pi|n|y) \left|\frac{n}{m}\right|^{\frac{1-k}{2}} \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K_{2-k}(-m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

2) If $n > 0$, then

$$c_m(n, y) = -2\pi i^k \Gamma(k) \left(\frac{n}{m}\right)^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K_{2-k}(-m, n, c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

3) If $n = 0$, then

$$c_m(0, y) = -2^k \pi^k i^k m^{k-1} \sum_{\substack{c>0 \\ c \equiv 0 \pmod N}} \frac{K_{2-k}(-m, 0, c)}{c^k}.$$

These three families of Poincaré series are closely related. The following theorem gives the explicit relationships between the cusp forms $P(m, k, N; z)$, the weakly holomorphic modular forms $P(-m, k, N; z)$, and the harmonic Maass forms $Q(-m, k, N; z)$. They are related by the ξ_{2-k} and D^{k-1} differential operators.

Theorem 5.5. *Suppose that $2 < k \in \frac{1}{2}\mathbb{Z}$, and that $m, N \geq 1$ (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). Then the following are true.*

(1) *Under the operator ξ_{2-k}, we have that*

$$\xi_{2-k}(Q(-m, k, N; z)) = (4\pi)^{k-1} m^{k-1}(k-1) \cdot P(m, k, N; z) \in S_k(N).$$

(2) *Under the operator D^{k-1}, if k is an integer, then we have that*

$$D^{k-1} Q(-m, k, N; z) = -m^{k-1} \Gamma(k) \cdot P(-m, k, N; z) \in M_k^!(N).$$

Proof. The claims follow easily from the explicit expansions in Theorems 5.3 and 5.4. □

5.4. **A family of weight $1/2$ harmonic Maass forms.** In his work [252, 253] (also see Section 3.2), Zwegers describes the holomorphic parts of certain harmonic Maass forms in an explicit way. These series are presented as reciprocals of theta functions multiplied with Lambert-type series, and are also given as generating functions for indefinite theta series.

Here we present an infinite family of Maass forms which arise from the Eulerian series

$$R(w; q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n}, \tag{5.8}$$

where, as before, we let

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).$$

To place this formal power series in context, we recall that we have already seen two important specializations in w. Thanks to Proposition 1.1, by setting $w = 1$ we have that

$$R(1; q) = \sum_{n=0}^{\infty} p(n) q^n.$$

By letting $w = -1$, we obtain the series

$$R(-1; q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2 (1+q^2)^2 \cdots (1+q^n)^2}.$$

This series is Ramanujan's third order mock theta function $f(q)$, which by Theorem 3.1 is essentially the holomorphic part of a weight $1/2$ harmonic Maass form. In [64], Bringmann and the author generalized this fact.

To make this precise, suppose that $0 < a < c$ are integers, and let $\zeta_c := e^{2\pi i/c}$. If $f_c := \frac{2c}{\gcd(c,6)}$, then define the theta function $\Theta\left(\frac{a}{c};\tau\right)$ by

$$\Theta\left(\frac{a}{c};\tau\right) := \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \cdot \theta\left(6m+1, 6f_c; \frac{\tau}{24}\right), \tag{5.9}$$

where

$$\theta(\alpha,\beta;\tau) := \sum_{n\equiv\alpha \pmod{\beta}} ne^{2\pi i\tau n^2}. \tag{5.10}$$

Throughout, let $\ell_c := \mathrm{lcm}(2c^2, 24)$, and let $\widetilde{\ell_c} := \ell_c/24$. It is well known that $\Theta\left(\frac{a}{c};\ell_c\tau\right)$ is a cusp form of weight $3/2$ (for example, see [223]).

Using this cuspidal theta function, we define the function $S_1\left(\frac{a}{c};z\right)$ by the period integral

$$S_1\left(\frac{a}{c};z\right) := \frac{-i\sin\left(\frac{\pi a}{c}\right)\ell_c{}^{\frac{1}{2}}}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\ell_c\tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{5.11}$$

Using this notation, define $D\left(\frac{a}{c};z\right)$ by

$$D\left(\frac{a}{c};z\right) := -S_1\left(\frac{a}{c};z\right) + q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c}). \tag{5.12}$$

Theorem 5.6. *If $0 < a < c$, where c is odd, then $D\left(\frac{a}{c};z\right)$ is a Maass form of weight $1/2$ on $\Gamma_1(144f_c^2\widetilde{\ell_c})$.*

Remark. For even c, these Maass forms also satisfy transformation laws with respect to Γ_1 congruence subgroups. Our proof of Theorem 5.6 in [64] is long and complicated. It is based on a modification of earlier work of Watson [239, 240]. Zagier [246] found a shorter proof of this result using a clever application of Zwegers's work on the $\widehat{\mu}$-function. His proof of Theorem 5.6 relies on the fact that

$$\frac{q^{-1/24}R(e(\alpha);q)}{e^{-\pi i\alpha} - e^{\pi i\alpha}} = \frac{\eta(3z)^3/\eta(z)}{\vartheta(3\alpha;3z)} - q^{-\frac{1}{6}}e(-\alpha)\mu(3\alpha,-z;3z) + q^{-\frac{1}{6}}e(\alpha)\mu(3\alpha,z;3z). \tag{5.13}$$

6. Dyson-Ramanujan theory of partition congruences

As mentioned in the introduction, the partition function has been a "testing ground" for the theory of modular forms. In number theory we are interested in the divisibility properties of $p(n)$ and the size of $p(n)$. Here we consider the congruence properties of the partition function $p(n)$. It will turn out that these harmonic Maass forms have arithmetic properties which may be studied using the theory of ℓ-adic Galois representations as developed by Deligne, Serre, and Swinnerton-Dyer [99, 218, 229]. These new congruences imply new partition congruences.

6.1. **Ramanujan's congruences.** Ramanujan proved the striking congruences

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11},$$

and conjectured generalizations where the moduli are powers of 5, 7 and 11. These conjectures have been resolved in works by Atkin and Watson [33, 241].

These congruences are just the tip of the iceberg. In the late 1960s, Atkin [34] proved further congruences for small prime moduli. More recently, the author proved [189] that there are infinitely many such congruences for any prime modulus $Q \geq 5$. This result was subsequently generalized by S. Ahlgren and the author [2, 3], and these results imply the following theorem.

Theorem 6.1. *If M is coprime to 6, then there are infinitely many non-nested arithmetic progressions $An + B$ for which*

$$p(An+B) \equiv 0 \pmod M.$$

Remark. By non-nested, we mean that there are infinitely many arithmetic progressions $An + B$, with $0 \leq B < A$, with the property that there are no progressions which contain another progression.

Remark. It is believed that there are no such congruences with moduli which are not coprime to 6 (for example, see [5, 190]).

Example. Here are the simplest examples of these congruences for the prime moduli $17 \leq M \leq 31$:

$$p(48037937n + 1122838) \equiv 0 \pmod{17},$$
$$p(1977147619n + 815655) \equiv 0 \pmod{19},$$
$$p(14375n + 3474) \equiv 0 \pmod{23},$$
$$p(348104768909n + 43819835) \equiv 0 \pmod{29},$$
$$p(4063467631n + 30064597) \equiv 0 \pmod{31}.$$

6.2. **Dyson's rank.** In an effort to provide a combinatorial explanation of Ramanujan's congruences modulo 5 and 7, Dyson introduced [108] the so-called "rank" of a partition. The rank of a partition is defined to be its largest part minus the number of its parts. Let $N(m, n)$ denote the number of partitions of n with rank m, and for integers $0 \leq r < t$ let $N(r, t; n)$ denote the number of partitions of n with rank congruent to r modulo t. Dyson noticed that ranks modulo 5 (resp. 7) apppeared to divide the partitions of $5n + 4$ (resp. $7n + 5$) into 5 (resp. 7) groups of equal size, thereby providing an elegant explanation for these two congruences of Ramanujan.

Atkin and Swinnerton-Dyer [36] confirmed Dyson's conjecture by proving the following result.

Theorem 6.2. *The following are true:*

(1) *If* $0 \le r < 5$, *then*

$$\sum_{n=0}^{\infty} \left(N(r, 5; 5n+4) - \frac{p(5n+4)}{5} \right) q^n = 0.$$

(2) *If* $0 \le r < 7$, *then*

$$\sum_{n=0}^{\infty} \left(N(r, 7; 7n+5) - \frac{p(7n+5)}{7} \right) q^n = 0.$$

Remark. Dyson [108] also noted that the rank fails to explain the Ramanujan congruence with modulus 11. A short calculation reveals that the first failure already occurs with $p(6) = 11$. In view of this difficulty, Dyson further postulated the existence of another statistic, the so-called "crank", that could be used to provide an explanation for all three Ramanujan congruences. In 1988, Andrews and Garvan [28] found the crank, and they confirmed Dyson's speculation that it explains the three Ramanujan congruences. Recent work of Mahlburg [178] establishes that the Andrews-Dyson-Garvan crank plays an even more central role in the theory partition congruences. His work establishes congruences modulo arbitrary powers of all primes ≥ 5. Other work by Garvan, Kim and Stanton [121] gives a different "crank" for several other Ramanujan congruences.

6.3. **Maass form generating functions for Dyson's ranks.** Although the rank does not explain Ramanujan's partition congruence with modulus 11, it turns out that it indeed plays a central role thanks to the arithmetic properties of certain harmonic Maass forms. Indeed, it turns out that the generating functions for $N(r, t; n)$ are related to harmonic Maass forms. To make this connection, one simply generalizes the proof of Proposition 1.1 to obtain the following proposition concerning the power series

$$R(w; q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n}$$

considered earlier in (5.8).

Proposition 6.3. *As a formal power series, we have that*

$$R(w; q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n.$$

Thanks to Theorem 5.6, it is not difficult to relate the functions $N(r, t; n)$ to the coefficients of weight $1/2$ harmonic Maass forms. In this direction, Bringmann and the author proved the following theorem in [64], which also sheds further light on Theorem 6.2.

Theorem 6.4. *Assume the notation from Section 5.4. If* $0 \le r < t$ *are integers, where* t *is odd, then*

$$\sum_{n=0}^{\infty} \left(N(r, t; n) - \frac{p(n)}{t} \right) q^{\ell_t n - \widetilde{\ell_t}}$$

is the holomorphic part of a weight 1/2 harmonic Maass form on $\Gamma_1\left(144f_t^2\widetilde{\ell_t}\right)$. *Moreover, the non-holomorphic part of this Maass form is*

$$\frac{i\ell_t^{\frac{1}{2}}}{t\sqrt{3}}\sum_{m \pmod{f_t}}\sum_{n\equiv 6m+1 \pmod{6f_t}} A(r,t,m)\cdot\gamma(t,y;n)q^{-\widetilde{\ell_t}n^2}.$$

Here we have that

$$A(r,t,m) := (-1)^m\sum_{j=1}^{t-1}\zeta_t^{-rj}\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{\pi j(6m+1)}{t}\right), \tag{6.1}$$

and we have that

$$\gamma(t,y;n) := \frac{i}{\sqrt{2\pi\widetilde{\ell_t}}}\cdot\Gamma\left(\frac{1}{2};4\pi\widetilde{\ell_t}n^2y\right).$$

Proof. This result follows easily from a standard argument involving the orthogonality of roots of unity. In particular, observe that

$$\sum_{n=0}^{\infty}N(r,t;n)q^n = \frac{1}{t}\sum_{n=0}^{\infty}p(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\cdot R(\zeta_t^j;q). \tag{6.2}$$

By Theorem 5.6, it then follows that

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{\ell_t n-\frac{\ell_t}{24}}$$

is the holomorphic part of a Maass form of weight 1/2, one which is given as an appropriate weighted sum of the Maass forms $D\left(\frac{a}{t};z\right)$.

The result then follows from the explicit description of the non-holomorphic parts of these forms. For integers $0<a<t$, we have

$$\begin{aligned} D\left(\frac{a}{t};z\right) &= q^{-\frac{\ell_t}{24}} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}N(m,n)\zeta_t^{am}q^{\ell_t n-\frac{\ell_t}{24}} \\ &+ \frac{i\sin\left(\frac{\pi a}{t}\right)\ell_t^{\frac{1}{2}}}{\sqrt{3}}\sum_{m \pmod{f_t}}(-1)^m\sin\left(\frac{a\pi(6m+1)}{t}\right)\sum_{n\equiv 6m+1 \pmod{6f_t}}\gamma(t,y;n)q^{-\widetilde{\ell_t}n^2}. \end{aligned} \tag{6.3}$$

This expansion follows easily from

$$\begin{aligned} -S_1\left(\frac{a}{t};z\right) &= \frac{i\sin\left(\frac{\pi a}{t}\right)\ell_t^{\frac{1}{2}}}{\sqrt{3}}\sum_{m \pmod{f_t}}(-1)^m\sin\left(\frac{a\pi(6m+1)}{t}\right) \\ &\qquad\times\sum_{n\equiv 6m+1 \pmod{6f_t}}\int_{-\overline{z}}^{i\infty}\frac{ne^{2\pi in^2\widetilde{\ell_t}\tau}}{\sqrt{-i(\tau+z)}}\,d\tau, \end{aligned}$$

and the identity (recall (4.8))

$$\int_{-\overline{z}}^{i\infty} \frac{ne^{2\pi in^2\widetilde{\ell}_t\tau}}{\sqrt{-i(\tau+z)}}\,d\tau = \gamma(t,y;n)\cdot q^{-\widetilde{\ell}_t n^2}.$$

□

Remark. It is not difficult to deduce Theorem 6.2 from Theorem 6.4 using Atkin's U-operator. In short, one must merely prove that certain harmonic Maass forms, which turn out to be weakly holomorphic modular forms, are identically 0.

Using quadratic twists, one can use Theorem 6.4 to obtain weakly holomorphic modular forms. Theorem 1.4 of [64] asserts the following result.

Theorem 6.5. *If $0 \le r < t$ are integers, where t is odd, and $\mathcal{P} \nmid 6t$ is prime, then*

$$\sum_{\substack{n\geq 1\\ \left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-24\widetilde{\ell}_t}{\mathcal{P}}\right)}} \left(N(r,t;n)-\frac{p(n)}{t}\right) q^{\ell_t n-\frac{\ell_t}{24}}$$

is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1\left(144f_t^2\widetilde{\ell}_t\mathcal{P}^4\right)$.

Proof. The non-holomorphic parts of the Maass forms in Theorem 6.4 have the property that their coefficients are supported on a fixed square class, numbers of the form $-\widetilde{\ell}_t n^2$. This square class is easily annihilated by taking linear combinations of quadratic twists. In particular, suppose that $\mathcal{P} \nmid 6t$ is prime. For this prime $\mathcal{P}$, let

$$g := \sum_{v=1}^{\mathcal{P}-1} \left(\frac{v}{\mathcal{P}}\right) e^{\frac{2\pi iv}{\mathcal{P}}}$$

be the usual Gauss sum with respect to $\mathcal{P}$. Define the function $D\left(\frac{a}{t};z\right)_{\mathcal{P}}$ by

$$D\left(\frac{a}{t};z\right)_{\mathcal{P}} := \frac{g}{\mathcal{P}}\sum_{v=1}^{\mathcal{P}-1}\left(\tfrac{v}{\mathcal{P}}\right) D\left(\frac{a}{t};z\right)\ |_{\frac{1}{2}}\left(\begin{smallmatrix}1 & -\frac{v}{\mathcal{P}}\\ 0 & 1\end{smallmatrix}\right), \tag{6.4}$$

where $|_{\frac{1}{2}}$ is the usual slash operator (4.4). By construction, $D\left(\frac{a}{t};z\right)_{\mathcal{P}}$ is the $\mathcal{P}$-quadratic twist of $D\left(\frac{a}{t};z\right)$. In other words, the nth coefficient in the q-expansion of $D\left(\frac{a}{t};z\right)_{\mathcal{P}}$ is $\left(\frac{n}{\mathcal{P}}\right)$ times the nth coefficient of $D\left(\frac{a}{t};z\right)$. For the non-holomorphic part, this follows from the fact that the factors $\gamma(t,y;n)$ appearing in (6.3) are fixed by the transformations in (6.4).

Generalizing classical facts about twists of modular forms, $D\left(\frac{a}{t};z\right)_{\mathcal{P}}$ is a Maass form of weight $1/2$ on $\Gamma_1(144f_t^2\widetilde{\ell}_t\mathcal{P}^2)$. By (6.3), it follows that

$$D\left(\frac{a}{t};z\right) - \left(\frac{-\widetilde{\ell}_t}{\mathcal{P}}\right) D\left(\frac{a}{t};z\right)_{\mathcal{P}} \tag{6.5}$$

is a Maass form of weight $1/2$ on $\Gamma_1(144f_c^2\widetilde{\ell}_t\mathcal{P}^2)$ with the property that its non-holomorphic part is supported on summands of the form $*q^{-\widetilde{\ell}_t\mathcal{P}^2n^2}$. These terms are annihilated by taking

the $\mathcal{P}$-quadratic twist of this Maass form. Consequently, we obtain a weakly holomorphic modular form of weight $1/2$ on $\Gamma_1(144f_t^2\widetilde{\ell}_t\mathcal{P}^4)$. Combining these observations with (6.2), we obtain the theorem. □

Remark. All of the results in this section continue to hold for even t thanks to Theorem 3.3 and (5.13).

6.4. **Implications for partition congruences.** Here we apply Theorem 6.5 to shed light on the congruence properties of Dyson's functions $N(r,t;n)$. We shall employ a method first used by the author in [189] in his work on $p(n)$ to show that Dyson's rank partition functions themselves uniformly satisfy Ramanujan-type congruences. Bringmann and the author have proved the following theorem (see Theorem 1.5 of [64], as well as [53]).

Theorem 6.6. *Let t be a positive odd integer, and let M be a positive integer coprime to 6. Then there are infinitely many non-nested arithmetic progressions $An+B$ such that for every $0 \leq r < t$ we have*

$$N(r,t;An+B) \equiv 0 \pmod{M}.$$

Remark. Since we have

$$p(An+B) = \sum_{r=0}^{t-1} N(r,t;An+B),$$

Theorem 6.6 provides a combinatorial decomposition of the partition function congruence

$$p(An+B) \equiv 0 \pmod{M}.$$

Sketch of the proof of Theorem 6.6. For simplicity, we suppose that $M = Q^j$, where $5 \leq Q \nmid t$ is prime, and j is a positive integer. The proof depends on Theorem 6.5, the observation that certain "sieved" partition rank generating functions are weakly holomorphic modular forms. This result then reduces the proof of Theorem 6.6 to the fact that any finite number of half-integral weight cusp forms with integer coefficients are annihilated, modulo a fixed prime power, by a positive proportion of half-integral weight Hecke operators.

To be precise, suppose that $f_1(z), f_2(z), \dots, f_s(z)$ are half-integral weight cusp forms where

$$f_i(z) \in S_{\lambda_i+\frac{1}{2}}(\Gamma_1(4N_i)) \cap \mathcal{O}_K[[q]],$$

and where $\mathcal{O}_K$ is the ring of integers of a fixed number field K. If Q is prime and $j \geq 1$ is an integer, then the set of primes L for which

$$f_i \mid T_{\lambda_i+\frac{1}{2}}(L) \equiv 0 \pmod{Q^j}, \tag{6.6}$$

for each $1 \leq i \leq s$, has positive Frobenius density. As before, here $T_{\lambda_i+\frac{1}{2}}(L)$ (see Section 4.5) denotes the usual L^2 index Hecke operator of weight $\lambda_i + \frac{1}{2}$.

Suppose that $\mathcal{P} \nmid 6tQ$ is prime. By Theorem 6.5, for every $0 \leq r < t$

$$F(r,t,\mathcal{P};z) = \sum_{n=1}^{\infty} a(r,t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)} \left(N(r,t;n) - \frac{p(n)}{t}\right) q^{\ell_t n - \frac{\ell_t}{24}} \tag{6.7}$$

is a weakly holomorphic modular form of weight $1/2$ on $\Gamma_1(144f_t^2\widetilde{\ell}_t\mathcal{P}^4)$. Furthermore, by the work of Ahlgren and the author [3], it follows that

$$P(t,\mathcal{P};z)=\sum_{n=1}^{\infty}p(t,\mathcal{P};n)q^n:=\sum_{(\frac{24\ell_tn-\ell_t}{\mathcal{P}})=-\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)}p(n)q^{\ell_tn-\frac{\ell_t}{24}} \tag{6.8}$$

is a weakly holomorphic modular form of weight $-1/2$ on $\Gamma_1(576\widetilde{\ell}_t\mathcal{P}^4)$. In particular, all of these forms are modular with respect to $\Gamma_1(576f_t^2\widetilde{\ell}_t\mathcal{P}^4)$.

Since $Q\nmid 576f_t^2\widetilde{\ell}_t\mathcal{P}^4$, a result of Treneer (see Theorem 3.1 of [233]), generalizing earlier observations of Ahlgren and the author [3, 189], implies that there is a sufficiently large integer m for which

$$\sum_{Q\nmid n}a(r,t,\mathcal{P};Q^mn)q^n,$$

for all $0\le r<t$, and

$$\sum_{Q\nmid n}p(t,\mathcal{P};Q^mn)q^n$$

are all congruent modulo Q^j to forms in the graded ring of half-integral weight cusp forms with algebraic integer coefficients on $\Gamma_1(576f_t^2\widetilde{\ell}_t\mathcal{P}^4Q^2)$.

The system of simultaneous congruences (6.6), in the case of these forms, guarantees that a positive proportion of primes L have the property that these forms modulo Q^j are annihilated by the index L^2 half-integral weight Hecke operators. Theorem 6.6 now follows *mutatis mutandis* as in the proof of Theorem 1 of [189]. One merely interprets this annihilation, using (4.15), in terms of the $N(r,t;n)$. □

Two remarks.
1) The simultaneous system (6.6) of congruences follows from a straightforward generalization of a classical observation of Serre (see Section 6 of [219]).
2) Treneer states her result for weakly holomorphic modular forms on $\Gamma_0(4N)$ with Nebentypus. We are using a straightforward extension of her result to $\Gamma_1(4N)$.

6.5. **Related recent works.** Recently there have been many works on the relationship between Dyson's rank and the Andrews-Garvan crank [28]. Several years ago, Atkin and Garvan [35] discovered a striking partial differential equation which related the rank and crank generating functions. Their work then inspired the recent investigation of Andrews on Durfee symbols and higher partition statistics [20].

Armed with the new understanding of partition ranks in terms of harmonic Maass forms, much light has been shed recently on these results from the perspective of modular forms and Maass forms. In this direction there are papers by Bringmann, Garvan, Lovejoy, Mahlburg, and Osburn [54, 58, 61], in which theorems concerning the automorphic properties of partition generating functions, asymptotics, and congruences are obtained for various partition statistics such as those arising in the work of Andrews [20] on rank moments. In a recent paper, Bringmann and Zwegers [69] investigate further the phenomenon of the the Atkin-Garvan "crank-rank" PDE from the perspective of non-holomorphic Jacobi forms.

In another recent development, M. Monks, an undergraduate at MIT, has discovered a deep connection between the power series $R(\pm i; q^{-1})$, some false theta functions of Rogers, and the generating function for Dyson's ranks restricted to partitions into distinct parts. Her work [183] allows one to determine the behavior of the holomorphic part of the Maass form $D(\pm 1/4; z)$ in Theorem 5.6 for complex z in the lower half of the complex plane.

7. Eulerian series as modular forms

Eulerian series are combinatorial formal power series which are constructed from basic hypergeometric series. As described in Section 1.2, there are famous examples of Eulerian series which essentially are modular forms. For example, we have the celebrated Rogers-Ramanujan identities

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$
$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

These identities provide series expansions of infinite products which correspond to weight 0 modular forms. As another example, we have Proposition 1.1, which asserts that the partition number generating function satisfies

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}.$$

Since this series is essentially the reciprocal of Dedekind's weight $1/2$ modular form, this provides another example of an Eulerian series which is a modular form.

The literature on such identities is extensive, and the pursuit of further identities and their interpretations remains an active area of research largely due to applications in combinatorics, Lie theory, number theory and physics (for example, see [14], [155], and [172] to name a few).

Here we illustrate how one may easily obtain modular forms from Eulerian series by forcing the cancellation of non-holomorphic parts of harmonic Maass forms. To this end, we again consider Dyson's generating function $R(w; q)$

$$R(w; q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n},$$

and the functions $N(r, t; n)$.

By conjugating Ferrers boards, we have the trivial relation

$$N(r, t; n) = N(t - r, t; n). \tag{7.1}$$

However, these functions also satisfy [36] some highly non-trivial sporadic identities such as

$$N(1, 7; 7n + 1) = N(2, 7; 7n + 1) = N(3, 7; 7n + 1). \tag{7.2}$$

Atkin and Swinnerton-Dyer [36] proved some surprising further identities such as (see also (5.19) of [145])

$$-\frac{(q;q^7)_\infty^2(q^6;q^7)_\infty^2(q^7;q^7)_\infty^2}{(q;q)_\infty} = \sum_{n=0}^{\infty}\left(N(0,7;7n+6) - N(1,7;7n+6)\right)q^n. \tag{7.3}$$

This identity expresses a weight $1/2$ modular form as a linear combination of Eulerian series.

This example is a special case of a much more general phenomenon which gives highly nontrivial relationships between Dyson's partition rank functions. Generalizing it, Bringmann, Rhoades, and the author [66] used the theory of harmonic Maass forms to obtain several infinite families of modular forms as Eulerian series. Here we present examples which arise directly from Theorem 6.4.

Theorem 7.1. *Suppose that $t \geq 5$ is prime, $0 \leq r_1, r_2 < t$ and $0 \leq d < t$. Then the following are true:*

(1) *If $\left(\frac{1-24d}{t}\right) = -1$, then*

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d) - N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(576t^6)$.

(2) *Suppose that $\left(\frac{1-24d}{t}\right) = 1$. If $r_1, r_2 \not\equiv \frac{1}{2}(\pm 1 \pm \alpha) \pmod{t}$, where α is any integer for which $0 \leq \alpha < 2t$ and $1 - 24d \equiv \alpha^2 \pmod{2t}$, then*

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d) - N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(576t^6)$.

Remark. Example (7.3) is the $t = 7$ and $d = 6$ case of Theorem 7.1 (2). In this case, the only choices of r_1 and r_2 satisfying the hypotheses are $0, 1$, and 6. Since $N(1,7;n) = N(6,7;n)$, (7.3) is the only nontrivial example of Theorem 7.1 (2) in this case. The proof of the theorem will show, for all other pairs of r_1 and r_2 (apart from the trivial examples such as those arising from (7.1)), that

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d) - N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is not a weakly holomorphic modular form. In other words, the corresponding Maass forms turn out to have non-trivial non-holomorphic parts.

Remark. The "Mock theta Conjectures" (see Section 1.5), which were proved by Hickerson, are instances of pairs of harmonic Maass forms which differ by a weakly holomorphic modular form. Using arguments along the lines of the proofs in this section, A. Folsom has provided [115] a "Maass form" proof of these conjectures.

Remark. In a recent paper, S.-Y. Kang [158] has further developed the relationship between basic hypergeometric series and non-holomorphic Jacobi forms using the work of Zwegers. In particular, she gives (see Theorems 1.2 and 1.3 of [158]) two explicit families of weight $1/2$

modular forms, described in terms of theta functions and Dedekind's eta-function, which are sums of basic hypergeometric series.

Theorem 7.1 gives modular forms as differences of the generating functions for the functions $N(r,t;tn+d)$. There are similar theorems where the rank modulus t is independent of the modulus of the arithmetic progression of the sizes of the partitions. As before, for integers $0<a<t$, let $f_t := \frac{2t}{\gcd(t,6)}$, $\ell_t := \operatorname{lcm}(2t^2, 24)$, and let $\widetilde{\ell_t} := \ell_t/24$. We then have the following theorem which is contained in [66].

Theorem 7.2. *Suppose that $t>1$ is an odd integer. If $0 \le r_1, r_2 < t$ are integers, and $\mathcal{P} \nmid 6t$ is prime, then*

$$\sum_{\substack{n\geq 1\\ (\frac{24\ell_t n-\ell_t}{\mathcal{P}})=-\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)}} \left(N(r_1,t;n)-N(r_2,t;n)\right) q^{\ell_t n-\frac{\ell_t}{24}}$$

is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1\left(144 f_t^2 \widetilde{\ell_t}\mathcal{P}^4\right)$.

Theorems 7.1 and 7.2 depend on Theorem 6.4. However, one may obtain many further results of this type by making use of results which are analogous to Theorem 6.4. This is done in [66], where we obtain several further infinite families of Eulerian modular forms. These families give rise to further identities such as

$$2\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^4} = \sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} + \sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}^2}.$$

7.1. **Proofs of Theorems 7.1 and 7.2.** Theorem 7.2 follows immediately from Theorem 6.5 since the $p(n)/t$ summands cancel when taking the difference of the relevant generating functions.

Now we turn to the proof of Theorem 7.1. By Theorem 6.4, for any $0 \le r < t$ we have

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24t^2n-t^2} + \sum_{n\in\mathbb{Z}} \widetilde{A}(r,t,n)\cdot\gamma(t,y;n) q^{-t^2n^2}$$

is a weight $1/2$ harmonic Maass form on $\Gamma_1(576t^4)$. Here $\widetilde{A}(r,t,n)$ is a complex number given by

$$\widetilde{A}(r,t,n) = i\sqrt{8} \sum_{\substack{m \pmod{2t}\\ 6m+1\equiv n \pmod{12t}}} A(r,t,m), \tag{7.4}$$

where $\gamma(t,y;n)$ and $A(r,t;m)$ are defined in Theorem 6.4. Applying the Atkin $U(t^2)$ operator, we have, by a straightforward generalization of Proposition 1.5 of [223], that

$$\mathfrak{R}(r,t;z) := \sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24n-1} + \sum_{n\in\mathbb{Z}} \widetilde{A}(r,t,n)\cdot\gamma(t,y;n) q^{-n^2} \tag{7.5}$$

is a weight $1/2$ harmonic Maass form on $\Gamma_1(576t^4)$.

Now we prove Theorem 7.1 (1). By a straightforward generalization of the classical argument on twists of modular forms (for example, see Proposition 22 of [160]), the quadratic twist of $\mathfrak{R}(r,t;z)$ by $\left(\frac{\cdot}{t}\right)$, say $\mathfrak{R}(r,t;z)_t$, is a weight $1/2$ harmonic Maass form on $\Gamma_1(576t^6)$. In particular, $\left(\frac{-1}{t}\right)\mathfrak{R}(r,t;z)_t$ has an expansion of the form

$$\left(\frac{-1}{t}\right)\mathfrak{R}(r,t;z)_t = \sum_{n=0}^{\infty}\left(\frac{1-24n}{t}\right)\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24n-1}+\sum_{\substack{n\in\mathbb{Z}\\ t\nmid n}}\widetilde{A}(r,t,n)\cdot\gamma(t,y;n)q^{-n^2}.$$

We find that $\mathfrak{R}(r,t;z)-\left(\frac{-1}{t}\right)\mathfrak{R}(r,t;z)_t$ is on $\Gamma_1(576t^6)$, and its non-holomorphic part is supported on terms of the form $q^{-t^2n^2}$. By taking the quadratic twist of this form again by $\left(\frac{\cdot}{t}\right)$, to annihilate these non-holomorphic terms, one then finds that

$$\sum_{\substack{n\geq 0,\\ \left(\frac{1-24n}{t}\right)=-1}}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24n-1}$$

is weight $1/2$ weakly holomorphic modular form. Using the orthogonality of Dirichlet characters modulo t, and facts about twists again, it follows that

$$\sum_{n=0}^{\infty}\left(N(r,t;tn+d)-\frac{p(tn+d)}{t}\right)q^{24(tn+d)-1}$$

is a weight $1/2$ weakly holomorphic modular form. Theorem 7.1 (1) follows by taking the difference of these forms when $r=r_1$ and r_2. Since taking twists of twists can be viewed as a single twist by the trivial character, we find that the resulting form is on $\Gamma_1(576t^6)$.

Now we turn to the proof of Theorem 7.1 (2). Here we argue directly with (6.1) and (7.5). Using the theory of twists of Maass forms again, we see that the restriction of $\mathfrak{R}(r_1,t;z)-\mathfrak{R}(r_2,t;z)$ to forms whose holomorphic parts are supported on exponents of the form $24(tn+d)-1$, is a weight $1/2$ harmonic Maass form on $\Gamma_1(576t^6)$.

It suffices to show that the non-holomorphic part of this form is zero under the given hypotheses on r_1 and r_2. By (7.5), one sees that the non-holomorphic part is supported on terms of the form q^{-n^2}. By construction, these n satisfy $n\equiv\alpha \pmod{2t}$, for some $0\leq\alpha<2t$ with $1-24d\equiv\alpha^2 \pmod{2t}$. Therefore, by (6.1) and (7.5), it suffices to show that

$$A(r_1,t,m)-A(r_2,t,m)=0,$$

when $6m+1\equiv\alpha \pmod{12t}$. By (6.1), we have

$$A(r_1,t,m)-A(r_2,t,m)=(-1)^m\sum_{j=1}^{t-1}\left(\zeta_t^{-r_1j}-\zeta_t^{-r_2j}\right)\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{\pi j\alpha}{t}\right).$$

Since $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$, we have that

$$\sum_{j=1}^{t-1}\Big(e^{\frac{\pi ij}{t}(1-2r_1+\alpha)} - e^{\frac{\pi ij}{t}(1-2r_1-\alpha)} - e^{\frac{\pi ij}{t}(-1-2r_1+\alpha)} + e^{\frac{\pi ij}{t}(-1-2r_1-\alpha)}$$
$$-e^{\frac{\pi ij}{t}(1-2r_2+\alpha)} + e^{\frac{\pi ij}{t}(1-2r_2-\alpha)} + e^{\frac{\pi ij}{t}(-1-2r_2+\alpha)} - e^{\frac{\pi ij}{t}(-1-2r_2-\alpha)}\Big) = 0.$$

This follows since $\pm 1 - 2r_i \pm \alpha$, for $i = 1$ and 2, are even and coprime to t.

We conclude this section further infinite families of Eulerian series which are weakly holomorphic modular forms of weight $1/2$. Instead of specializing a single multivariable Eulerian series, these modular forms will be presented as linear combinations of different pairs of Eulerian series. More precisely, we present the next theorem to illustrate that two seemingly unrelated Eulerian series can be the holomorphic parts of distinct Maass forms with equal non-holomorphic parts.

Define the series $H'(a, c, w; z)$, $K'(w; z)$, $K''(w; z)$, with $0 < a < c$, by

$$H'(a, c, w; z) := \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}(-q)_n}{(wq^{\frac{a}{c}})_{n+1}(wq^{1-\frac{a}{c}})_{n+1}}, \tag{7.6}$$

$$K'(w; z) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(wq^2;q^2)_n(w^{-1}q^2;q^2)_n}, \tag{7.7}$$

$$K''(w; z) := \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_{n-1}}{(wq;q^2)_n(w^{-1}q;q^2)_n}. \tag{7.8}$$

Let $\zeta_c := e^{2\pi i/c}$ and $f_c := 2c/\gcd(c, 4)$. For $0 < a < c$, we let

$$\widetilde{H}(a, c; z) := q^{\frac{a}{c}(1-\frac{a}{c})}\left(H'(a, c, 1; z) + H'(a, c, -1; z)\right), \tag{7.9}$$

$$\widetilde{K}(a, c; z) := \frac{1}{4}\csc\left(\pi\frac{a}{c}\right)q^{-\frac{1}{8}}K'(\zeta_c^a; z) + \sin\left(\pi\frac{a}{c}\right)q^{-\frac{1}{8}}K''(\zeta_c^a; z). \tag{7.10}$$

The following theorem is proved by Bringmann, Rhoades, and the author [66].

Theorem 7.3. *Let $0 < a < c$. In the notation above, $\widetilde{H}(a, c; 4f_c^2 z)$ is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(64f_c^4)$ and $\widetilde{K}(a, c; 2f_c^2 z)$ is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(64f_c^4)$.*

Example. By letting $a = 1$ and $c = 2$ in Theorem 7.3, it turns out that one may obtain

$$\sum_{n\geq 0}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} + \sum_{n\geq 0}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}^2} = 2\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^4}.$$

Remark. There are many more examples of Eulerian series which have been related to modular forms and holomorphic parts of harmonic Maass forms. The interested reader can see the recent work by Kang [158] and also the work of Gordon and McIntosh [128] which defines two "universal" mock theta functions.

8. Exact formulas

Armed with the theory of harmonic Maass forms, one may obtain exact formulas for the coefficients of generating functions which turn out to be holomorphic parts of such forms. By employing the method of Poincaré series as described in Section 5.3, one obtains such results. Here we first describe the recent resolution of the Andrews-Dragonette Conjecture, and then we give exact formulas for generic harmonic Maass forms with weight $\leq 1/2$.

8.1. **The Andrews-Dragonette Conjecture.** Rademacher famously employed the modularity of the generating function for $p(n)$ to perfect the Hardy-Ramanujan asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}} \tag{8.1}$$

to obtain his "exact" formula. To state his formula, let $I_s(x)$ be the usual I-Bessel function of order s. Furthermore, if $k \geq 1$ and n are integers, then let

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right), \tag{8.2}$$

where the sum runs over the residue classes modulo $24k$, and where

$$\chi_{12}(x) := \left(\frac{12}{x}\right). \tag{8.3}$$

If n is a positive integer, then Rademacher's formula reads [196]

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

Here we address the problem of obtaining such exact formulas for $N_e(n)$ (resp. $N_o(n)$), the number of partitions of n with even (resp. odd) rank. To obtain these results, we begin with the simple observation that

$$f(q) = R(-1;q) = 1 + \sum_{n=1}^{\infty} (N_e(n) - N_o(n))q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \tag{8.4}$$

In view of (2.4) and (8.4), since

$$p(n) = N_e(n) + N_o(n),$$

our task is reduced to the problem of deriving exact formulas for the coefficients $\alpha(n)$ of the mock theta function

$$f(q) = R(-1;q) = 1 + \sum_{n=1}^{\infty} \alpha(n)q^n.$$

The problem of estimating the coefficients $\alpha(n)$ has a long history, one which even precedes Dyson's definition of partition ranks. Indeed, Ramanujan's last letter to Hardy already

includes the claim that

$$\alpha(n) = (-1)^{n-1}\frac{\exp\left(\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6}-\frac{1}{144}}\right)}{\sqrt{n-\frac{1}{24}}}\right).$$

Typical of his writings, Ramanujan offered no proof of this claim. Dragonette proved this claim in her 1951 Ph.D. thesis [101], and Andrews [8] subsequently improved upon Dragonette's work, and he proved[14] that

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}}\sum_{k=1}^{[\sqrt{n}]}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\left(n-\frac{k(1+(-1)^k)}{4}\right)}{k}\cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right)+O(n^{\epsilon}). \tag{8.5}$$

This result falls short of the problem of obtaining an exact formula for $\alpha(n)$. Andrews and Dragonette formulated the following conjecture (see page 456 of [8], and Section 5 of [18]) for the coefficients $\alpha(n)$.

Conjecture. (Andrews-Dragonette)
If n is a positive integer, then

$$\alpha(n) = \pi(24n-1)^{-\frac{1}{4}}\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}\left(n-\frac{k(1+(-1)^k)}{4}\right)}{k}\cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

Bringmann and the author have proved [62] the following theorem.

Theorem 8.1. *The Andrews-Dragonette Conjecture is true.*

Remark. In her Ph.D. thesis (see [119, 120]), S. Garthwaite generalized the proof of Theorem 8.1, and she obtained exact formulas for the coefficients of Ramanujan's mock theta function $\omega(q)$. These results will be further generalized in the next subsection.

Sketch of the proof of Theorem 8.1. By Theorem 3.1, we have that $D\left(\frac{1}{2};z\right)$ (see (5.12)) is a harmonic weight 1/2 Maass form on $\Gamma_0(144)$ with Nebentypus character $\chi_{12} = \left(\frac{12}{\cdot}\right)$.

The idea behind the proof is simple. We shall construct a Maass-Poincaré series which we shall show equals $D\left(\frac{1}{2};z\right)$. The method is analogous to material in Section 5.3. The proof of the conjecture then follows from the fact that the formulas in the Andrews-Dragonette Conjecture give the coefficients of this Maass-Poincaré series.

Suppose that $k \in \frac{1}{2}+\mathbb{Z}$. We define a class of Poincaré series $P_k(s;z)$. For matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2)$, with $c \geq 0$, define the character $\chi(\cdot)$ by

$$\chi\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) := \begin{cases} e\left(-\frac{b}{24}\right) & \text{if } c=0, \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)}e\left(-\frac{a+d}{24c}-\frac{a}{4}+\frac{3dc}{8}\right)\cdot\omega_{-d,c}^{-1} & \text{if } c>0, \end{cases} \tag{8.6}$$

where

$$\omega_{d,c} := e^{\pi i s(d,c)}. \tag{8.7}$$

Here $s(d,c)$ denotes the classical Dedekind sum.

[14]This is a reformulation of Theorem 5.1 of [8] using the identity $I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cdot\sinh(z)$.

Throughout, let $z = x + iy$, and for $s \in \mathbb{C}, k \in \frac{1}{2} + \mathbb{Z}$, and $y \in \mathbb{R} \setminus \{0\}$, and let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{8.8}$$

where $M_{\nu,\mu}(z)$ again is the M-Whittaker function. Furthermore, let

$$\varphi_{s,k}(z) := \mathcal{M}_s\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

Using this notation, define the Poincaré series $P_k(s; z)$ by

$$P_k(s; z) := \frac{2}{\sqrt{\pi}} \sum_{M \in \Gamma_\infty \backslash \Gamma_0(2)} \chi(M)^{-1}(cz + d)^{-k} \varphi_{s,k}(Mz). \tag{8.9}$$

Here Γ_∞ again is the subgroup of translations in $\mathrm{SL}_2(\mathbb{Z})$.

The defining series is absolutely convergent for $P_k\left(1 - \frac{k}{2}; z\right)$ for $k < 1/2$, and is conditionally convergent when $k = 1/2$. We are interested in $P_{\frac{1}{2}}\left(\frac{3}{4}; z\right)$, which we define by analytically continuing the Fourier expansion. This argument is not straightforward (see Theorem 3.2 and Corollary 4.2 of [62]). Thanks to the properties of $M_{\nu,\mu}$, we find that $P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right)$ is a Maass form of weight $1/2$ for $\Gamma_0(144)$ with Nebentypus χ_{12}.

A long calculation gives the following Fourier expansion

$$P_{\frac{1}{2}}\left(\frac{3}{4}; z\right) = \left(1 - \pi^{-\frac{1}{2}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi y}{6}\right)\right) \cdot q^{-\frac{1}{24}} + \sum_{n=-\infty}^{0} \gamma_y(n) q^{n-\frac{1}{24}} + \sum_{n=1}^{\infty} \beta(n) q^{n-\frac{1}{24}}, \tag{8.10}$$

where for positive integers n we have

$$\beta(n) = \pi(24n - 1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{8.11}$$

The Poincaré series $P_{\frac{1}{2}}\left(\frac{3}{4}; z\right)$ was defined so that (8.11) coincides with the conjectured expressions for the coefficients $\alpha(n)$.

For convenience, we let

$$P(z) := P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right). \tag{8.12}$$

Canonically decompose $P(z)$ into a non-holomorphic and a holomorphic part

$$P(z) = P^-(z) + P^+(z). \tag{8.13}$$

In particular, we have that

$$P^+(z) = q^{-1} + \sum_{n=1}^{\infty} \beta(n) q^{24n-1}.$$

Since $P(z)$ and $D\left(\frac{1}{2}; z\right)$ are Maass forms of weight $1/2$ for $\Gamma_0(144)$ with Nebentypus χ_{12}, (8.10) and (8.11) imply that the proof of the conjecture reduces to proving that these forms are equal. This conclusion is obtained after a lengthy and somewhat complicated argument. □

8.2. **Exact formulas for harmonic Maass forms with weight $\leq 1/2$.** Generalizing the results of the previous section, Bringmann and the author have obtained exact formulas for the coefficients of the holomorphic parts of harmonic Maass forms with weight $2-k \leq 1/2$ [68]. Suppose that f is in $H_{2-k}(N,\chi)$, the space of weight $2-k$ harmonic Maass forms on $\Gamma_0(N)$ with Nebentypus character χ, where we assume that $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$. As usual, we denote its Fourier expansion by

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n. \tag{8.14}$$

It is our objective to determine exact formulas for the coefficients $c_f^+(n)$ of the holomorphic part of f.

We now define the functions which are required for these exact formulas. Throughout, we let $k \in \frac{1}{2}\mathbb{Z}$, and we let χ be a Dirichlet character modulo N, where $4 \mid N$ whenever $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Using this character, for a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we let

$$\Psi_k(M) := \begin{cases} \chi(d) & \text{if } k \in \mathbb{Z}, \\ \chi(d) \left(\frac{c}{d}\right) \epsilon_d^{2k} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \tag{8.15}$$

where ϵ_d is defined by (4.2), and where $\left(\frac{c}{d}\right)$ is the usual extended Legendre symbol. In addition, if $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then we let

$$\mu(T;z) := (cz+d)^{2-k}. \tag{8.16}$$

Moreover, for pairs of matrices $S, T \in \mathrm{SL}_2(\mathbb{Z})$, we then let

$$\sigma(T,S) := \frac{\mu(T;Sz)\mu(S;z)}{\mu(TS;z)}. \tag{8.17}$$

Using this notation, we now define certain generic Kloosterman sums which are naturally associated with cusps of $\Gamma_0(N)$.

Suppose that $\rho = \frac{a_\rho}{c_\rho} = L^{-1}\infty$, $(L \in \mathrm{SL}_2(\mathbb{Z}))$ is a cusp of $\Gamma_0(N)$ with $c_\rho | N$ and $\gcd(a_\rho, N) = 1$. Let t_ρ and κ_ρ be the cusp width and parameter of ρ with respect to $\Gamma_0(N)$ (see 8.20). Suppose that $c > 0$ with $c_\rho | c$ and $\frac{N}{c_\rho} \nmid c$. Then for integers n and m we have the Kloosterman sum

$$K_c(2-k, \rho, \chi, m, n) := \sum_{\substack{0<d<c \\ 0<a<ct \\ a_\rho a \equiv -\frac{c}{c_\rho} \pmod{\frac{N}{c_\rho}} \\ (ad,c)=1}} \frac{\sigma(L^{-1},S)}{\Psi_k(L^{-1}S)} \cdot \exp\left(\frac{2\pi i}{c}\left(\frac{(m+\kappa_\rho)a}{t_\rho} + nd\right)\right), \tag{8.18}$$

where $S := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is the unique matrix defined using the integers a, c, and d. Using properties of σ and Ψ_k, one can easily show that (8.18) is well-defined.

For convenience, we let $\mathcal{S}_N$ be a subset of $\mathrm{SL}_2(\mathbb{Z})$ with the property that

$$\left\{ S^{-1}\infty \ : \ S \in \mathcal{S}_N \right\}$$

represents the set of inequivalent cusps of $\Gamma_0(N)$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define

$$f_M(z) := (cz+d)^{k-2} f\left(\frac{az+b}{cz+d}\right), \tag{8.19}$$

where $\sqrt{z}$ is the principal branch of the holomorphic square root. Using this notation, we have the Fourier expansion of a form f at a cusp ρ. More precisely, if $L \in \mathcal{S}_N$ with $\rho = L^{-1}\infty$, then we have

$$f_\rho(z) = \sum_{n \in \mathbb{Z}} a_\rho^+(n) q^{\frac{n+\kappa_\rho}{t_\rho}} + f_\rho^-(\tau). \tag{8.20}$$

We define the *principal part* of f at ρ by

$$P_{f,\rho}(z) := \sum_{m+\kappa_\rho<0} a_\rho^+(m) q^{\frac{m+\kappa_\rho}{t_\rho}}. \tag{8.21}$$

We shall use the principal parts of a form f to determine our exact formulas. To this end, we identify, for each cusp ρ, its contribution to the exact formula. To make this precise, let $M = L^{-1}$ and $\mu = L\infty$. For positive n, we then define

(8.22)

$$\mathcal{A}(N, 2-k, \chi, \rho, m, c; n) :=$$
$$-\frac{i^k 2\pi}{t_\mu} \left|\frac{(-m+\kappa_\mu)}{t_\mu n}\right|^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c_\mu | c, \frac{N}{c_\mu} \nmid c}} \frac{K_c(2-k, \mu, \chi, -m, -n)}{c} \cdot I_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{n|-m+\kappa_\mu|}{t_\mu}}\right).$$

Here t_μ and κ_μ are the cusp parameters for μ as in the notation above.

Using this notation, we define the order $\mathcal{N}$ *Kloosterman approximation* of $c_f^+(n)$ by

$$\mathcal{C}(f, \mathcal{N}; n) := \sum_{L \in \mathcal{S}_N} \sum_{m+\kappa_\rho<0} a_\rho^+(m) \sum_{c=1}^{\mathcal{N}} \mathcal{A}(N, 2-k, \chi, \rho, m, c; n). \tag{8.23}$$

Moreover, we define $\mathcal{C}(f, \infty; n)$ in the obvious way.

Remark. We stress again that L and ρ are related (throughout this section) by the formula $\rho = L^{-1}\infty$.

Theorem 8.2. *If $f \in H_{2-k}(N, \chi)$ with $2 \le k \in \frac{1}{2}\mathbb{Z}$, then for positive n we have*

$$c_f^+(n) = \mathcal{C}(f, \infty; n).$$

Remark. Using the asymptotic behavior of I-Bessel functions, an inspection of the principal parts of f gives a minimal $\mathcal{N}$ for which

$$\mathcal{C}(f, \mathcal{N}; n) \sim c_f^+(n).$$

Remark. Theorem 8.2 includes the classic results of Rademacher and Zuckerman [197, 198, 250, 251] in the very special case of those $f \in H_{2-k}(N, \chi)$ for which $2-k<0$ and $f^- = 0$. It also recovers some results of Bruinier, Hejhal, and Niebur [71, 143, 187, 188] for harmonic Maass forms of non-positive weight. Recent work by Bruinier and Strömberg [81] addresses the problem of efficiently computing coefficients of harmonic Maass forms.

For weight $2-k=\frac{1}{2}$, we have a conditional result. To make it precise, we say that a form $f \in H_{\frac{1}{2}}(N,\chi)$ is *good* if the Maass-Poincaré series corresponding to nontrivial terms in the principal parts of f are individually convergent.

Theorem 8.3. *If $f \in H_{\frac{1}{2}}(N,\chi)$ is good, then there is a finite set $S_\Theta(f)$ of complex numbers such that for positive n we have*

$$c_f^+(n) = \mathcal{C}(f,\infty;n) + \mu$$

for some $\mu \in S_\Theta(f)$. Moreover, if $n \neq dm^2$ for some $d \mid N$ and $m \in \mathbb{Z}^+$, then $\mu = 0$.

Remark. We believe that all $f \in H_{\frac{1}{2}}(N,\chi)$ are good. In earlier work we deduced convergence of such Maass-Poincaré series by making using of relationships between Kloosterman sums and Salié sums (see Section 4 of [62]), and by generalizing work of Goldfeld and Sarnak [123] on sums of Kloosterman sums (see [117]). It seems likely that a careful application of these ideas will prove that each such f is indeed good.

Sketch of the proof of Theorem 8.2 and 8.3. By the general theory of Maass-Poincaré series, which generalizes Section 5.3, we have an explicit linear combination of Maass-Poincaré series, say $\mathfrak{f} \in H_{2-k}(N,\chi)$, whose principal parts agree with the principal parts of f up to additive constants. The complex number $\mathcal{C}(f,\infty;n)$ is the nth coefficient of the holomorphic part of this linear combination. For the weight $2-k=1/2$, one must argue further (for example, using the equidistribution of CM points) to establish that these formulas are convergent.

There are then three possibilites:

Case 1. We have that $f-\mathfrak{f}$ is a holomorphic modular form. It can only be nonzero when $2-k=\frac{1}{2}$, in which case the Serre-Stark Basis Theorem [220] implies that $f-\mathfrak{f}$ is a linear combination of theta functions, giving Theorem 8.3.

Case 2. We have that $f-\mathfrak{f}$ is a weakly holomorphic modular form which is not a holomorphic modular form. Such a form must have a pole at a cusp. However, this cannot happen since we constructed $\mathfrak{f}$ so that the principal parts of $f-\mathfrak{f}$ are constant.

Case 3. We have that $f-\mathfrak{f}$ is a harmonic Maass form with a non-trivial non-holomorphic part. However, Lemma 4.6 shows that all such harmonic Maass forms have at least one principal part which is nonconstant. Therefore, this possibility never occurs. □

9. Applications to modular forms

Here we use the theory of harmonic Maass forms to address two classical problems in the theory of modular forms:

- Linear relations among Poincaré series
- Vanishing of Hecke eigenvalues.

We are able to obtain new results by making use of Lemma 4.4, the fact that

$$\xi_{2-k} : H_{2-k}(\Gamma) \longrightarrow S_k(\Gamma).$$

These classical problems, which pertain to properties of the Fourier coefficients of cusp forms in $S_k(\Gamma)$, can be investigated by leveraging information from harmonic Maass forms in $H_{2-k}(\Gamma)$.

9.1. **Relations among classical Poincaré series.** It is well known that the Poincaré series (see Section 5.3)

$$\{P(m,k,N;z) \ : \ m \geq 1\}$$

span the space $S_k(N)$. Since the space $S_k(N)$ is finite dimensional, there exist many relations among these Poincaré series. In his classic text on automorphic forms, Iwaniec [151] states the following seemingly simple problem.

Problem. *Find all the linear relations between $P(m,k,N;z)$.*

Remark. Iwaniec states this problem for more general multiplier systems, but the techniques here also work in that setting.

The next theorem, due to Fay [113] and Rhoades (see [206, 208]), follows from the theory of harmonic Maass forms.

Theorem 9.1. *Suppose that $2 \leq k \in \frac{1}{2}\mathbb{Z}$, and suppose that $\mathcal{I}$ is a finite set of positive integers. Then we have that*

$$\sum_{m \in \mathcal{I}} \alpha_m P(m,k,N;z) \equiv 0$$

if and only if there is a form in $M^!_{2-k}(N)$ whose principal part at ∞ is

$$\sum_{m \in \mathcal{I}} \frac{\alpha_m}{m^{k-1}} \cdot q^{-m}$$

with trivial principal parts at all other cusps.

Example. In practice it is not difficult to implement Theorem 9.1. For example, consider the two dimensional space $S_{24}(1)$. Obviously, there is a linear relation between the Poincaré series $P(1,24,1;z), P(2,24,1;z)$ and $P(3,24,1;z)$. Using the formulas in Theorem 5.3, we find that

$$\begin{aligned} P(1,24,1;z) \sim & 1.00100852 \cdot q + 132.988977 \cdot q^2 + 189296.261 \cdot q^3 + \cdots \\ P(2,24,1;z) \sim & 0.00001585 \cdot q + 2.45743136 \cdot q^2 + 114.854805 \cdot q^3 + \cdots \\ P(3,24,1;z) \sim & 0.00000201 \cdot q + 0.01023411 \cdot q^2 + 0.88465633 \cdot q^3 + \cdots . \end{aligned}$$

From these numerics, we find that

$$-0.00000207832 \cdot P(1,24,1;z) + 0.00427703 \cdot P(2,24,1;z) + P(3,24,1;z) \sim 0.$$

Although it is not possible to make this approximation precise by using explicit Fourier expansions, Theorem 9.1 reduces this problem to a simple calculation. We find the exact relation thanks to the existence of the modular form E_{14}/Δ^3

$$\frac{E_{14}(z)}{\Delta(z)^3} = \frac{1}{q^3} + \frac{48}{q^2} - \frac{195660}{q} + \cdots .$$

Therefore, we then find that the two coefficients in the linear combination above are exactly

$$\frac{-195660}{3^{23}} \approx -0.00000207832\ldots \quad \text{and} \quad \frac{48 \cdot 2^{23}}{3^{23}} \approx 0.00427703\ldots .$$

Theorem 9.1 is a simple consequence of the theory of harmonic Maass forms.

Proof of Theorem 9.1. Assume that $\sum_{m\in\mathcal{I}} \alpha_m P(m,k,N;z) \equiv 0$. Define a weight $2-k$ harmonic Maass form f by

$$f = \sum_{m\in\mathcal{I}} \frac{\alpha_m}{m^{k-1}} \cdot Q(-m,k,N;z) = \sum_{n\gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1,4\pi|n|y)q^n.$$

By Lemma 4.4 and Theorem 5.5, we find that

$$\xi_{2-k}(f) = (4\pi)^{k-1}(k-1)\sum_{m\in\mathcal{I}} \alpha_m P(m,k,N;z) = -(4\pi)^{k-1}(k-1)\sum_{n\geq 1} \overline{c_f^-(-n)} n^{k-1} q^n.$$

By our assumption on the sum over the Poincaré series, we know that $c_f^-(n) = 0$ for all $n<0$, and so $f \in M_{2-k}^!(N)$. Namely, in terms of the Maass-Poincaré series $Q(-m,k,N;z)$ in Section 5.3, we have that

$$\sum_{m\in I} \frac{\alpha_m}{m^{k-1}} \cdot Q^+(-m,k,N;z) \in M_{2-k}^!(N)$$

is the weakly holomorphic form that we desire.

Conversely, assume that f is such a weakly holomorphic modular form. From the expansion for the coefficients of $Q^+(-m,k,N;z)$ in Theorem 5.4, we find that

$$\widetilde{f} := -f + \frac{1}{\Gamma(k)} \cdot \sum_{m\in\mathcal{I}} \frac{\alpha_m}{m^{k-1}} \cdot Q(-m,k,N;z) \in H_{2-k}(N)$$

is a harmonic Maass form with trivial principal parts at every cusp. By Lemma 4.6, it then follows that $\widetilde{f}$ is in $M_{2-k}^!(N)$. Theorem 5.5 then implies that $0 = \xi_{2-k}(\widetilde{f}) = \sum_{m\in\mathcal{I}} \alpha_m P(m,k,N;z)$. □

9.2. **Algebraicity and the vanishing of Hecke eigenvalues.** Despite the fact that we have a fairly complete theory of algebraicity for forms in $M_k^!(N,\chi)$, thanks to the q-expansion principle, the theory of Eisenstein series and newforms, the question of algebraicity for harmonic Maass forms remains open. As we shall now see, these questions (in the case of integer weight) are intimately related to the vanishing of eigenvalues of Hecke operators.

Remark. The algebraicity of the coefficients of half-integral weight harmonic Maass forms is much more mysterious. For certain weight $1/2$ forms, we shall relate algebraicity to the vanishing of derivatives of quadratic twists of weight 2 modular L-functions (see Section 12).

To make this precise, we shall restrict our attention to those $f \in H_{2-k}(N,\chi)$ for which $\xi_{2-k}(f) \in S_k(N,\overline{\chi})$ is a Hecke eigenform. For reasons which will become apparent, we shall concentrate on those forms for which

$$\xi_{2-k}(f) = \frac{g}{\|g\|^2}, \tag{9.1}$$

where g is a normalized newform and $\|g\|$ denotes its usual Petersson norm.

To illustrate the nature of our result, we consider two examples of Maass-Poincaré series which are not weakly holomorphic modular forms. The Maass-Poincaré series $f := \frac{1}{11!} \cdot$

$Q(-1,12,1;z) \in H_{-10}(1)$ satisfies (9.1) for $g = \Delta(z)$, the unique normalized weight 12 cusp form on the full modular group. The first few coefficients[15] of its holomorphic part are

$$\frac{1}{11!} \cdot Q^+(-1,12,1;z) \sim q^{-1} - \frac{65520}{691} - 1842.89472q - 23274.07545q^2 - \cdots .$$

There is little reason to believe that these coefficients are rational or algebraic. On the other hand, we shall prove that the Maass-Poincaré series $\frac{1}{3!} \cdot Q(-1,4,9;z) \in H_{-2}(9)$ has the property that $\frac{1}{3!} \cdot Q^+(-1,4,9;z)$ has rational coefficients. Its first few terms are

$$\frac{1}{3!} \cdot Q^+(-1,4,9;z) = q^{-1} - \frac{1}{4}q^2 + \frac{49}{125}q^5 - \frac{48}{512}q^8 - \frac{771}{1331}q^{11} + \cdots , \tag{9.2}$$

and $f := \frac{1}{3!} \cdot Q(-1,4,9;z)$ satisfies (9.1) for the unique normalized newform in $S_4(9)$.

Our next theorem explains the distinction between these two cases. To make this precise, let $g \in S_k(N,\overline{\chi})$ be a normalized newform, and let F_g be the number field obtained by adjoining the coefficients of g to $\mathbb{Q}$.

Definition 9.2. Assuming the hypotheses above, we say that a harmonic Maass form $f \in H_{2-k}(N,\chi)$ is *good for* g if it satisfies the following properties:

(i) The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$.
(ii) The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant.
(iii) We have that $\xi_{2-k}(f) = \|g\|^{-2} g$.

Remark. For every such g, there is an f which is good for g. Moreover, such an f is unique up to a weakly holomorphic form in $M^!_{2-k}(N,\chi)$ with coefficients in F_g. Such f can be constructed explicitly using Poincaré series.

In joint work with Bruinier and Rhoades [79], the author has obtained the following result which explains this phenomenon.

Theorem 9.3. *Let $g \in S_k(N,\overline{\chi})$ be a normalized newform with complex multiplication (see (9.6)). If $f \in H_{2-k}(N,\chi)$ is good for g, then all coefficients of f^+ are in $F_g(\zeta_M)$, where $\zeta_M := e^{2\pi i/M}$ and $M = ND$, where D is the discriminant of the field of complex multiplication.*

Remark.
i) The rationality of $Q^+(-1,4,9;z)$ in (9.2) is an example of Theorem 9.3. In this case $Q(-1,4,9;z)$ is good for $\eta(3z)^8$, the unique CM newform in $S_4(9)$.
ii) The field F_g in Theorem 9.3 is explicit, and is determined by Hecke characters.
iii) Let $g \in S_k(N,\overline{\chi})$ and $f \in H_{2-k}(N,\chi)$ be as in Theorem 9.3. The proof of the theorem implies that all of the coefficients of f^+ belong to $F_g(c_f^+(1))$. In fact, the proof gives this conclusion for any newform g, not just those with CM.
iv) In the examples we know, it turns out that the coefficients of f^+ are actually contained in F_g. It is natural to ask whether this is true in general.

The proof of Theorem 9.3 relies on the fact that some Hecke eigenvalues of g vanish. A simple generalization of the proof of Theorem 9.3 can be used to detect the vanishing of the Fourier coefficients of a newform.

[15]This corrects a typographical error for the constant term in [79].

Theorem 9.4. *Suppose that $g = \sum_{n=1}^{\infty} c_g(n)q^n \in S_k(N, \overline{\chi})$ is a normalized newform, and suppose that $f \in H_{2-k}(N, \chi)$ is good for g. If $p \nmid N$ is a prime for which $c_g(p) = 0$, then $c_f^+(pn)$ is algebraic for every positive n coprime to p.*

As in Remark 9.2 iv), it seems possible that the algebraic coefficients of f^+ are always in F_g. We address this in the next example when $N = 1$.

Example. Here we consider Lehmer's Conjecture on the nonvanishing of Ramanujan's τ-function, where

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n.$$

This example generalizes easily to all level 1 Hecke eigenforms.

Although Theorem 9.4 relates Lehmer's Conjecture to the alleged transcendence of the coefficients, say $a_\Delta(n)$, of $\frac{1}{11!} \cdot Q^+(-1, 12, 1; z)$, it turns out that more is true. Lehmer's Conjecture is implied by the irrationality of these coefficients.

We make use of explicit formulas. Using the classical Eisenstein series E_4 and E_6, and the classical j-function $j(z)$, we define polynomials $J_m(x) \in \mathbb{Z}[x]$ by

$$\sum_{m=0}^{\infty} J_m(x)q^m := \frac{E_4(z)^2 E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - x} = 1 + (x - 744)q + \cdots . \tag{9.3}$$

For each m we then let $j_m(z) = J_m(j(z))$. If p is prime, then define the modular functions

$$A_p(z) := \frac{24}{B_{12}}(1 + p^{11}) + j_p(z) - 264 \sum_{m=1}^{p} \sigma_9(m) j_{p-m}(z), \tag{9.4}$$

$$B_p(z) := -\tau(p) \left(-264 + \frac{24}{B_{12}} + j_1(z) \right). \tag{9.5}$$

Here $B_{12} = -\frac{691}{2730}$ is the 12th Bernoulli number. Using the principal part of $Q(-1, 12, 1; z)$ combined with the fact that $\Delta(z)$ is an eigenform of the Hecke algebra, Theorem 4.11 implies (also see [193]), for primes p, that

$$\sum_{n=-p}^{\infty} \left(p^{11} a_\Delta(pn) - \tau(p) a_\Delta(n) + a_\Delta(n/p) \right) q^n = \frac{A_p(z) + B_p(z)}{E_4(z) E_6(z)}.$$

These weight -10 modular forms have integer coefficients. Now suppose that $\tau(p) = 0$ for a prime p. Then $a_\Delta(np)$ is rational for every n coprime to p.

Remark. Results similar to the example above have also been obtained recently by Boylan and Guerzhoy [51, 135].

Here we prove Theorem 9.3 by combining facts about ξ_{2-k}, with Hecke theory and the theory of complex multiplication. We first begin with an important proposition.

Proposition 9.5. *Let $g = \sum_{n=1}^{\infty} b(n)q^n \in S_k(N, \overline{\chi})$ be a normalized newform with integer weight $k \geq 2$, and let F_g be the number field obtained by adjoining the coefficients of g to $\mathbb{Q}$. Then there is a harmonic Maass form $f \in H_{2-k}(N, \chi)$ which satisfies:*

(i) *The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$.*

(ii) *The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant.*
(iii) *We have that $\xi_{2-k}(f) = \|g\|^{-2}g$.*

Proof. Let $H_{2-k,\infty}(N,\chi)$ be the subspace of those $f \in H_{2-k}(N,\chi)$ whose principal parts at the cusps other than ∞ are constant. Note that

$$H_{2-k}(N,\chi) = H_{2-k,\infty}(N,\chi) + M^!_{2-k}(N,\chi).$$

Arguing as in Section 3 of [73], the restriction of ξ_{2-k} to $H_{2-k,\infty}(N,\chi)$ defines a surjective map to $S_k(N,\overline{\chi})$. One now argues as in the proof of Lemma 7.3 of [77] using the pairing $\{g,f\} = (g,\xi_{2-k}(f))$, where $f \in H_{2-k,\infty}(N,\chi)$ and $g \in S_k(N,\overline{\chi})$. □

Remark. The harmonic Maass form f satisfying (i)–(iii) is unique up to the addition of a weakly holomorphic form in $M^!_{2-k}(N,\chi)$ with coefficients in F_g and a pole possibly at infinity and constant principal part at all other cusps.

For completeness, here we briefly recall the notion of a newform with complex multiplication (for example, see Chapter 12 of [151] or Section 1.2 of [191]). Let $D < 0$ be the fundamental discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let O_K be the ring of integers of K, and let $\chi_K := \left(\frac{D}{\bullet}\right)$ be the usual Kronecker character associated to K. Let $k \geq 2$, and let c be a Hecke character of K with exponent $k-1$ and conductor $\mathfrak{f}_c$, a non-zero ideal of O_K. By definition, this means that

$$c : I(\mathfrak{f}_c) \longrightarrow \mathbb{C}^\times$$

is a homomorphism, where $I(\mathfrak{f}_c)$ denotes the group of fractional ideals of K prime to $\mathfrak{f}_c$, and

$$c(\alpha O_K) = \alpha^{k-1}$$

for $\alpha \in K^\times$ for which $\alpha \equiv 1 \bmod^\times \mathfrak{f}_c$. To c we naturally associate a Dirichlet character ω_c defined, for every integer n coprime to $\mathfrak{f}_c$, by

$$\omega_c(n) := \frac{c(nO_K)}{n^{k-1}}.$$

We then let

$$\Phi_{K,c}(z) := \sum_{\mathfrak{a}} c(\mathfrak{a}) q^{N(a)}, \tag{9.6}$$

where $\mathfrak{a}$ varies over the ideals of O_K prime to $\mathfrak{f}_c$, and where $N(\mathfrak{a})$ is the usual ideal norm. It is well known that $\Phi_{K,c}(z) \in S_k(|D| \cdot N(\mathfrak{f}_c), \chi_K \cdot \omega_c)$ is a normalized newform. These are newforms with *complex multiplication*. By construction, if we let

$$\Phi_{K,c}(z) = \sum_{n=1}^{\infty} b(n) q^n,$$

then

$$b(n) = 0 \quad \text{whenever } \chi_K(n) = -1. \tag{9.7}$$

This follows since every prime p for which $\chi_K(p) = -1$ is inert.

Proof of Theorem 9.3. Suppose that f is good for a CM form $g = \sum_{n=1}^{\infty} b(n)q^n$, and let $D = D_g$ be the fundamental discriminant of the associated imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. By Lemma 4.4, we then have that

$$\xi_{2-k}(f) = \|g\|^{-2} g = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n.$$

Since g has complex multiplication, (9.7) implies that $c_f^-(n) = 0$ when $\chi_K(-n) = -1$. Because $D < 0$, this means that

$$c_f^-(n) = 0 \text{ when } \chi_K(n) = 1. \tag{9.8}$$

Let $M = ND$. We write χ_0 for the trivial character modulo $|D|$. Since $D \mid N$, a simple generalization of standard facts about twists of modular forms implies that

$$u := f \otimes \chi_0 + f \otimes \chi_K$$

is in $H_{2-k}(M, \chi)$. The Fourier expansion of $u = u^+ + u^-$ is given by

$$u^+(z) = 2 \sum_{\substack{n \gg -\infty \\ \chi_K(n)=1}} c_f^+(n) q^n,$$

$$u^-(z) = 2 \sum_{\substack{n < 0 \\ \chi_K(n)=1}} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

The non-holomorphic part u^- vanishes, and so u is a weakly holomorphic modular form.

We now claim that for any integer b, $f(z + b/D)$ has principal parts at all cusps in $F_g(\zeta_M)[q^{-1}]$. To see this, we let $\gamma \in \Gamma(1)$ and consider the cusp $\gamma\infty$. There exists a $\widetilde{\gamma} \in \Gamma(1)$ and $\alpha, \beta, \delta \in \mathbb{Z}$ such that

$$\begin{pmatrix} D & b \\ 0 & D \end{pmatrix} \gamma = \widetilde{\gamma} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}.$$

Hence, the Fourier expansion of $f(z + b/D)$ at the cusp $\gamma\infty$ is given by

$$f \mid \widetilde{\gamma} \mid \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}.$$

By the assumption of f, it is holomorphic at the cusp ∞, unless $\widetilde{\gamma} \in \Gamma_0(N)$, in which case it is equal to

$$f \mid \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}.$$

Since $\delta \mid D^2 \mid M$, the principal part at ∞ of this modular form is contained in $F_g(\zeta_M)[q^{-1}]$, proving the claim. This implies that the twists $f \otimes \chi_0$, $f \otimes \chi_D$, have principal parts at all cusps in $F_g(\zeta_M)[q^{-1}]$. Therefore, the same is true for u.

The action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\zeta_N))$ commutes with the action of $SL_2(\mathbb{Z})$ on modular functions for $\Gamma(N)$ (for example, see Theorem 6.6 in Chapter 6.2 and the diagram before Remark 6.7 in Shimura's book [224]). Using the action of $\mathrm{Aut}(\mathbb{C}/F_g(\zeta_M))$ on weakly holomorphic modular forms, we see that u^σ has the same properties for any $\sigma \in \mathrm{Aut}(\mathbb{C}/F_g(\zeta_M))$. Moreover, u^σ has the same principal parts as u at all cusps. Hence the difference $u - u^\sigma$ is a holomorphic

modular form which vanishes at the cusp ∞. Since $2-k \leq 0$, this implies that $u = u^{\sigma}$. Consequently, u is defined over $F_g(\zeta_M)$. So for all $n \in \mathbb{Z}$ with $\chi_K(n) = 1$, we have that $c_f^+(n) \in F_g(\zeta_M)$. In particular, $c_f^+(1) \in F_g(\zeta_M)$.

We now use the Hecke action on f and g. By Theorem 4.11, we have that

$$f\,|_{2-k}\,T(m) = m^{1-k}b(m)f + f',$$

where $f' \in M^!_{2-k}(\Gamma_0(N), \chi)$ is a weakly holomorphic form with coefficients in F_g. In view of the formula for the action of the Hecke operators on the Fourier expansion, we obtain for any prime p that

$$c_f^+(pn) + \chi(p)p^{1-k}c_f^+(n/p) = p^{1-k}b(p)c_f^+(n) + c_{f'}^+(n),$$

where $c_{f'}^+(n) \in F_g$. Hence an inductive argument shows that all coefficients $c_f^+(n)$ are contained in the extension $F_g(c_f^+(1))$. This concludes the proof of the theorem since we have already established that $c_f^+(1)$ is in $F_g(\zeta_M)$. $\square$

The proof of Theorem 9.4 is similar to the proof of Theorem 9.3, and so we only give a brief indication of how the proof must be modified.

Sketch proof of Theorem 9.4. If $p \nmid N$ is a prime, then for every positive integer m we have that

$$c_g(p)c_g(p^m) = c_g(p^{m+1}) + \overline{\chi(p)}p^{k-1}c_g(p^{m-1}).$$

Therefore, if $p \nmid N$ is a prime for which $c_g(p) = 0$, then we have that

$$c_g(p^{m+1}) = -\overline{\chi(p)}p^{k-1}c_g(p^{m-1}),$$

which in turn implies that

$$c_g(p^m) = \begin{cases} \left(-\overline{\chi(p)}p^{k-1}\right)^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

A standard argument gives a harmonic Maass form whose Fourier coefficients are supported on terms whose exponents n have the property that p exactly divides n. By the multiplicativity of the Fourier coefficients of newforms, it then follows by the observation above that the non-holomorphic part of this form is identically zero. In other words, this particular harmonic Maass form is a weakly holomorphic modular form with suitable principal parts at cusps. The proof now follows *mutatis mutandis* as in the proof of Theorem 9.3. $\square$

10. Generating functions for singular moduli

Let $j(z)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots .$$

The values of modular functions such as $j(z)$ at imaginary quadratic arguments in $\mathbb{H}$ are known as *singular moduli*. Singular moduli are algebraic integers which play many roles in number theory. For example, they generate class fields of imaginary quadratic fields, and they parameterize isomorphism classes of elliptic curves with complex multiplication.

In an important paper, Zagier [245] proved that the traces of these numbers are Fourier coefficients of certain weight 3/2 modular forms. To illustrate his results, we begin by comparing the classical evaluations

$$\frac{j\left(\frac{-1+\sqrt{-3}}{2}\right)-744}{3}=-248, \qquad \frac{j(i)-744}{2}=492, \qquad j\left(\frac{1+\sqrt{-7}}{2}\right)-744=-4119,$$

with the Fourier coefficients of the modular form

$$g(z):=-\frac{\eta(z)^2\cdot E_4(4z)}{\eta(2z)\eta(4z)^6}=-q^{-1}+2-248q^3+492q^4-4119q^7+7256q^8-\cdots. \tag{10.1}$$

The appearance of singular moduli as the initial coefficients of the modular form $g(z)$ is not a coincidence.

We now make this more precise. For integers λ, let $\mathcal{M}^!_{\lambda+\frac{1}{2}}(4)$ be the space of weight $\lambda+\frac{1}{2}$ weakly holomorphic modular forms on $\Gamma_0(4)$ satisfying the "Kohnen plus-space" condition. A form satisfies this condition if its q-expansion has the form

$$\sum_{(-1)^\lambda n\equiv 0,1 \pmod 4} a(n)q^n. \tag{10.2}$$

Throughout let $d\equiv 0,3 \pmod 4$ be a positive integer, let $H(d)$ be the Hurwitz-Kronecker class number for the discriminant $-d$, and let $\mathcal{Q}_d$ be the set of positive definite integral binary quadratic forms (note. including imprimitive forms)

$$Q(x,y)=[a,b,c]=ax^2+bxy+cy^2$$

with discriminant $D_Q=-d=b^2-4ac$. For each Q, let τ_Q be the unique root in $\mathbb{H}$ of $Q(x,1)=0$. The singular modulus $f(\tau_Q)$, for any modular invariant $f(z)$, depends only on the equivalence class of Q under the action of $\Gamma:=\mathrm{PSL}_2(\mathbb{Z})$. If $\omega_Q\in\{1,2,3\}$ is given by

$$\omega_Q:=\begin{cases}2 & \text{if } Q\sim_\Gamma [a,0,a],\\ 3 & \text{if } Q\sim_\Gamma [a,a,a],\\ 1 & \text{otherwise,}\end{cases} \tag{10.3}$$

then, for a modular invariant $f(z)$, define the trace $\mathrm{Tr}(f;d)$ by

$$\mathrm{Tr}(f;d):=\sum_{Q\in\mathcal{Q}_d/\Gamma}\frac{f(\tau_Q)}{\omega_Q}. \tag{10.4}$$

Theorems 1 and 5 of Zagier's paper [245] imply the following theorem.

Theorem 10.1. *If $f(z)\in\mathbb{Z}[j(z)]$ has a Fourier expansion with constant term 0, then there is a finite principal part $A_f(z)=\sum_{n\le 0}a_f(n)q^n$ for which*

$$A_f(z)+\sum_{0<d\equiv 0,3 \pmod 4}\mathrm{Tr}(f;d)q^d\in\mathcal{M}^!_{\frac{3}{2}}(4).$$

Remark. The earlier claim in (10.1) is the $f(z)=j(z)-744$ case of this theorem.

Remark. Zagier's paper [245] includes many generalizations of this theorem. One of his results proves that the generating functions of the traces of $j_m(z) := m\,(j(z) - 744)\,|T(m)$ are also explicit weight 3/2 weakly holomorphic modular forms. This result is of particular significance because it reduces the computation of Hilbert class polynomials to a straightforward calculation.

10.1. **Further generating functions for traces of singular moduli.** Zagier's work has inspired an enormous number of works [63, 67, 74, 75, 92, 93, 103, 104, 106, 116, 153, 154, 181] by the author, Bringmann, Bruinier, D. Choi, W. Duke, A. Folsom, J. Funke, O. Imamoḡlu, P. Jenkins, D. Jeon, S.-Y. Kang, C. Kim, R. Masri, A. Miller, A. Pixton, J. Rouse, and A. Toth. These papers give theorems related to the phenomenon that the coefficients of modular forms and harmonic Maass forms are often the "traces" of some sort of singular moduli. For brevity, here we discuss one construction which is based on Maass-Poincaré series.

For fundamental discriminants D_1, let χ_{D_1} denote the associated genus character for positive definite binary quadratic forms whose discriminants are multiples of D_1. If λ is an integer and D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then define the twisted trace of a modular invariant $f(z)$, say $\mathrm{Tr}_{D_1}(f; D_2)$, by

$$\mathrm{Tr}_{D_1}(f; D_2) := \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma} \frac{\chi_{D_1}(Q) f(\tau_Q)}{\omega_Q}. \tag{10.5}$$

Many of the works mentioned above prove that such traces are often coefficients of Maass forms and weakly holomorphic modular forms. Here we give one illustrative example of this phenomenon using some of the Poincaré series constructed in Section 5.3. For convenience, we briefly recall their construction. Let $k := \lambda + \frac{1}{2}$, where λ is an arbitrary integer, and let $M_{\nu,\mu}(z)$ be the usual M-Whittaker function. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|).$$

Suppose that $m \geq 1$ is an integer with $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$. As before, let

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx),$$

where $z = x + iy$ and $e(w) := e^{2\pi i w}$, and let Γ_∞ denote the translations in $\mathrm{SL}_2(\mathbb{Z})$. With this notation, define the Poincaré series

$$\mathcal{F}_\lambda(-m, s; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \,|_k\, A)(z) \tag{10.6}$$

for $\mathrm{Re}(s) > 1$. Here $|_k$ denotes the usual half-integral weight k "slash operator". If pr_λ is Kohnen's projection operator (see page 250 of [162]) to the weight $\lambda + \frac{1}{2}$ plus-space for $\Gamma_0(4)$, then for $\lambda \notin \{0, 1\}$ define $F_\lambda(-m; z)$ by

$$F_\lambda(-m; z) := \begin{cases} \frac{3}{2}\mathcal{F}_\lambda\left(-m, \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \geq 2, \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \leq -1. \end{cases} \tag{10.7}$$

Remark. For $\lambda = 0$ or 1 we also have series $F_\lambda(-m; z)$. Their construction requires more care due to questions of convergence (see [63, 75]).

Arguing as in Section 5.3, if $\lambda \geq -6$ with $\lambda \neq -5$, then $F_\lambda(-m;z) \in \mathcal{M}^!_{\lambda+\frac{1}{2}}(4)$. For such λ, we denote the corresponding Fourier expansions by

$$F_\lambda(-m;z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n \in \mathcal{M}^!_{\lambda+\frac{1}{2}}(4). \tag{10.8}$$

For other λ, namely $\lambda = -5$ or $\lambda \leq -7$, it turns out that the $F_\lambda(-m;z)$ are Maass forms in $H_{\lambda+\frac{1}{2}}(4)$. We denote their expansions by

$$F_\lambda(-m;z) = B_\lambda(-m;z) + q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n, \tag{10.9}$$

where $B_\lambda(-m;z)$ is the "non-holomorphic" part of $F_\lambda(-m;z)$.

Example. If $\lambda = 1$ and $-m = -1$, then we find the form in (10.1)

$$-F_1(-1;z) = g(z) = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 - \cdots .$$

Generalizing Zagier's results, we have that the coefficients $b_\lambda(-m;n)$ of the $F_\lambda(-m;z)$ are traces of singular moduli for functions defined by Niebur [187]. If $I_s(x)$ denotes the usual I-Bessel function, and if $\lambda > 1$, then let

$$\mathfrak{F}_\lambda(z) := \pi \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi \mathrm{Im}(Az)) e(-\mathrm{Re}(Az)). \tag{10.10}$$

For $\lambda = 1$, a similar construction holds. Indeed, Niebur [187] showed that $\mathfrak{F}_1(z) = \frac{1}{2}(j(z) - 744)$, where this function is the analytic continuation, as $s \to 1^+$, of

$$-12 + \pi \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi \mathrm{Im}(Az)) e(-\mathrm{Re}(Az)).$$

We then have the following theorem.

Theorem 10.2. (Bringmann and Ono; Theorem 1.2 of [63])
If $\lambda, m \geq 1$ are integers for which $(-1)^{\lambda+1}m$ is a fundamental discriminant (note. which includes 1), then for each positive integer n with $(-1)^\lambda n \equiv 0, 1 \pmod 4$ we have

$$b_\lambda(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]} n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda; n).$$

Remark. A version of Theorem 10.2 holds for integers $\lambda \leq 0$. This follows from a duality (see Theorem 1.1 of [63]) of Fourier expansions. Suppose that $\lambda \geq 1$, and that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0, 1 \pmod 4$. For every positive integer n with $(-1)^\lambda n \equiv 0, 1 \pmod 4$, this duality implies that

$$b_\lambda(-m;n) = -b_{1-\lambda}(-n;m).$$

Example. For $\lambda = 1$, Theorem 10.2 relates $b_1(-m;n)$ to traces and twisted traces of $\mathfrak{F}_1(z) = \frac{1}{2}(j(z) - 744)$. These are Theorems 1 and 6 of Zagier's paper [245].

Theorem 10.2 is obtained by reformulating, as traces of singular moduli, exact expressions for the coefficients $b_\lambda(-m;n)$. The proof follows from the classical fact that certain half-integral weight Kloosterman sums are essentially Salié sums. To define these sums, suppose that $0 \neq D_1 \equiv 0, 1 \pmod 4$. If λ is an integer, $D_2 \neq 0$ is an integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, and N is a positive multiple of 4, then define the generalized Salié sum $S_\lambda(D_1, D_2, N)$ by

$$S_\lambda(D_1, D_2, N) := \sum_{\substack{x \pmod N \\ x^2 \equiv (-1)^\lambda D_1 D_2 \pmod N}} \chi_{D_1}\left(\frac{N}{4}, x, \frac{x^2 - (-1)^\lambda D_1 D_2}{N}\right) e\left(\frac{2x}{N}\right), \tag{10.11}$$

where $\chi_{D_1}(a, b, c)$, for a binary quadratic form $Q = [a, b, c]$, is given by

$$\chi_{D_1}(a, b, c) := \begin{cases} 0 & \text{if } (a, b, c, D_1) > 1, \\ \left(\frac{D_1}{r}\right) & \text{if } (a, b, c, D_1) = 1 \text{ and } Q \text{ represents } r \text{ with } (r, D_1) = 1. \end{cases} \tag{10.12}$$

Remark. If $D_1 = 1$, then χ_{D_1} is trivial. Therefore, if $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then

$$S_\lambda(1, D_2, N) = \sum_{\substack{x \pmod N \\ x^2 \equiv (-1)^\lambda D_2 \pmod N}} e\left(\frac{2x}{N}\right).$$

Half-integral weight Kloosterman sums are essentially equal to such Salié sums, a fact which plays a fundamental role throughout the theory of half-integral weight modular forms. The following proposition is due to Kohnen (see Proposition 5 of [162]).

Proposition 10.3. *Suppose that N is a positive multiple of 4. If λ is an integer, and D_1 and D_2 are non-zero integers for which $D_1, (-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then*

$$N^{-\frac{1}{2}}(1 - (-1)^\lambda i)(1 + \delta_{\text{odd}}(N/4)) \cdot K_\lambda((-1)^\lambda D_1, D_2, N) = S_\lambda(D_1, D_2, N).$$

As a consequence, we may rewrite the formulas for the $b_\lambda(-m;n)$ using Salié sums. The following proposition describes these Salié sums as Poincaré-type series over CM points.

Proposition 10.4. *Suppose that λ is an integer, and that D_1 is a fundamental discriminant. If D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then for every positive integer a we have*

$$S_\lambda(D_1, D_2, 4a) = 2 \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma} \frac{\chi_{D_1}(Q)}{\omega_Q} \sum_{\substack{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Im}(A\tau_Q) = \frac{\sqrt{|D_1 D_2|}}{2a}}} e\left(-\mathrm{Re}\left(A\tau_Q\right)\right).$$

Proof. For every integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2$$

of discriminant $(-1)^\lambda D_1 D_2$, let $\tau_Q \in \mathbb{H}$ be as before. Clearly τ_Q is equal to

$$\tau_Q = \frac{-b + i\sqrt{|D_1 D_2|}}{2a}, \tag{10.13}$$

and the coefficient b of Q solves the congruence

$$b^2 \equiv (-1)^\lambda D_1 D_2 \pmod{4a}. \tag{10.14}$$

Conversely, every solution of (10.14) corresponds to a quadratic form with an associated CM point thereby providing a one-to-one correspondence between the solutions of

$$b^2 - 4ac = (-1)^\lambda D_1 D_2 \qquad (a, b, c \in \mathbb{Z}, a, c > 0)$$

and the points of the orbits

$$\bigcup_Q \left\{ A\tau_Q : \ A \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_{\tau_Q} \right\},$$

where Γ_{τ_Q} denotes the isotropy subgroup of τ_Q in $\mathrm{SL}_2(\mathbb{Z})$, and where Q varies over the representatives of $\mathcal{Q}_{|D_1 D_2|}/\Gamma$. The group Γ_∞ preserves the imaginary part of such a CM point τ_Q, and preserves (10.14). However, it does not preserve the middle coefficient b of the corresponding quadratic forms modulo $4a$. It identifies the congruence classes $b, b+2a \pmod{4a}$ appearing in the definition of $S_\lambda(D_1, D_2, 4a)$. Since $\chi_{D_1}(Q)$ is fixed under the action of Γ_∞, the corresponding summands for such pairs of congruence classes are equal. Proposition 10.4 follows since $\#\Gamma_{\tau_Q} = 2\omega_Q$, and since both Γ_{τ_Q} and Γ_∞ contain the negative identity matrix. □

Sketch of the proof of Theorem 10.2. Here we prove the cases where $\lambda \geq 2$. The argument when $\lambda = 1$ is identical. For $\lambda \geq 2$, the exact formula for $b_\lambda(-m; n)$ is

$$b_\lambda(-m; n) = (-1)^{[(\lambda+1)/2]} \pi \sqrt{2} (n/m)^{\frac{\lambda}{2} - \frac{1}{4}} (1 - (-1)^\lambda i) \\ \times \sum_{\substack{c > 0 \\ c \equiv 0 \pmod 4}} (1 + \delta_{\mathrm{odd}}(c/4)) \frac{K_\lambda(-m, n, c)}{c} \cdot I_{\lambda - \frac{1}{2}} \left(\frac{4\pi\sqrt{mn}}{c} \right).$$

Using Proposition 10.3, where $D_1 = (-1)^{\lambda+1} m$ and $D_2 = n$, for integers $N = c$ which are positive multiples of 4, we have

$$c^{-\frac{1}{2}} (1 - (-1)^\lambda i)(1 + \delta_{\mathrm{odd}}(c/4)) \cdot K_\lambda(-m, n, c) = S_\lambda((-1)^{\lambda+1} m, n, c).$$

These identities, combined with the change of variable $c = 4a$, give

$$b_\lambda(-m; n) = \frac{(-1)^{[(\lambda+1)/2]} \pi}{\sqrt{2}} (n/m)^{\frac{\lambda}{2} - \frac{1}{4}} \sum_{a=1}^{\infty} \frac{S_\lambda((-1)^{\lambda+1} m, n, 4a)}{\sqrt{a}} \cdot I_{\lambda - \frac{1}{2}} \left(\frac{\pi\sqrt{mn}}{a} \right).$$

Using Proposition 10.4, this becomes

$$b_\lambda(-m; n) = \frac{2(-1)^{[(\lambda+1)/2]} \pi}{\sqrt{2}} (n/m)^{\frac{\lambda}{2} - \frac{1}{4}} \sum_{Q \in \mathcal{Q}_{nm}/\Gamma} \frac{\chi_{(-1)^{\lambda+1} m}(Q)}{\omega_Q} \\ \sum_{a=1}^{\infty} \sum_{\substack{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Im}(A\tau_Q) = \frac{\sqrt{mn}}{2a}}} \frac{I_{\lambda - \frac{1}{2}}(2\pi \mathrm{Im}(A\tau_Q))}{\sqrt{a}} \cdot e(-\mathrm{Re}(A\tau_Q)).$$

The definition of $\mathfrak{F}_\lambda(z)$ in (10.10), combined with the obvious change of variable relating $1/\sqrt{a}$ to $\mathrm{Im}(A\tau_Q)^{\frac{1}{2}}$, gives

$$\begin{aligned} b_\lambda(-m;n) &= \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}}\cdot\pi\sum_{Q\in\mathcal{Q}_{nm}/\Gamma}\frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q} \\ &\qquad \sum_{A\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})}\mathrm{Im}(A\tau_Q)^{\frac{1}{2}}\cdot I_{\lambda-\frac{1}{2}}(2\pi\mathrm{Im}(A\tau_Q))e(-\mathrm{Re}(A\tau_Q)) \\ &= \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}}\cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda;n). \end{aligned}$$

□

10.2. **The "24-Theorem".** Exact formulas for traces of singular moduli can lead to nice number theoretic consequences. Here we mention one such application which is related to the classical observation that

$$e^{\pi\sqrt{163}} = 262537412640768743.9999999999992\ldots \tag{10.15}$$

is nearly an integer. To make this precise, we recall some classical facts. A primitive positive definite binary quadratic form Q is *reduced* if $|B|\le A\le C$, and $B\ge 0$ if either $|B|=A$ or $A=C$. If $-d<-4$ is a fundamental discriminant, then there are $H(d)$ reduced forms with discriminant $-d$. The set of such reduced forms, say $\mathcal{Q}_d^{\mathrm{red}}$, is a complete set of representatives for $\mathcal{Q}_d/\mathrm{SL}_2(\mathbb{Z})$. Moreover, each such reduced form has $1\le A\le\sqrt{d/3}$ (see page 29 of [97]), and has the property that τ_Q lies in the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{F}=\left\{-\frac{1}{2}\le \mathrm{Re}(z)<\frac{1}{2}\text{ and }|z|>1\right\}\cup\left\{-\frac{1}{2}\le\mathrm{Re}(z)\le 0\ \text{ and }\ |z|=1\right\}. \tag{10.16}$$

Since $J_1(z):=j(z)-744=q^{-1}+196884q+\cdots$, it follows that if $G^{\mathrm{red}}(d)$ is defined by

$$G^{\mathrm{red}}(d)=\sum_{Q=(A,B,C)\in\mathcal{Q}_d^{\mathrm{red}}} e^{\pi Bi/A}\cdot e^{\pi\sqrt{d}/A}, \tag{10.17}$$

then $\mathrm{Tr}(d)-G^{\mathrm{red}}(d)$ is "small", where $\mathrm{Tr}(d):=\mathrm{Tr}(J_1;d)$. In other words, q^{-1} provides a good approximation for $J_1(z)$ for most points z. This is illustrated by (10.15) where $H(163)=1$.

It is natural to investigate the "average value"

$$\frac{\mathrm{Tr}(d)-G^{\mathrm{red}}(d)}{H(d)},$$

which for $d=163$ is $-0.0000000000008\ldots$. Armed with the exact formulas for $\mathrm{Tr}(d)$, it turns out that a uniform picture emerges for a slightly perturbed average, one including some non-reduced quadratic forms. For each positive integer A, let $\mathcal{Q}_{A,d}^{\mathrm{old}}$ denote the set

$$\mathcal{Q}_{A,d}^{\mathrm{old}}=\{Q=(A,B,C)\ :\ \text{non-reduced with } D_Q=-d \text{ and } |B|\le A\}. \tag{10.18}$$

Define $G^{\text{old}}(d)$ by

$$G^{\text{old}}(d) = \sum_{\substack{\sqrt{d}/2 \le A \le \sqrt{d/3} \\ Q \in \mathcal{Q}^{\text{old}}_{A,d}}} e^{\pi B i/A} \cdot e^{\pi\sqrt{d}/A}. \tag{10.19}$$

The non-reduced forms Q contributing to $G^{\text{old}}(d)$ are those primitive discriminant $-d$ forms for which τ_Q is in the bounded region obtained by connecting the two endpoints of the lower boundary of $\mathcal{F}$ with a horizontal line. Since $\mathrm{Tr}(d)$ is subexponential in $|d|$ and $H(d) \ll |d|^{\frac{1}{2}+\epsilon}$, the following numerics are quite surprising:

$$\frac{\mathrm{Tr}(d) - G^{\text{red}}(d) - G^{\text{old}}(d)}{H(d)} = \begin{cases} -24.672\ldots & \text{if } d = 1931, \\ -24.483\ldots & \text{if } d = 2028, \\ -23.458\ldots & \text{if } d = 2111. \end{cases}$$

Recently, Duke has proved [103] a result which implies the following theorem.

Theorem 10.5. *As $-d$ ranges over negative fundamental discriminants, we have*

$$\lim_{-d \to -\infty} \frac{\mathrm{Tr}(d) - G^{\text{red}}(d) - G^{\text{old}}(d)}{H(d)} = -24.$$

Here we explain the source of -24 in the limit

$$\lim_{-d \to -\infty} \frac{\mathrm{Tr}(d) - G^{\text{red}}(d) - G^{\text{old}}(d)}{H(d)} = -24. \tag{10.20}$$

Using Theorem 10.2 and Propositions 10.3 and 10.4, it is not difficult to reformulate the exact formulas for $\mathrm{Tr}(d)$ to read

$$\mathrm{Tr}(d) = -24H(d) + \sum_{\substack{c>0 \\ c \equiv 0 \ (4)}} S(d,c) \sinh(4\pi\sqrt{d}/c),$$

where $S(d,c)$ is the Salié sum

$$S(d,c) = \sum_{x^2 \equiv -d \pmod{c}} e(2x/c).$$

As a consequence, the "24 Theorem" is equivalent to the assertion that

$$\sum_{\substack{c > \sqrt{d/3} \\ c \equiv 0 \ (4)}} S(d,c) \sinh\left(\frac{4\pi}{c}\sqrt{d}\right) = o\left(H(d)\right).$$

This follows from the fact the sum over $c \le \sqrt{d/3}$ is essentially $G^{\text{red}}(d) + G^{\text{old}}(d)$. The sinh factor contributes the size of q^{-1} in the Fourier expansion of a singular modulus, and the summands in the Kloosterman sum provide the corresponding "angles". The contribution $G^{\text{old}}(d)$ arises from the fact that the Kloosterman sum cannot distinguish between reduced and non-reduced forms. In view of Siegel's theorem that $H(d) \gg_\epsilon d^{\frac{1}{2}-\epsilon}$, (10.20) follows from a bound for such sums of the form $\ll d^{\frac{1}{2}-\gamma}$, for some $\gamma > 0$. Such bounds are implicit

in Duke's proof of this result [103], and more generally in his famous work on bounding coefficients of half-integral weight cusp forms [102].

10.3. **Singular moduli for certain non-holomorphic modular functions.** Here we briefly touch on recent work which gives further similar theorems on traces of singular moduli. For brevity we only give a brief indication of the flavor of these results. The reader should consult the cited references for further details.

Along these lines, Bruinier and the author [78], Bruinier have recently proved that the coefficients of certain weight $-1/2$ harmonic Maass forms are traces of singular moduli over Heegner divisors for weak Maass forms We have the following theorem.

Theorem 10.6 (Theorem 3.6 of [78])**.** *Let $f \in H_{-2}(N)$ and put $\partial f := \frac{1}{4\pi} R_{-2,z} f$. For $m \in \mathbb{Q}_{>0}$, the (m,h)-th Fourier coefficient of the holomorphic part of the theta lift $\Lambda(\tau, f)$ equals*

$$\mathrm{tr}_f(m,h) = -\frac{1}{2m} \sum_{z \in Z(m,h)} \partial f(z).$$

Remark. This theorem is stated using the notion of vector-valued harmonic Maass forms. This notion will be introduced in the next section in connection with an eye towards Borcherds products and modular L-functions. We also note that this result has recently be generalized by Alfes [7].

As a special case of this theorem, we recall Theorem 2.1 proved by Bruinier and the author in [78]. To state this result we let

$$F(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 10 - 29q - \dots,$$

and then we define

$$P(z) := -\left(\frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z) = \left(1 - \frac{1}{2\pi y}\right) q^{-1} + \frac{5}{\pi y} + \left(29 + \frac{29}{2\pi y}\right) q + \dots.$$

Theorem 10.7. *If n is a positive integer, then*

$$H_n^{\mathrm{part}}(x) = x^{h(1-24n)} - (24n-1)p(n)x^{h(1-24n)-1} + \dots,$$

where $p(n)$ is the partition function. Moreover, we that

$$p(n) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_{6,1-24n,1}} P(\alpha_Q),$$

and we have that each $(24n-1)P(\alpha_Q)$ is an algebraic integer.

Remark. It is interesting to compare this formula for $p(n)$ with Rademacher's infinite series expansion (2.4). The sum above is a finite sum of algebraic numbers with bounded denominators. Bruinier, Sutherland, and the author have obtained Chinese Remainder Theorem based algorithms to compute class polynomials such as $H_n^{\mathrm{part}}(x)$ in [80].

Example. Using Algorithm 3 of [80], we have computed $H_n^{\mathrm{part}}(x)$ for $n \leq 750$. Some small examples are listed below.

n	$(24n-1)p(n)$	$H_n^{\text{part}}(x)$
1	23	$x^3 - 23x^2 + \frac{3592}{23}x - 419$
2	94	$x^5 - 94x^4 + \frac{169659}{47}x^3 - 65838x^2 + \frac{1092873176}{47^2}x + \frac{1454023}{47}$
3	213	$x^7 - 213x^6 + \frac{1312544}{71}x^5 - 723721x^4 + \frac{44648582886}{71^2}x^3 + \frac{9188934683}{71}x^2 + \frac{166629520876208}{71^3}x + \frac{2791651635293}{71^2}$
4	475	$x^8 - 475x^7 + \frac{9032603}{95}x^6 - 9455070x^5 + \frac{3949512899743}{95^2}x^4 - \frac{97215753021}{19}x^3 + \frac{9776785708507683}{95^3}x^2 - \frac{53144327916296}{19^2}x - \frac{134884469547631}{5^4 \cdot 19}$.

Sketch of the proof of Theorem 2.1. The function $\eta(z)^{-1}$ can be viewed as a component of a vector valued modular form, and direct calculations shows that Theorem 10.6 applies.

The main issue is that the singular moduli $(24n-1)P(\alpha_Q)$ are algebraic integers. Using the classical theory of complex multiplication, Bruinier and the author proved in [78] that the only denominators in these algebraic numbers could be a divisor of 6. Larson and Rolen [170] recently proved that there are no such contributions. □

11. Borcherds Products

Here we describe work of the author and Bruinier [77] on Borcherds products. To describe the context of this work, we first recall a famous result of Borcherds on infinite product expansions of modular forms possessing a Heegner divisor.

11.1. **The "classical" Borcherds products.** We begin by recalling that $\Delta(z)$, the unique normalized weight 12 cusp form with respect to $\mathrm{SL}_2(\mathbb{Z})$, has a Fourier expansion given by the infinite product

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{11.1}$$

Most modular forms do not possess simple infinite product expansions. It was widely believed that such elegant product expansions only belong to the domain of those modular forms whose divisors are supported at cusps, such as the modular units (for example, see [165]).

Then in the 1990s, Borcherds [48, 49, 50] provided a striking description for the exponents in the naive infinite product expansions of a much larger class of modular forms, those modular forms possessing a *Heegner divisor*. For example, let $c(n)$ denote the integer exponents

one obtains by expressing $E_4(z)$ as an infinite product:

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n = (1-q)^{-240}(1-q^2)^{26760}\cdots = \prod_{n=1}^{\infty}(1-q^n)^{c(n)}. \tag{11.2}$$

Borcherds's theory implies that there is a weight $1/2$ meromorphic modular form

$$G(z) = \sum_{n\geq -3} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 + \cdots - 4096240q^9 + \dots$$

with the property that $c(n) = b(n^2)$ for every positive integer n. Notice that the product in (11.1) also satisfies this phenomenon, where the product exponents are the coefficients of the weight $1/2$ modular form

$$12\Theta_0(z) = 12\sum_{n\in\mathbb{Z}} q^{n^2} = 12 + 24q + 24q^4 + 24q^9 + \cdots .$$

These two examples illustrate his general result that a modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a Heegner divisor has an infinite product expansion where the exponents are coefficients of certain weight $1/2$ weakly holomorphic modular forms.

Remark. The work of Borcherds [48, 49, 50] is given in the more general context of automorphic forms on orthogonal groups.

Remark. It is possible to derive explicit formulas for the exponents in the infinite product expansions of generic modular forms, not just those with a Heegner divisor (for example, see [76]). However, these formulas are in general quite complicated.

We now briefly recall the most classical case of this work of Borcherds. We shall refer to a complex number $\tau \in \mathbb{H}$ of the form

$$\tau = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

with $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and $b^2 - 4ac < 0$ as a *CM point*, and we denote its discriminant by the integer $d_\tau := b^2 - 4ac$. A meromorphic modular form $f(z)$ on $\mathrm{SL}_2(\mathbb{Z})$ is said to have a *Heegner divisor* if its zeros and poles are supported at the cusp at infinity and at CM points.

To state Borcherds's results, we require a special sequence of weight $1/2$ modular forms in $\mathcal{M}^!_{\frac{1}{2}}(4)$. These forms will be distinguished by their q-expansions. For each nonnegative integer $d \equiv 0, 3 \pmod 4$ let $f_d(z) \in \mathcal{M}^!_{\frac{1}{2}}(4)$ be the unique modular form with a Fourier expansion of the form

$$f_d(z) = q^{-d} + \sum_{D>0} A(D, d)q^D.$$

That these forms are well defined follows from Lemma 14.2 of [49]. Moreover, they form a basis of $\mathcal{M}^!_{\frac{1}{2}}(4)$.

The form $f_0(z)$ is the classical Jacobi theta-function

$$f_0(z) = \Theta_0(z) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots , \tag{11.3}$$

and the form $f_3(z)$ is given by the expression

$$\begin{aligned} f_3(z) &= \frac{D(\Theta_0(z))E_{10}(4z)}{2\Delta(4z)} - \frac{\Theta_0(z)\,(D(E_{10}(z))|V(4))}{10\Delta(4z)} - \frac{152}{5}\Theta_0(z) \\ &= q^{-3} - 248q + 26752q^4 - 85995q^5 + \cdots, \end{aligned} \tag{11.4}$$

where $D := q \cdot \frac{d}{dq}$.

Remark. It is straightforward to inductively compute the $f_d(z)$ using $f_0(z)$ and $f_3(z)$. To compute $f_d(z)$ for $d \geq 4$, one computes $f_{d-4}(z)j(4z)$, and then iteratively subtracts multiples of those $f_j(z)$ with $0 \leq j \leq d-4$.

For completeness, we include the initial terms of the first few $f_d(z)$ below.

$$\begin{aligned} f_0(z) &= 1 + 2q + 2q^4 + 2q^9 + \cdots, \\ f_3(z) &= q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 - 4096248q^9 + \cdots, \\ f_4(z) &= q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + 51180012q^9 + \cdots, \\ f_7(z) &= q^{-7} - 4119q + 8288256q^4 - 52756480q^5 + \cdots, \\ f_8(z) &= q^{-8} + 7256q + 26124256q^4 + 190356480q^5 + \cdots. \end{aligned} \tag{11.5}$$

Using these forms, we may now state Borcherds's famous theorem on the infinite product expansion of those meromorphic forms on $\mathrm{SL}_2(\mathbb{Z})$ possessing a Heegner divisor. Let $\mathcal{M}_H$ denote the set of integer weight meromorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with a Heegner divisor, integer coefficients, and leading coefficient 1. Obviously, $\mathcal{M}_H$ is closed under multiplication.

If $H(-n)$ denotes the usual Hurwitz class number of discriminant $-n$, then define $\widetilde{H}(z)$ by

$$\begin{aligned} \widetilde{H}(z) :&= -\frac{1}{12} + \sum_{1<n\equiv 0,3 \pmod 4} H(-n)q^n \\ &= -\frac{1}{12} + q^3/3 + q^4/2 + q^7 + q^8 + q^{11} + 4q^{12}/3 + \cdots. \end{aligned} \tag{11.6}$$

If $f(z) = \sum_{n \geq n_0} A(n)q^n \in \mathcal{M}^!_{\frac{1}{2}}(4)$, then define $\Psi(f(z))$ by

$$\Psi(f(z)) := q^{-h}\prod_{n=1}^{\infty}(1-q^n)^{A(n^2)}, \tag{11.7}$$

where h is the constant term of $f(z)\widetilde{H}(z)$. Borcherds [48] proved the following theorem.

Theorem 11.1. *The map Ψ satisfies*

$$\Psi : \mathcal{M}^!_{\frac{1}{2}}(4) \longrightarrow \mathcal{M}_H.$$

Furthermore, it is an isomorphism. The weight of $\Psi(f)$ is $A(0)$, and the multiplicity of the zero of $\Psi(f)$ at a CM point of discriminant $D < 0$ is

$$\sum_{n>0} A(Dn^2).$$

Remark. The isomorphism Ψ respects the additive structure of $\mathcal{M}^!_{\frac{1}{2}}(4)$ and the multiplicative structure of $\mathcal{M}_H$.

Example. If $f(z) = 12f_0(z) = 12\Theta_0(z)$, then

$$f(z) = \sum_{n=0}^{\infty} A(n)q^n = 12 + 24q + 24q^4 + 24q^9 + 24q^{16} + \cdots .$$

Therefore, it follows that

$$\Psi(f(z)) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Obviously, $\Psi(f(z))$ is the classical Δ-function. Its weight is $A(0) = 12$, and its divisor is supported at the cusp at infinity.

Example. If $f(z) = 3f_3(z)$, then

$$f(z) = \sum_{n=-3}^{\infty} A(n)q^n = 3q^{-3} - 744q + 80256q^4 - 257985q^5 + 5121792q^8 - 12288744q^9 + \cdots .$$

By Theorem 11.1, it follows that $\Psi(f(z))$ is a weight $A(0) = 0$ meromorphic modular form whose divisor is supported at the cusp at infinity and a triple zero at $\omega = e^{2\pi i/3}$. Therefore, it follows that $\Psi(f(z))$ must be $j(z)$, and this is confirmed by

$$\begin{aligned}\Psi(f(z)) &= q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744}\cdots \\ &= q^{-1} + 744 + 196884q + 21493760q^3 + \cdots .\end{aligned}$$

Example. If $f(z) = 4f_0(z) + f_3(z)$, then

$$f(z) = \sum_{n=-3}^{\infty} A(n)q^n = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + \cdots - 4096240q^9 + \cdots .$$

Theorem 11.1 implies that $\Psi(f(z))$ is a holomorphic modular form of weight 4 with leading coefficient 1. Therefore, it must be that $\Psi(f(z)) = E_4(z)$. This explains (11.2).

11.2. **Harmonic Maass forms and Generalized Borcherds products.** Motivated by questions related to derivatives of modular L-functions, the author and Bruinier [77] derived Borcherds-type products using harmonic Maass forms. These results are quite technical, and a thorough treatment would occupy more space than is warranted in this expository paper. For brevity, we will be content with a brief indication of the flavor of these results, followed by two illustrative examples.

These results are phrased in terms of vector-valued weight $1/2$ harmonic Maass forms. Let (V, Q) be a non-degenerate rational quadratic space of signature (b^+, b^-), and let $L \subset V$ be an even lattice with dual L'. The discriminant group L'/L, together with the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by Q, is called the *discriminant form* of the lattice L.

As usual, we let $\mathrm{Mp}_2(\mathbb{R})$ denote the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$. The elements of this group are pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The group law is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the integral metaplectic group, the inverse image of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ under the covering map, by $\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$. It is well known that $\widetilde{\Gamma}$ is generated by $T := (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$, and $S := (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$. One has the relations $S^2 = (ST)^3 = Z$, where $Z := \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\widetilde{\Gamma}$. We let $\widetilde{\Gamma}_\infty := \langle T \rangle \subset \widetilde{\Gamma}$.

We now recall the Weil representation associated with the discriminant form L'/L (for example, see [49], [71]). It is a representation of $\widetilde{\Gamma}$ on the group algebra $\mathbb{C}[L'/L]$. We denote the standard basis elements of $\mathbb{C}[L'/L]$ by $\mathfrak{e}_h$, $h \in L'/L$, and write $\langle \cdot, \cdot \rangle$ for the standard scalar product (antilinear in the second entry) such that $\langle \mathfrak{e}_h, \mathfrak{e}_{h'} \rangle = \delta_{h,h'}$. The *Weil representation* ρ_L associated with the discriminant form L'/L is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)(\mathfrak{e}_h) := e(h^2/2)\mathfrak{e}_h, \tag{11.8}$$

$$\rho_L(S)(\mathfrak{e}_h) := \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h, h'))\mathfrak{e}_{h'}. \tag{11.9}$$

If $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is a function, we write $f = \sum_{\lambda \in L'/L} f_h \mathfrak{e}_h$ for its decomposition in components with respect to the standard basis of $\mathbb{C}[L'/L]$. Let $k \in \frac{1}{2}\mathbb{Z}$, and let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms of weight k and type ρ_L for the group $\widetilde{\Gamma}$.

Now assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *vector-valued harmonic Maass form* of weight k with respect to $\widetilde{\Gamma}$ and ρ_L if it satisfies:

(i) $f\,(M\tau) = \phi(\tau)^{2k}\rho_L(M, \phi)f(\tau)$ for all $(M, \phi) \in \widetilde{\Gamma}$;
(ii) $\Delta_k f = 0$;
(iii) We have that

$$f(\tau) = P_f(\tau) + O(e^{-\varepsilon v}),$$

as $v \to \infty$, for some Fourier polynomial

$$P_f(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(h) \\ -\infty \ll n \leq 0}} c^+(n, h)e(n\tau)\mathfrak{e}_h$$

and some $\varepsilon > 0$.

We write H_{k,ρ_L} for the vector space of such harmonic Maass forms.

Generalizing Lemma 4.2, where the modular variable is now $\tau = u + iv$, we have that any $f \in H_{k,\rho_L}$ decomposes as $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n, h)e(n\tau)\mathfrak{e}_h, \tag{11.10a}$$

$$f^-(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c^-(n, h)W(2\pi nv)e(n\tau)\mathfrak{e}_h, \tag{11.10b}$$

and $W(x) = W_k(x) := \int_{-2x}^{\infty} e^{-t}t^{-k}\,dt = \Gamma(1 - k, 2|x|)$ for $x < 0$.

In [77], the theory of theta liftings is applied to such harmonic Maass forms to obtain Borcherds products with twisted Heegner divisors on $X_0(N)$. To this end, let N be a positive integer. We consider the rational quadratic space

$$V := \{X \in \mathrm{Mat}_2(\mathbb{Q}) : \ \mathrm{tr}(X) = 0\} \tag{11.11}$$

with the quadratic form $Q(X) := -N \det(X)$. We let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : \quad a, b, c \in \mathbb{Z} \right\}. \tag{11.12}$$

Then the dual lattice is given by

$$L' := \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} : \quad a, b, c \in \mathbb{Z} \right\}. \tag{11.13}$$

We identify L'/L with $\mathbb{Z}/2N\mathbb{Z}$. Here the quadratic form on L'/L is identified with the quadratic form $x \mapsto x^2$ on $\mathbb{Z}/2N\mathbb{Z}$.

If $D \in \mathbb{Z}$, let L_D be the set of vectors $X \in L'$ with $Q(X) = D/4N$. Notice that L_D is empty unless D is a square modulo $4N$. For $r \in L'/L$ with $r^2 \equiv D \pmod{4N}$ we define

$$L_{D,r} := \{X \in L' : \quad Q(X) = D/4N \text{ and } X \equiv r \pmod{L}\}.$$

If $X = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix} \right) \in L_{D,r}$, then the matrix

$$\psi(X) := \begin{pmatrix} a & b/2 \\ b/2 & Nc \end{pmatrix} = X \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix} \tag{11.14}$$

defines an integral binary quadratic form of discriminant $D = b^2 - 4Nac = 4NQ(X)$ with $b \equiv r \pmod{2N}$. We have that $\Gamma_0(N)$ acts on $L_{D,r}$, and the number of orbits of $L_{D,r}$ is finite if $D \neq 0$.

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. Following [133], we define a generalized genus character for $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix} \right) \in L'$ by putting

$$\chi_\Delta(\lambda) = \chi_\Delta([a, b, Nc]) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta \mid b^2 - 4Nac \text{ and } (b^2 - 4Nac)/\Delta \text{ is a} \\ & \text{square modulo } 4N \text{ and } \gcd(a, b, c, \Delta) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here $[a, b, Nc]$ is the integral binary quadratic form corresponding to λ, and n is any integer prime to Δ represented by one of the quadratic forms $[N_1 a, b, N_2 c]$ with $N_1 N_2 = N$ and $N_1, N_2 > 0$.

The cusps of $\Gamma_0(N)$ correspond to $\Gamma_0(N)$-classes of primitive isotropic vectors in L. Here we let $\ell, \ell' \in L$ be the isotropic vectors

$$\ell = \begin{pmatrix} 0 & 1/N \\ 0 & 0 \end{pmatrix}, \qquad \ell' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The 1-dimensional lattice

$$K = L \cap \ell'^{\perp} \cap \ell^{\perp}$$

is positive definite. We find that L splits into

$$L = K \oplus \mathbb{Z}\ell' \oplus \mathbb{Z}\ell, \tag{11.15}$$

and $K'/K \cong L'/L$. Then we have $K = \mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. For $\lambda \in K \otimes \mathbb{R}$, we write $\lambda > 0$ if λ is a positive multiple of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

We now define twisted Heegner divisors on the modular curve $X_0(N)$. Let Δ be a fundamental discriminant and let $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. For any vector $\lambda \in L'$ of negative norm, the orthogonal complement $\lambda^\perp \subset V(\mathbb{R})$ defines a point $H(\lambda)$ in $\mathrm{Gr}(V) \cong \mathbb{H}$. For $h \in L'/L$ and a negative rational number $m \in \mathbb{Z} + \mathrm{sgn}(\Delta)Q(h)$, we consider the twisted Heegner divisor

$$H_{\Delta,r}(m,h) := \sum_{\lambda \in L_{d\Delta,hr}/\Gamma_0(N)} \frac{\chi_\Delta(\lambda)}{w(\lambda)} H(\lambda) \in \mathrm{Div}(X_0(N))_{\mathbb{Q}}, \tag{11.16}$$

where $d := 4Nm\,\mathrm{sgn}(\Delta) \in \mathbb{Z}$. Note that d is a discriminant which is congruent to a square modulo $4N$ and which has the opposite sign as Δ. Here $w(\lambda)$ is the order of the stabilizer of λ in $\Gamma_0(N)$. We also require the degree zero divisor

$$y_{\Delta,r}(m,h) := H_{\Delta,r}(m,h) - \deg(H_{\Delta,r}(m,h)) \cdot \infty. \tag{11.17}$$

We have $y_{\Delta,r}(f) = H_{\Delta,r}(f)$ when $\Delta \neq 1$. By the theory of complex multiplication, the divisor $H_{\Delta,r}(m,h)$ is defined over $\mathbb{Q}(\sqrt{\Delta})$.

Recall that $\tilde{\rho}_L = \rho_L$ for $\Delta > 0$, and $\tilde{\rho}_L = \bar{\rho}_L$ for $\Delta < 0$. Suppose that $f \in H_{1/2,\tilde{\rho}_L}$ is a harmonic weak Maass form of weight $1/2$ with representation $\tilde{\rho}_L$. We denote the coefficients of $f = f^+ + f^-$ by $c^\pm(m,h)$ as in (11.10). Using the Fourier coefficients of the principal part of f, we finally define the twisted Heegner divisor associated to f by

$$H_{\Delta,r}(f) := \sum_{h \in L'/L} \sum_{m<0} c^+(m,h) H_{\Delta,r}(m,h) \in \mathrm{Div}(X_0(N))_{\mathbb{R}}, \tag{11.18}$$

$$y_{\Delta,r}(f) := \sum_{h \in L'/L} \sum_{m<0} c^+(m,h) y_{\Delta,r}(m,h) \in \mathrm{Div}(X_0(N))_{\mathbb{R}}. \tag{11.19}$$

Notice that $y_{\Delta,r}(f) = H_{\Delta,r}(f)$ when $\Delta \neq 1$.

Theorem 11.2. *Assume the notation and hypotheses above. Let $f \in H_{1/2,\tilde{\rho}_L}$ be a harmonic Maass form with real coefficients $c^+(m,h)$ for all $m \in \mathbb{Q}$ and $h \in L'/L$. Moreover, assume that $c^+(n,h) \in \mathbb{Z}$ for all $n \leq 0$. Then infinite product*

$$\Psi_{\Delta,r}(z,f) = e((\rho_{f,\ell},z)) \prod_{\substack{\lambda \in K' \\ \lambda > 0}} \prod_{b\,(\Delta)} \left[1 - e((\lambda,z) + b/\Delta)\right]^{\left(\frac{\Delta}{b}\right) c^+(|\Delta|\lambda^2/2, r\lambda)},$$

where $\rho_{f,\ell}$ is a certain Weyl vector (see (4.8) of [77]), converges for y sufficiently large and has a meromorphic continuation to all of $\mathbb{H}$ with the following properties.

(i) *It is a meromorphic modular form for $\Gamma_0(N)$ with a unitary character σ which may have infinite order.*
(ii) *The weight of $\Psi_{\Delta,r}(z,f)$ is $c^+(0,0)$ when $\Delta = 1$, and is 0 when $\Delta \neq 1$.*
(iii) *The divisor of $\Psi_{\Delta,r}(z,f)$ on $X_0(N)$ is given by $H_{\Delta,r}(f) + C_{\Delta,r}(f)$, where $C_{\Delta,r}(f)$ is supported at cusps.*

(iv) *The "regularized theta integral" of f defined in (5.5) of* [77]*, satisfies*

$$\Phi_{\Delta,r}(z,f)=\begin{cases}-c^+(0,0)(\log(4\pi N)+\Gamma'(1))-4\log|\Psi_{\Delta,r}(z,f)y^{c^+(0,0)/2}|, & \text{if } \Delta=1,\\ 2\sqrt{\Delta}c(0,0)L(1,\chi_\Delta)-4\log|\Psi_{\Delta,r}(z,f)|, & \text{if } \Delta\neq 1.\end{cases}$$

Unlike the results in [49], such generalized Borcherds products typically transform with a multiplier system of infinite order under $\Gamma_0(N)$. The following criterion is obtained which determines when the multiplier system has finite order.

Theorem 11.3. *Suppose that $\Delta\neq 1$. Let $f\in H_{1/2,\tilde{\rho}_L}$ be a harmonic Maass form with real coefficients $c^+(m,h)$ for all $m\in\mathbb{Q}$ and $h\in L'/L$. Moreover, assume that $c^+(n,h)\in\mathbb{Z}$ for all $n\leq 0$. The following are equivalent.*

(i) *The character σ of the function $\Psi_{\Delta,r}(z,f)$ defined in Theorem 11.2 is of finite order.*
(ii) *The coefficients $c^+(|\Delta|\lambda^2/2,r\lambda)$ are rational for all $\lambda\in K'$.*

Remark. Theorem 11.3 is particularly simple to implement for weakly holomorphic modular forms f (for example, see Lemma 6.5 of [77]). In this case the rationality of all the coefficients is implied by the rationality of the principal part, and so Theorem 11.3 implies that the σ for the corresponding generalized Borcherds product has finite order.

Example (Twisted modular polynomials)**.** Here we use Theorems 11.2 and 11.3 to deduce the infinite product expansion of twisted modular polynomials found by Zagier (see Section 7 of [245]).

The weight $1/2$ harmonic Maass forms in question are the same weakly holomorphc modular forms $f_d(z)$ which appear in (11.5). Theorems 11.2 and 11.3 then give a meromorphic modular form $\Psi_\Delta(z,f_d):=\Psi_{\Delta,r}(z,f_d)$ of weight 0 for the group $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ whose divisor on $X(1)$ is given by

$$H_\Delta(d):=H_{\Delta,r}(d/4,d/2)=\sum_{\lambda\in L_{\Delta d}/\Gamma}\frac{\chi_\Delta(\lambda)}{w(\lambda)}\cdot H(\lambda).$$

Classically, the quotient $L_{\Delta d}/\Gamma$ corresponds to the Γ-classes of integral binary quadratic forms of discriminant Δd. Moreover, for sufficiently large $\mathrm{Im}(z)$, we have the product expansion

$$\Psi_\Delta(z,f_d)=\prod_{n=1}^{\infty}\prod_{b\ (\Delta)}[1-e(nz+b/\Delta)]^{\left(\frac{\Delta}{b}\right)c_d(\Delta n^2)}. \tag{11.20}$$

From these properties it follows that

$$\Psi_\Delta(z,f_d)=\prod_{\lambda\in L_{\Delta d}/\Gamma}\big(j(z)-j(H(\lambda))\big)^{\chi_\Delta(\lambda)}. \tag{11.21}$$

As an example, let $\Delta:=5$ and $d:=-3$. There are two classes of binary quadratic forms of discriminant -15, represented by $[1,1,4]$ and $[2,1,2]$, and their corresponding CM points are $\frac{-1+\sqrt{-15}}{2}$ and $\frac{-1+\sqrt{-15}}{4}$. It is well known that the singular moduli of $j(\tau)$ of these points are $-\frac{191025}{2}-\frac{85995}{2}\sqrt{5}$, and $-\frac{191025}{2}+\frac{85995}{2}\sqrt{5}$. The function f_{-3} has the Fourier expansion

$$f_{-3}=q^{-3}-248\,q+26752\,q^4-85995\,q^5+1707264\,q^8-4096248\,q^9+\dots.$$

Multiplying out the product over b in (11.20), we obtain the infinite product expansion

$$\Psi_5(z, f_{-3}) = \frac{j(z) + \frac{191025}{2} + \frac{85995}{2}\sqrt{5}}{j(z) + \frac{191025}{2} - \frac{85995}{2}\sqrt{5}} = \prod_{n=1}^{\infty} \left(\frac{1 + \frac{1-\sqrt{5}}{2} q^n + q^{2n}}{1 + \frac{1+\sqrt{5}}{2} q^n + q^{2n}} \right)^{c_{-3}(5n^2)}.$$

Example (A generalized Borcherds product for Ramanujan's $\omega(q)$)**.** Here we give an example of a generalized Borcherds product arising from Ramanujan's mock theta function

(11.22)
$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} = \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \cdots.$$

Zwegers's work (see Theorem 3.1) completes this q-series in a way which allows him to produce a vector valued harmonic Maass form.

Define integers $a(n)$ by

$$-2q^{1/3}\big(\omega(q^{1/2}) + \omega(-q^{1/2})\big) =: \sum_{n \in \mathbb{Z}+1/3} a(n) q^n = -4\,q^{1/3} - 12\,q^{4/3} - 24\,q^{7/3} - 40\,q^{10/3} - \ldots.$$

If we let

$$P(X) := \frac{1 + \sqrt{-2}X - X^2}{1 - \sqrt{-2}X - X^2},$$

then Theorems 11.2 and 11.3 imply that

$$\Psi(z) = \prod_{n=1}^{\infty} P\left(q^n\right)^{\left(\frac{n}{3}\right) a(n^2/3)} \tag{11.23}$$

is a modular function on $\Gamma_0(6)$.

Here we work out an expression for $\Psi(z)$. We use the Hauptmodul for $\Gamma_0^*(6)$, the extension of $\Gamma_0(6)$ by all Atkin-Lehner involutions, which is

$$j_6^*(z) = \left(\frac{\eta(z)\eta(2z)}{\eta(3z)\eta(6z)} \right)^4 + 4 + 3^4 \left(\frac{\eta(3z)\eta(6z)}{\eta(z)\eta(2z)} \right)^4 = q^{-1} + 79\,q + 352\,q^2 + 1431\,q^3 + \ldots.$$

Let α_1 and α_2 be the Heegner points

$$\alpha_1 := \frac{-2 + \sqrt{-2}}{6} \quad \text{and} \quad \alpha_2 := \frac{2 + \sqrt{-2}}{6}.$$

We have $j_6^*(\alpha_1) = j_6^*(\alpha_2) = -10$. Hence $j_6^*(z) + 10$ is a rational function on $X_0(6)$ whose divisor consists of the 4 cusps with multiplicity -1 and the points α_1, α_2 with multiplicity 2. The unique normalized cusp form of weight 4 for $\Gamma_0^*(6)$ is

$$\delta(z) := \eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2 = q - 2\,q^2 - 3\,q^3 + 4\,q^4 + 6\,q^5 + 6\,q^6 - 16\,q^7 - 8\,q^8 + \ldots.$$

Using these functions, we find that

$$\phi(z) := \Psi(z) \cdot (j_6^*(z) + 10)\delta(z)$$

is a holomorphic modular form of weight 4 for $\Gamma_0(6)$ with divisor $4(\alpha_1)$. Using the classical Eisenstein series, it turns out that

$$\begin{aligned}450\phi(z) = (3360 - 1920\sqrt{-2})\delta(z) + (1 - 7\sqrt{-2})E_4(z) + (4 - 28\sqrt{-2})E_4(2z)\\ + (89 + 7\sqrt{-2})E_4(3z) + (356 + 28\sqrt{-2})E_4(6z).\end{aligned}$$

Putting this all together, (11.23) becomes

$$\begin{aligned}\prod_{n=1}^{\infty}\left(\frac{1+\sqrt{-2}q^n - q^{2n}}{1-\sqrt{-2}q^n - q^{2n}}\right)^{\left(\frac{n}{3}\right)a(n^2/3)} &= \frac{\phi(z)}{(j_6^*(z)+10)\delta(z)}\\ = 1 - 8\sqrt{-2}q - (64 - 24\sqrt{-2})q^2 + (384 + 168\sqrt{-2})q^3 + (64 - 1768\sqrt{-2})q^4 + \cdots .\end{aligned}$$

12. Derivatives and values of modular L-functions

Harmonic Maass forms are very useful for investigating the central values and derivatives of weight 2 modular L-functions. Recent works by the author, Bruinier, and Yang involve the interplay between generalized Borcherds products, harmonic Maass forms, and earlier celebrated theorems of Gross and Zagier, Kohnen and Zagier, and Waldspurger.

To explain these results, we first recall some standard definitions. Let

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}^{\text{new}}(M)$$

be a newform of even integral weight $2k$ on $\Gamma_0(M)$. For $\text{Re}(s) \gg 0$, let

$$L(f,s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s} \tag{12.1}$$

be its L-function. Let D denote the fundamental discriminant of a quadratic field, and let $\chi_D = \left(\frac{D}{\bullet}\right)$ denote the Kronecker character for the field $\mathbb{Q}(\sqrt{D})$. The *D-quadratic twist of* $f(z)$, denoted $f_D(z)$, is given by

$$f_D(z) = \sum_{n=1}^{\infty}\chi_D(n)a(n)q^n,$$

and for $\text{Re}(s) \gg 0$ its L-function is given by

$$L(f,\chi_D,s) = \sum_{n=1}^{\infty}\frac{\chi_D(n)a(n)}{n^s}.$$

These L-functions have analytic continuations to $\mathbb{C}$ and satisfy well known functional equations. If we define $\Lambda(f,s)$ by

$$\Lambda(f,s) := (2\pi)^{-s}\Gamma(s)M^{s/2}L(f,s),$$

then there is an $\epsilon \in \{\pm 1\}$ for which

$$\Lambda(f,s) = \epsilon \cdot \Lambda(f, 2k-s).$$

Furthermore, if D is a fundamental discriminant which is coprime to M, then

$$\Lambda(f_D, s) = \epsilon \cdot \chi_D(-M)\Lambda(f_D, 2k - s).$$

Remark. The number ϵ above is referred to as the *sign of the functional equation* of $L(f, s)$. If $f(z) \in S_{2k}^{\text{new}}(M)$ is a newform, then

$$\epsilon = (-1)^k \lambda_M,$$

where λ_M is the eigenvalue of $f(z)$ under the Fricke involution $W(M)$. In other words, we have

$$(f \mid_{2k} W(M))(z) = \lambda_M f(z).$$

The values $L(f, \chi_D, k)$ are the *central critical values* of the L-functions $L(f, \chi_D, s)$.

Our motivating problem is to describe the behavior of the values $L(f, \chi_D, k)$, as D varies. Notice that if $\chi_D(-M)\epsilon = -1$, then $L(f, \chi_D, k) = 0$. Therefore at least half of these $L(f, \chi_D, k)$ are trivially zero. As we shall see, the "nontrivial zeros" are much more mysterious.

Much of the interest in central critical values of modular L-functions follows from their connection to the Birch and Swinnerton-Dyer. Here we briefly recall important facts and results.

Suppose that $E/\mathbb{Q}$ is an elliptic curve of conductor $N(E)$ (see [225] for background on elliptic curves), and let, for $\text{Re}(s) \gg 0$,

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \tag{12.2}$$

be its Hasse-Weil L-function. In particular, if $p \nmid N(E)$ is prime, then we have

$$N_E(p) = p + 1 - a_E(p),$$

where $N_E(p)$ denotes the number of points on the reduction of E modulo p.

By the work of Breuil, Conrad, Diamond, Taylor, and Wiles [52, 96, 230, 242], we have the following deep result which confirmed the *Shimura-Taniyama Conjecture*.

Theorem 12.1. *If $E/\mathbb{Q}$ is an elliptic curve of conductor $N(E)$, then there is a newform $f_E(z) \in S_2^{\text{new}}(N(E))$ for which*

$$L(E, s) = L(f_E, s).$$

For elliptic curves $E/\mathbb{Q}$, Theorem 12.1 implies that $L(E, s)$ has an analytic continuation to $\mathbb{C}$, and so the analytic behavior of $L(E, s)$ at $s = 1$ is well defined. The Birch and Swinnerton-Dyer Conjecture (see Appendix C of [225]) gives arithmetic significance to this behavior.

Conjecture. *If $E/\mathbb{Q}$ is an elliptic curve, and if $\text{rk}(E)$ is the Mordell-Weil rank of E over $\mathbb{Q}$, then*

$$\text{ord}_{s=1}(L(E, s)) = \text{rk}(E).$$

Remark. This is the "weak" form of the Birch and Swinnerton-Dyer Conjecture. The strong conjecture gives an arithmetic formula for

$$\lim_{s\to 1}\left((s-1)^{-\mathrm{rk}(E)}\cdot L(E,s)\right)$$

in terms of arithmetic invariants associated to E.

Although this conjecture remains open, we have the following strong result which follows from the work of Kolyvagin, Gross, and Zagier [134, 164][16].

Theorem 12.2. *If $E/\mathbb{Q}$ is an elliptic curve for which $\mathrm{ord}_{s=1}(L(E,s))\in\{0,1\}$, then*

$$\mathrm{ord}_{s=1}(L(E,s))=\mathrm{rk}(E).$$

Now we consider quadratic twists of elliptic curves and modular forms. Let $E/\mathbb{Q}$ be an elliptic curve given by the Weierstrass equation

$$E:\quad y^2=x^3+ax^2+bx+c, \tag{12.3}$$

where a,b and c are integers. If D is a square-free integer, then let $E(D)$ denote the *$D-$quadratic twist of E* given by

$$E(D):\quad y^2=x^3+aDx^2+bD^2x+cD^3. \tag{12.4}$$

The curves E and $E(D)$ are isomorphic over $\mathbb{Q}(\sqrt{D})$.

Suppose that $E/\mathbb{Q}$ is an elliptic curve, and that

$$f_E(z)=\sum_{n=1}^{\infty}a_E(n)q^n\in S_2^{\mathrm{new}}(N(E))$$

is the weight two newform associated to E by Theorem 12.1. Similarly, if D is a fundamental discriminant, then let

$$f_{E(D)}(z)=\sum_{n=1}^{\infty}a_{E(D)}(n)q^n$$

be the newform associated to $E(D)$. If $p\nmid DN(E)$ is prime, then it is simple to check that

$$a_{E(D)}(p)=\left(\frac{D}{p}\right)a_E(p).$$

Consequently, the newform $f_{E(D)}(z)$ is the unique newform whose Hecke eigenvalues, for primes $p\nmid DN(E)$, equal those of the quadratic twist f_E by χ_D.

In view of the Birch and Swinnerton-Dyer Conjecture, and Theorem 12.2, we are compelled to study central values and derivatives of weight 2 modular L-functions. In this direction we have the celebrated works of Kohnen, Zagier and Waldspurger, and also the work of Gross and Zagier [134].

It turns out that the Fourier coefficients of half-integral weight cusp forms often interpolate the "square-roots" of the central critical values of the L-functions of the quadratic twists of even weight newforms.

[16]Coates and Wiles [94] earlier proved that $L(E,1)\neq 0\Longrightarrow \mathrm{rk}(E)=0$ for elliptic curves $E/\mathbb{Q}$ with complex multiplication.

Here we make this precise. First suppose that N is odd and square-free, and suppose further that

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S^{\text{new}}_{k+\frac{1}{2}}(4N) \tag{12.5}$$

is a Kohnen newform[17]. There is a unique newform, say $f(z) \in S^{\text{new}}_{2k}(N)$, associated to $g(z)$ under Shimura's correspondence. The coefficients of $g(z)$ determine the central critical values of many of the quadratic twists $L(f, \chi_D, s)$. Let $\nu(N)$ denote the number of distinct prime divisors of N, and let $\langle f, f\rangle$ (resp. $\langle g, g\rangle$) denote the Petersson inner product on $S_{2k}(N)$ (resp. $S_{k+\frac{1}{2}}(4N)$). Generalizing a result in [163], Kohnen proved the following important theorem in [162].

Theorem 12.3. *Assume the notation in the preceding discussion. If $\ell \mid N$ is prime, then let $\lambda_\ell \in \{\pm 1\}$ be the eigenvalue of the Atkin-Lehner involution*

$$(f \mid_{2k} W(Q_\ell))(z) = \lambda_\ell f(z).$$

If $(-1)^k D > 0$ and D has the property that $\left(\frac{D}{\ell}\right) = \lambda_\ell$ for each prime $\ell \mid N$, then

$$L(f, \chi_D, k) = \frac{\langle f, f\rangle \cdot \pi^k}{2^{\nu(N)}(k-1)!|D|^{k-\frac{1}{2}}\langle g, g\rangle} \cdot |b(|D|)|^2.$$

For all other fundamental discriminants D with $(-1)^k D > 0$ we have $b(|D|) = 0$.

Example. If $f(z) = \Delta(z) \in S_{12}$, then it turns out that the associated Kohnen newform in $S^{\text{new}}_{13/2}(4)$ is

$$\begin{aligned} g(z) = \sum_{n=1}^{\infty} b(n)q^n &= \frac{E_4(4z) \cdot D(\Theta_0(z))}{2} - \frac{D(E_4(4z)) \cdot \Theta_0(z)}{16} \\ &= q - 56q^4 + 120q^5 - \dots. \end{aligned}$$

Again, here D denotes the differential operator $D := q\frac{d}{dq}$. For positive fundamental discriminants D, we then have that

$$L(\Delta, \chi_D, 6) = \frac{\langle \Delta, \Delta\rangle \cdot \pi^6}{120 D^{\frac{11}{2}}\langle g, g\rangle} \cdot b(D)^2.$$

Waldspurger proved a similar theorem which holds in greater generality. For every fundamental discriminant D, define D_0 by

$$D_0 := \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases} \tag{12.6}$$

The following is a convenient reformulation of Waldspurger's theorem [238].

[17]These are the half-integral weight cuspidal Hecke eigenforms in Kohnen's paper [162].

Theorem 12.4. *If $f(z)=\sum_{n=1}^{\infty}a(n)q^n\in S_{2k}^{\text{new}}(M)$ is an even weight newform and $\delta\in\{\pm 1\}$ is the sign of the functional equation of $L(f,s)$, then there is a positive integer N with $M\mid N$, a Dirichlet character χ modulo $4N$, a nonzero complex number Ω_f, and a nonzero half-integral weight Hecke eigenform*

$$g_f(z)=\sum_{n=1}^{\infty}b_f(n)q^n\in S_{k+\frac{1}{2}}(4N,\chi)$$

with the property that there are arithmetic progressions of fundamental discriminants D coprime to $4N$ for which $\delta D>0$ and

$$L(f,\chi_D,k)=\epsilon_D\cdot\frac{b_f(D_0)^2}{D_0^{k-\frac{1}{2}}}\cdot\Omega_f,$$

where ϵ_D is algebraic. For all other D with $\delta D>0$, we have $b_f(D_0)=0$. Moreover, the coefficients $a(n),b_f(n)$ and the values of χ are in $\mathcal{O}_K$, the ring of integers of some fixed number field K.

12.1. **Extension of the Kohnen-Waldspurger Theorem.** The author and Bruinier [77] have generalized this theorem of Waldspurger and Kohnen to prove that the Fourier coefficients of weight 1/2 harmonic Maass forms encode the vanishing and nonvanishing of both the central values and derivatives of quadratic twists of weight 2 modular L-functions.

Here we describe a special case of the main result of [77].

Theorem 12.5. *Suppose that*

$$G(z)=\sum_{n=1}^{\infty}B_G(n)q^n\in S_2(p)$$

is a weight 2 newform with prime level p. In addition, suppose that the sign of the functional equation of $L(G,s)$ is $\epsilon(G)=-1$. Then there is a weight 1/2 harmonic Maass form $f(z)$ on $\Gamma_0(4p)$, say

$$f(z)=\sum_{n\gg-\infty}c_g^+(n)q^n+\sum_{n<0}c_g^-(n)\Gamma(1/2;4\pi|n|y)q^n,$$

which satisfies the following:

(1) *If $\Delta<0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then*

$$L(G,\chi_\Delta,1)=\alpha_G\cdot\sqrt{|\Delta|}\cdot c_g^-(\Delta)^2,$$

where α_G is an explicit non-zero constant.

(2) *If $\Delta>0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then $L'(G,\chi_\Delta,1)=0$ if and only if $c_g^+(\Delta)$ is algebraic.*

Remark. Theorem 12.5 is a special case of the general result which holds for all levels, and any arbitrary sign. Notice that the result concerns all weight 2 newforms, not just those which correspond to modular elliptic curves.

Remark. The Maass form $f(z)$ in Theorem 12.5 is unique up to the addition of a weight $1/2$ weakly holomorphic modular form with coefficients in F_G, the number field obtained by adjoining the coefficients of $G(z)$ to $\mathbb{Q}$. In view of this ambiguity, it would be very interesting to pin down a choice of $f(z)$ which in turn gives a precise formula relating $L'(G, \chi_\Delta, 1)$ to $c_g^+(\Delta)$ in Theorem 12.5 (2).

Theorem 12.5 is obtained by combining the Gross-Zagier formula with Borcherds products arising from harmonic Maass forms and general transcendence theorems for canonical differentials of the third kind on modular curves. Here we give a brief sketch of the proof of this theorem.

Sketch of the proof of Theorem 12.5. By Kohnen's theory, there is a half-integral weight newform

$$g(z) = \sum_{n=1}^{\infty} b_g(n) q^n \in S^+_{\frac{3}{2}}(4p), \tag{12.7}$$

unique up to a multiplicative constant, which lifts to G under the Shimura correspondence. We can (and do) choose g so that its coefficients are in F_G, the totally real number field obtained by adjoining the Fourier coefficients of G to $\mathbb{Q}$. We prove that there is a weight $1/2$ harmonic Maass form on $\Gamma_0(4p)$ in the plus space, say

$$f_g(z) = \sum_{n \gg -\infty} c_g^+(n) q^n + \sum_{n<0} c_g^-(n) \Gamma(1/2, 4\pi|n|y) q^n, \tag{12.8}$$

whose principal part P_{f_g} has coefficients in F_G, which also enjoys the property that $\xi_{\frac{1}{2}}(f_g) = \|g\|^{-2} g$, where $\|g\|$ denotes the usual Petersson norm.

By Lemma 4.4, if $n > 0$, then

$$b_g(n) = -4\sqrt{\pi n} \|g\|^2 \cdot c_g^-(-n). \tag{12.9}$$

Theorem 12.5 (1) now follows from Theorem 12.3.

The proof of Theorem 12.5 (2) is more difficult, and it involves a detailed study of Heegner divisors. We establish that the algebraicity of the coefficients $c_g^+(\Delta)$ is dictated by the vanishing of certain twisted Heegner divisors in the Jacobian of $X_0(p)$, which when combined with the work of Gross and Zagier [134] and Scholl and Waldschmidt, then implies Theorem 12.5 (2).

To make this precise, let $d < 0$ and $\Delta > 0$ be fundamental discriminants which are both squares modulo p. Let $\mathcal{Q}_{d,p}$ be the set of discriminant $d = b^2 - 4ac$ integral binary quadratic forms $aX^2 + bXY + cY^2$ with the property that $p \mid a$. For these pairs of discriminants, we define the *twisted Heegner divisor* $H_\Delta(d)$ by

$$H_\Delta(d) := \sum_{Q \in \mathcal{Q}_{\Delta d, p}/\Gamma_0(p)} \chi_\Delta(Q) \cdot \frac{\tau_Q}{w_Q}, \tag{12.10}$$

where χ_Δ denotes the generalized genus character corresponding to the decomposition $\Delta \cdot d$ as in [133], τ_Q is the unique root of $Q(x, 1)$ in $\mathbb{H}$ as before, and w_Q denotes the order of the

stabilizer of Q in $\Gamma_0(p)$. Then $H_\Delta(d)$ is a divisor on $X_0(p)$ defined over $\mathbb{Q}(\sqrt{\Delta})$. We use these twisted Heegner divisors to define the degree 0 divisor

$$y_\Delta(d) := H_\Delta(d) - \deg(H_\Delta(d)) \cdot \infty. \tag{12.11}$$

Finally, we associate a divisor to f_g by letting

$$y_\Delta(f_g) := \sum_{n<0} c_g^+(n) y_\Delta(n) \in \mathrm{Div}^0(X_0(p)) \otimes F_G. \tag{12.12}$$

Let J be the Jacobian of $X_0(p)$, and let $J(F)$ denote the points of J over a number field F. The Hecke algebra acts on $J(F) \otimes \mathbb{C}$, which by the Mordell-Weil Theorem is a finite dimensional vector space. We show that the point corresponding to $y_\Delta(f_g)$ in $J(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{C}$ is in its G-isotypical component. Moreover, we show that the following are equivalent:

(i) The Heegner divisor $y_\Delta(f_g)$ vanishes in $J(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{C}$.
(ii) The coefficient $c_g^+(\Delta)$ is algebraic.
(iii) The coefficient $c_g^+(\Delta)$ is contained in F_G.

To obtain these results, we explicitly construct modular functions with prescribed Heegner divisors using results related to Theorems 11.2 and 11.3. By work of Scholl and Waldschmidt, the vanishing of the points corresponding to these divisors is equivalent to the algebraicity of the Fourier expansions of these modular functions. Using the Hecke algebra and the explicit formulas provided by the generalized Borcherds product, we then find that this algebraicity is dictated by single coefficients of the form $c_g^+(\Delta)$.

We then obtain the following generalization of the well known Gross-Kohnen-Zagier theorem [133]. Namely, we show that

$$\sum_{n>0} y_\Delta^G(-n) q^n = g(\tau) \otimes y_\Delta(f_g) \in S^+_{\frac{3}{2}}(4p) \otimes J(\mathbb{Q}(\sqrt{\Delta})),$$

where $y_\Delta^G(-n)$ denotes the projection of $y_\Delta(-n)$ onto its G-isotypical component. This result, when combined with the Gross-Zagier theorem [134], gives the conclusion that the Heegner divisor $y_\Delta(f_g)$ vanishes in $J(\mathbb{Q}(\sqrt{\Delta})) \otimes \mathbb{C}$ if and only if $L'(G, \chi_\Delta, 1) = 0$, thereby proving Theorem 12.5 (2).

□

12.2. **Recent work of Bruinier and Yang.** Building on the results of the last subsection, it is natural to seek exact formulas for derivatives of L-functions using techniques arising from the theory of harmonic Maass forms. Theorem 12.5 makes the connection between derivatives and coefficients of harmonic Maass forms, but it does not give exact formulas. This problem remains open. Nevertheless, the question of fundamental interest really should be: Can one directly obtain exact arithmetic formulas (perhaps in terms of heights) for certain derivatives of L-functions?

To this end, Bruinier and Yang [82] have recently proven some striking theorems about Faltings heights of CM cycles and derivatives of L-functions. In some important cases they are able to explicitly evaluate derivatives of certain Rankin-type L-functions in terms of Faltings heights. Their work depends critically on Borcherds lifts and the theory of harmonic Maass forms.

It is impossible to survey their work in any detail here. Instead, we give the flavor of their exciting program. Loosely speaking, they use the principal parts of suitable harmonic Maass forms f to construct Heegner divisors on certain Shimura varieties. This depends on earlier work of Kudla, recent work on Greens functions, and Borcherds lifts. For the cusp forms $\xi(f)$, they investigate a certain "Rankin-type" L-function, and they conjecture an exact formula relating its derivative at $s = 0$ to the Faltings height pairing of a Heegner divisor with a CM cycle. The conjectured formula is of the form

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_{Fal} = \kappa \cdot L'(\xi(f), U, 0), \tag{12.13}$$

where κ is explicit.

They have obtained several deep results in the direction of this conjecture. Here we highlight one implication of their work in the classical setting of the original formula of Gross and Zagier. Suppose that G is a normalized newform of weight 2 for $\Gamma_0(N)$ whose Hecke L-function $L(G, s)$ satisfies an odd functional equation. Then there is a weight $3/2$ cusp form g, which corresponds to G under the Shimura correspondence, and it turns out that the "Rankin-type" L-function is proportional to $L(G, s+1)$. Their conjecture therefore then implies a Faltings height pairing formula for $L'(G, 1)$.

As in the work of the author and Bruinier [77] (see Theorem 2.2), there is a weight $1/2$ harmonic Maass form f with vanishing constant term such that $\xi(f) = \|g\|^{-2}g$, with the additional property that the coefficients of its principal part are in the number field generated by the eigenvalues of G. In this case the Heegner divisor $Z(f)$ defines an explicit point in the Jacobian of $X_0(N)$, which lies in the G isotypical component (see the sketch of the proof of Theorem 12.5). In this setting, they prove the following comprehensive theorem.

Theorem 12.6 (Bruinier and Yang [82])**.** *Assuming the notation and hypotheses above, we have that the Neron-Tate height of $Z(f)$ is given by*

$$\langle Z(f), Z(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi\|g\|^2} L'\big(G, 1\big).$$

This beautiful theorem directly gives an arithmetic formula for the central derivative of weight 2 modular L-functions with odd sign. It is natural to ask how Theorem 12.6 relates to the classical Gross-Zagier formula. The short answer is that it implies it.

To see this, suppose that E is an elliptic curve over $\mathbb{Q}$. Assume that its L-function $L(E, s)$ has an odd functional equation so that the central critical value $L(E, 1)$ vanishes. Let $N = N(E)$ be the conductor of E, and let $X_0(N)$ be the moduli space of cyclic isogenies of degree N of generalized elliptic curves. Let K be an imaginary quadratic field such that N is the norm of an integral ideal of K, and write D for the discriminant of K. We may consider the divisor $Z(D)$ on $X_0(N)$ given by elliptic curves with complex multiplication by the maximal order of K. By the theory of complex multiplication, this divisor is defined over K, and its degree h is given by the class number of K. Hence the divisor $y(D) = \mathrm{tr}_{K/\mathbb{Q}}(Z(D) - h \cdot (\infty))$ has degree zero and is defined over $\mathbb{Q}$. By the modularity of elliptic curves, we obtain a rational point $y^E(D)$ on E using a modular parametrization $X_0(N) \to E$. Arguing a little further (so as to produce $L(E, \chi_D, 1)$), one can show that Theorem 12.6 then implies the Gross-Zagier theorem as stated below.

Theorem 12.7 (Gross-Zagier [134]). *Assume the notation and hypotheses above. Then the canonical height of $y^E(D)$, denoted $\langle y^E(D), y^E(D)\rangle_{NT}$, satisfies*

$$\langle y^E(D), y^E(D)\rangle_{NT} = C\sqrt{|D|}L'(E,1)L(E,\chi_D,1).$$

Here C is an explicit non-zero constant which is independent of K, and χ_D is the Kronecker character for $K/\mathbb{Q}$.

To place Theorem 12.6 in proper context, we stress that the work of Gross and Zagier has inspired an enormous body of further research on height pairings of algebraic cycles on Shimura varieties. For instance, Zhang considered heights of Heegner type cycles on Kuga-Sato fiber varieties over modular curves in [248], and the heights of Heegner points on compact Shimura curves over totally real fields in [249]. Gross and Keating discovered a connection between arithmetic intersection numbers of Hecke correspondences on the product of two copies of the modular curve $X(1)$ over $\mathbb{Z}$ and the coefficients of the derivative of the Siegel-Eisenstein series of genus three and weight two [132]. This has inspired the extensive program of Kudla, Rapoport and Yang which relates Arakelov intersection numbers on Shimura varieties of orthogonal type to derivatives of Siegel-Eisenstein series and modular L-functions (for example, see [166], [167], [168]).

In all of these works, the connection between a height pairing and the derivative of an automorphic L-function arises in an indirect way. The idea has been to identify the local height pairings in the Fourier coefficients of a suitable integral kernel function (often given by an Eisenstein series), which takes an automorphic form ϕ to the special value of the derivative of an L-function associated to ϕ.

This recent work of Bruinier and Yang gives a new approach for obtaining identities between certain height pairings on Shimura varieties of orthogonal type and derivatives of automorphic L-functions. As described above, it is based on the Borcherds lift [50] and its generalization in [71, 73, 77]. Their approach directly gives formulas for the Faltings height pairing of arithmetic Heegner divisors and CM cycles.

13. Ramanujan's last letter revisited

Ramanujan's last letter to Hardy was about his discovery of the mock theta functions. In this letter he introduces his vague notion, and he considers the radial limits of some of his so-called mock theta functions. Here we begin by recalling Theorem 2.5 on the radial limits which relate the rank and crank generating functions to the unimodal rank generating function $U(w;q)$.

Theorem. *Let $1 \le a < b$ and $1 \le h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b \mid k$. If h' is an integer satisfying $hh' \equiv -1 \pmod{k}$, then, as q approaches ζ_k^h radially within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\Big(R\left(\zeta_b^a;q\right) - \zeta_{b^2}^{-a^2h'k}C\left(\zeta_b^a;q\right)\Big) = -(1-\zeta_b^a)(1-\zeta_b^{-a})\cdot U(\zeta_b^a;\zeta_k^h).$$

Sketch of the Proof of Theorem 2.5. The proof requires the theory of modular units and mock modular forms. In particular, we employ the modular properties of Dedekind's η-function and certain Klein forms $\mathfrak{t}_{(r,s)}(z) = \mathfrak{t}^{(N)}_{(r,s)}(z)$ [165]. We also require the function

$\mu(u, v; z)$ defined by Zwegers in his thesis (see Section 3). The μ-function satisfies the following beautiful bilateral series identity due to Choi [91] Let $q = e^{2\pi iz}$, where $z \in \mathbb{H}$. For suitable complex numbers $\alpha = e^{2\pi iu}$ and $\beta = e^{2\pi iv}$, we have

$$\sum_{n=0}^{\infty} \frac{(\alpha\beta)^n q^{n^2}}{(\alpha q; q)_n (\beta q; q)_n} + \sum_{n=1}^{\infty} q^n (\alpha^{-1}; q)_n (\beta^{-1}; q)_n$$
$$= iq^{\frac{1}{8}}(1-\alpha)(\beta\alpha^{-1})^{\frac{1}{2}} \left(q\alpha^{-1}; q\right)_\infty \left(\beta^{-1}; q\right)_\infty \mu(u, v; z).$$

Remark. Choi's identity can also be obtained from Entry 3.4.7 of Ramanujan's "Lost Notebook" (see p.67 of [24]).

To make use of this identity, we study the function $A(u, v; z) := \vartheta(v; z)\mu(u, v; z)$ which was previously studied by Zwegers [254] and Folsom and Bringmann [55]. We employ the "completed" function $\widehat{A}(u, v; z)$, defined by Zwegers as

$$\widehat{A}(u, v; z) := A(u, v; z) + \frac{i}{2}\vartheta(v; z)R(u - v; z), \tag{13.1}$$

where

$$R(v; z) := \sum_{n\in\mathbb{Z}} \left\{ \operatorname{sgn}\left(n + \frac{1}{2}\right) - E\left(\left(n + \frac{1}{2} + \frac{\operatorname{Im}(v)}{\operatorname{Im}(z)}\right)\sqrt{2 \cdot \operatorname{Im}(z)}\right)\right\} (-1)^n q^{-\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{-2\pi iv\left(n+\frac{1}{2}\right)},$$

and for $w \in \mathbb{C}$ we have

$$E(w) := 2\int_0^w e^{-\pi u^2} du.$$

Using the transformation properties of the functions μ and ϑ, we have, for integers m, n, r, s and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, that

$$\widehat{A}\left(u + mz + n, v + rz + s; z\right) = (-1)^{m+n} e^{2\pi iu(m-r)} e^{-2\pi ivm} q^{\frac{m^2}{2} - mr} \widehat{A}(u, v; z), \tag{13.2}$$

$$\widehat{A}\left(\frac{u}{cz+d}, \frac{v}{cz+d}; \gamma z\right) = (cz+d) e^{\pi ic\frac{(-u^2+2uv)}{(cz+d)}} \widehat{A}(u, v; z). \tag{13.3}$$

The proof makes use of the modular transformation properties described above. We consider Choi's identity with $\alpha = \zeta_b^{-a}$ and $\beta = \zeta_b^a$, (hence $u = -\frac{a}{b}$, $v = \frac{a}{b}$), and q replaced by $e^{\frac{2\pi i}{k}(h+iz)}$, and we define

$$m(a, b; u) := ie^{\frac{\pi iu}{4}}(1 - \zeta_b^{-a})\zeta_b^a(\zeta_b^a e^{2\pi iu}; e^{2\pi iu})_\infty(\zeta_b^{-a}; e^{2\pi iu})_\infty. \tag{13.4}$$

To prove Theorem 2.5, noting that the function $U(\zeta_b^a; \zeta_k^h)$ is a finite convergent sum when $b|k$, it suffices to prove that upon appropriate specialization of variables, the "mixed mock" modular form $m \cdot \mu$ is asymptotic to a suitable multiple of the modular crank generating function C.

To be precise, let $b|k, \gcd(a, b) = 1, \gcd(h, k) = 1$, where a, b, h, k are positive integers. As $z \to 0^+$, we must prove that

$$m\left(a, b; \frac{1}{k}(h + iz)\right) \mu\left(-\frac{a}{b}, \frac{a}{b}; \frac{1}{k}(h + iz)\right) \sim \zeta_{b^2}^{-a^2h'k} C\left(\zeta_b^a; \frac{1}{k}(h + iz)\right). \tag{13.5}$$

Above and in what follows, we let $z \in \mathbb{R}^+$, and let $z \to 0^+$. This corresponds to the radial limit $q = e^{\frac{2\pi i}{k}(h+iz)} \to \zeta_k^h$ from within the unit disk.

The claim (13.5) is obtained by comparing separate asymptotic results for the crank function and the mixed mock modular form. To describe this, we let

$$q := e^{\frac{2\pi i}{k}(h+iz)}, \quad q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}. \tag{13.6}$$

For the mixed mock modular $m \cdot \mu$, we obtain the following asymptotics. Let $b|k, \gcd(a,b) = 1, \gcd(h,k) = 1$, where a, b, h, k are positive integers, and let b' and h' be positive integers such that $bb' = k$ and $hh' \equiv -1 \pmod{k}$. For $z \in \mathbb{R}^+$, as $z \to 0^+$, we established in Theorem 3.2 of [118] that there is an $\alpha > 0$ for which

$$m\left(a, b; \frac{1}{k}(h+iz)\right)\mu\left(-\frac{a}{b}, \frac{a}{b}; \frac{1}{k}(h+iz)\right)$$
$$= \left(\frac{i}{z}\right)^{\frac{1}{2}} (\psi(\gamma))^{-1} q^{\frac{1}{24}} q_1^{-\frac{1}{24}} (-1)^{ab'} \zeta_{2b}^{ah'-a} \zeta_{2b^2}^{-3a^2kh'} \frac{\zeta_b^a - 1}{1 - \zeta_b^{ah'}} (1 + O(q_1^\alpha)).$$

Here, $\gamma = \gamma(h,k) \in \mathrm{SL}_2(\mathbb{Z})$, and $\psi(\gamma)$ is a 24th root of unity, both of which are defined in [118].

Under the same hypotheses, we show in Proposition 3.3 of [118] that the modularity of Dedekind's eta-function and the Klein forms implies, for the crank function, that

$$C\left(\zeta_b^a; \frac{1}{k}(h+iz)\right) = \left(\frac{i}{z}\right)^{\frac{1}{2}} (\psi(\gamma))^{-1} q^{\frac{1}{24}} q_1^{-\frac{1}{24}} (-1)^{ab'} \zeta_{2b}^{ah'-a} \zeta_{2b^2}^{-a^2kh'} \frac{\zeta_b^a - 1}{1 - \zeta_b^{ah'}} (1 + O(q_1^\beta)), \tag{13.7}$$

for some $\beta > 0$. Combining these asymptotics gives (13.5), which then gives Theorem 2.5. □

It is important to explain the automorphic reasons which underlie the asymptotic relationship of Theorem 2.5. The rank generating function $R(w;q)$ and the crank generating function $C(w;q)$ are Jacobi forms with the same weight and multiplier. This coincidence explains why their radial asymptotics are closely related. However, they are not of the same index, and this difference accounts for the $(-1)^k$ in Theorem 1.2, and the $\zeta_{b^2}^{-a^2h'k}$ in Theorem 2.5. Of course, these facts alone do not directly lead to Theorem 2.5. To make this step requires a special case of Choi's identity, namely,

$$\sum_{n \in \mathbb{Z}} \frac{q^{n^2}}{(wq)_n (w^{-1}q)_n} = R(w;q) + (1-w)(1-w^{-1}) \sum_{n=0}^{\infty} q^{n+1} (wq)_n (w^{-1}q)_n.$$

It turns out that this series is related to a mixed-mock Jacobi form which has asymptotics resembling that of crank generating function. Although such coincidences are mysterious, it is not uncommon that such a bilateral series possesses better modular properties than either half of the series (See [127] for more dealing with bilateral series and mock theta functions).

Now we conclude this section with a discussion of Ramanujan's vague definition of a mock theta function. Theorem 2.5 shows that holomorphic parts of certain harmonic Maass forms can almost have the property that their exponential singularities can be cut out by a single weakly holomorphic modular form. The obstruction simply boils down to the need to

multiply the weakly holomorphic modular form by various roots of unity. Nevertheless, the question of whether Ramanujan's examples satisfy his own definition of a mock theta function remainded open despite the important work of Zwegers in 2002. The story is underscored by the recent comment by Berndt [43] that

"it has not been proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition."

Griffin, Rolen, and the author [131] have recently filled in this gap by proving Theorem 1.3.

Theorem. *Suppose that $M(z)$ is one of Ramanujan's mock theta functions, and let γ and δ be integers for which $q^\gamma M(\delta z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q \to \zeta} (q^\gamma M(\delta z) - g(z)) = O(1).$$

Sketch of the proof of Theorem 1.3. We begin by considering the following question whose solution would have been clear to him: If $f(z)$ is a weight k_1 weakly holomorphic modular form which has some exponential singularities at cusps, then can there be another weakly holomorphic modular form of different weight k_2, say $g(z)$, that exactly cuts out its singularities at roots of unity? The answer is *no*. If such a $g(z)$ existed, then both $f(z)$ and $g(z)$ must have the same principal parts at all cusps, and at least one of these must be nonconstant. Without loss of generality, suppose that the principal part at the cusp infnity is nonconstant, and then consider the function $h(z) := f(z) - g(z)$. By hypothesis, $h(z)$ has bounded radial limits as z approaches every root of unity. Now, since $f(z)$ and $g(z)$ are modular on some common subgroup $\Gamma_1(N')$, then if we take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N')$ with $cd \neq 0$, then we have

$$h\left(\tfrac{az+b}{cz+d}\right) = f\left(\tfrac{az+b}{cz+d}\right) - g\left(\tfrac{az+b}{cz+d}\right) = (cz+d)^{k_1} f(z) - (cz+d)^{k_2} g(z). \tag{13.8}$$

Letting $z \to i\infty$, we find that $f(z)$ and $g(z)$ cannot cut out the same exponential singularities at roots of unity because of the difference between the weights.

We have the following fact.

Fact 13.1. *Suppose that $M(z)$ is one of Ramanujan's alleged examples of a mock theta function. Thanks to Zwegers, there are integers γ and δ for which $q^\gamma M(\delta z) =: f^+(z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form $f(z)$ on a congruence subgroup $\Gamma_1(N)$. Moreover, the nonholomorphic part of this form is the period integral of a weight 3/2 unary theta function. In particular, there are finitely many positive integers $\delta_1, \dots, \delta_s$ for which $c_f^-(n) = 0$ unless $n = -\delta_i m^2$ for some $1 \leq i \leq s$ and some integer m.*

Using the work of Bruinier and Funke on the extended Petersson inner product (see Lemma 4.6), combined with the theory of Maass-Poincaré series (see Section 5.3), we have the following fact.

Fact 13.2. *Suppose that $f(z)$ is a weight 1/2 harmonic weak Maass form with a nonconstant principal part at some cusp. Let $f_P(z)$ be the weight 1/2 harmonic weak Maass form that is a linear combination of Maass-Poincaré series which matches, up to constants, the principal parts of $f(z)$ at all cusps. Then $f(z) - f_P(z)$ is a weight 1/2 holomorphic modular form,*

which, by the Serre-Stark Basis Theorem (for example, see [191]*), implies that $f(z) - f_P(z)$ is a linear combination of weight 1/2 unary theta functions. Therefore, the subexponential growth of the I-Bessel function, combined with the periodicity of the Kloosterman sums in n, when m and c are fixed, then implies that a positive proportion of the coefficients of the holomorphic part of $f^+(z)$ are nonzero. Indeed, this gives arithmetic progressions of coefficients with smooth asymptotic subexponential growth.*

Remark. Care must be taken when working with weight 1/2 Maass-Poincaré series. These series are not obviously well defined due to potential problems related to convergence. Results of Goldfeld and Sarnak in [123] and the spectral theory of automorphic forms apply in these cases.

Suppose that $M(z)$ is one of Ramanujan's alleged examples of a mock theta function. Then there are integers γ and δ for which $q^\gamma M(\delta z) =: f^+(z)$ is the holomorphic part of the weight 1/2 harmonic weak Maass form. Now suppose that $g(z)$ is a weakly holomorphic modular form of some weight k which cuts out the exponential singularities of $f(z)$. Following the proof of Theorem 6.5, we can use Fact 13.1, Fact 13.2, and the theory of quadratic (and trivial) twists to obtain a weight 1/2 weakly holomorphic modular form $\widehat{f}(z)$. By Fact 13.2, this can be done so that $\widehat{f}(z)$ is nontrivial and has nonconstant principal parts at some cusp. Applying the same procedure to $g(z)$ gives a weakly holomorphic modular form $\widehat{g}(z)$. We then have that $\widehat{f}(z)$ and $\widehat{g}(z)$ cut out exactly the same exponential singularities at all roots of unity. By the discussion related to (13.8), it then follows that $k = 1/2$. Therefore, if there is such a $g(z)$, then $f(z) - g(z)$ is a weight 1/2 harmonic weak Maass form which has a nonvanishing nonholomorphic part, which also has the property that $f^+(z) - g(z)$ has no exponential singularities at any roots of unity, and hence has constant principal parts at all cusps. However, this contradicts Lemma 4.6. □

14. Quantum modular forms

Here we sketch the proof of Theorem 2.11 which uses the theory of mock theta functions as a method for producing quantum modular forms. This theorem concerns two families of specializations of the Rogers-Fine basic hypergeometric function

$$F(\alpha, \beta, t; q) := \sum_{n=0}^{\infty} \frac{(\alpha q; q)_n t^n}{(\beta q; q)_n}.$$

Here we consider the q-series

$$G(a, b; z) := \frac{q^{\frac{a^2}{b^2}}}{1 - q^{\frac{a}{b}}} \cdot F\left(-q^{\frac{a}{b}-1}, q^{\frac{a}{b}}, -q^{\frac{a}{b}}; q\right),$$
$$H(a, b; z) := q^{\frac{1}{8}} \cdot F\left(\zeta_b^{-a} q^{-1}, \zeta_b^{-a}, \zeta_b^{-a} q; q^2\right).$$

We have the following false theta function identities, the second of which follows from equation (1) of [213], and the first of which is in [31].

$$G(a, b; z) = q^{\frac{a^2}{b^2}} \sum_{n=0}^{\infty} \frac{(-q^{\frac{a}{b}}; q)_n}{(q^{\frac{a}{b}}; q)_{n+1}} \cdot (-1)^n q^{n\frac{a}{b}} = \sum_{n=0}^{\infty} (-1)^n q^{\left(n+\frac{a}{b}\right)^2},$$

$$H(a,b;z) = q^{\frac{1}{8}} \sum_{n=0}^{\infty} \frac{(\zeta_b^{-a}q;q^2)_n}{(\zeta_b^{-a}q^2;q^2)_n} \cdot (\zeta_b^{-a}q)^n = \sum_{n=0}^{\infty} \zeta_b^{-an} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}.$$

For these series we recall Theorem 2.11.

Theorem. *Let $0 < a < b$ be coprime integers, with b even, and let*

$$Q_{a,b} := \left\{ \frac{h}{k} \in \mathbb{Q} : \gcd(h,k) = 1, h > 0, b \mid 2h, b \nmid h, k \equiv a \pmod{b}, k \geq a \right\}.$$

The following are true:
(1) The functions $G(a,b;z)$ and $H(a,b;z)$ converge for $z \in \mathbb{H}^+ \cup \mathbb{H}^-$.
(2) For $x \in Q_{a,b} \cup \mathbb{H}^+$, we have that

$$G(a,b;-x) + \frac{e^{-\frac{\pi i a}{b}}}{\sqrt{2ix}} \cdot H\left(a,b;\frac{1}{2x}\right) = -\int_0^{i\infty} \frac{(-iu)^{-\frac{3}{2}} T\left(a,b;-\frac{1}{u}\right) du}{\sqrt{-i(u+x)}},$$

where $T(a,b;z)$ is a weight $3/2$ modular form defined by

$$T(a,b;z) := i \sum_{n=-\infty}^{\infty} \left(n+\frac{1}{4}\right) \cosh\left(2\pi i \left(n+\frac{1}{4}\right)\left(\frac{2a}{b}-1\right)\right) q^{\left(n+\frac{1}{4}\right)^2}.$$

That is, $G(a,b;x)$ and $H(a,b;x)$ are weight 1/2 quantum modular forms on $Q_{a,b} \cup \mathbb{H}^+$.
(3) Let $B_r(n)$ be the rth Bernoulli polynomial. For $\frac{h}{k} \in Q_{a,b}$, as $t \to 0^+$, we have

$$G\left(a,b;\frac{-h}{k}+\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r,c_G) \cdot \frac{(-t)^r}{r! \cdot b^{2r}},$$
$$H\left(a,b;\frac{k}{2h}+\frac{it}{2\pi}\right) \sim \sum_{r=0}^{\infty} L(-2r,c_H) \cdot \frac{(-t)^r}{r! \cdot 8^r},$$

where

$$L(-r,c_G) = -\frac{(2kb^2)^r}{r+1} \sum_{n=1}^{2kb^2} c_G(n) B_{r+1}\left(\frac{n}{2kb^2}\right),$$
$$L(-r,c_H) = -\frac{(16h)^r}{r+1} \sum_{n=1}^{16h} c_H(n) B_{r+1}\left(\frac{n}{16h}\right),$$

$$c_G(n) := \begin{cases} \zeta_{kb^2}^{-hn^2}, & \text{if } n \equiv a \pmod{2b}, \\ -\zeta_{kb^2}^{-hn^2}, & \text{if } n \equiv a+b \pmod{2b}, \\ 0, & \text{otherwise}, \end{cases} \tag{14.1}$$

$$c_H(n) := \begin{cases} \zeta_b^{-a\left(\frac{n-1}{2}\right)} \zeta_{16h}^{kn^2}, & \text{if } n \equiv 1 \pmod{2}, \\ 0, & \text{otherwise}. \end{cases} \tag{14.2}$$

14.1. **Sketch of the proof of Theorem 2.11.** We begin with a simple lemma which is Lemma 2.1 of [118].

Lemma 14.1. *Let $0 < a < b$ be coprime integers, where b is even, and let $Q_{a,b}$ be as in the statement of Theorem 2.11. If $x = \frac{h}{k} \in Q_{a,b}$, then $G(a, b; -x)$ and $H\left(a, b; \frac{1}{2x}\right)$ converge. Moreover, they are explicitly given by the finite sums*

$$G\left(a, b; -\frac{h}{k}\right) = \zeta_{b^2k}^{-a^2h} \sum_{n=0}^{m} \frac{(-\zeta_{bk}^{-ah}; \zeta_k^{-h})_n (-\zeta_{bk}^{-ah})^n}{(\zeta_{bk}^{-ah}; \zeta_k^{-h})_{n+1}},$$

$$H\left(a, b; \frac{k}{2h}\right) = \zeta_{16h}^{k} \sum_{n=0}^{\ell} \frac{(\zeta_b^{-a}\zeta_{2h}^{k}; \zeta_h^{k})_n (\zeta_b^{-a}\zeta_{2h}^{k})^n}{(\zeta_b^{-a}\zeta_h^{k}; \zeta_h^{k})_n},$$

where the non-negative integers ℓ and m satisfy $b(2\ell+1) = 2h$ and $a + bm = k$, respectively.

To derive quantum modularity, we make use of earlier work of Bringmann, Rhoades, and the author. We summarize the required results from Theorem 4.3 and Lemma 4.5 of [66] in the theorem below. These results involve the q-hypergeometric functions[18]

$$g(a, b; z) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{\frac{n(n+1)}{2}}}{\left(q^{\frac{a}{b}}; q\right)_{n+1} \left(q^{-\frac{a}{b}} q; q\right)_{n+1}},$$

$$h(a, b; z) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(\zeta_b^a q^2; q^2)_n (\zeta_b^{-a} q^2; q^2)_n},$$

and the important theta function

$$T(a, b; z) := i \sum_{n=-\infty}^{\infty} \left(n + \frac{1}{4}\right) \cosh\left(2\pi i \left(n + \frac{1}{4}\right)\left(\frac{2a}{b} - 1\right)\right) q^{\left(n+\frac{1}{4}\right)^2}. \tag{14.3}$$

Theorem 14.2 (Theorem 4.3 and Lemma 4.5 of [66])**.** *Let $z \in \mathbb{H}$, and suppose that $0 < a < b$ are coprime integers. We have that*

$$q^{\frac{a}{b}\left(1-\frac{a}{b}\right)} g(a, b; z) = \sqrt{\frac{2i}{z}} \frac{e^{\frac{\pi i}{8z}}}{4 \sin\left(\pi \frac{a}{b}\right)} h\left(a, b; \frac{-1}{2z}\right) + \int_0^{i\infty} \frac{(-iu)^{-\frac{3}{2}} T\left(a, b; -\frac{1}{u}\right) du}{\sqrt{-i(u+z)}}.$$

Theorem 2.11 (3) relates asymptotic expansions to values of L-functions. To obtain our result in this direction, we make use of the following proposition of Lawrence and Zagier [171].

Proposition 14.3 (p. 98 of [171])**.** *Let $C : \mathbb{Z} \to \mathbb{C}$ be a periodic function with mean value 0. Then the associated L-series $L(s, C) = \sum_{n=1}^{\infty} C(n) n^{-s}$ ($Re(s) > 1$) extends holomorphically to all of $\mathbb{C}$ and the function $\sum_{n=1}^{\infty} C(n) e^{-n^2 t}$ ($t > 0$) has the asymptotic expansion*

$$\sum_{n=1}^{\infty} C(n) e^{-n^2 t} \sim \sum_{r=0}^{\infty} L(-2r, C) \cdot \frac{(-t)^r}{r!}$$

[18]These functions were also studied by Gordon and McIntosh in [127]. In [66] §4.1, the function $g(a, b; z)$ is shown to be equal to what the authors call $H(a, 0, b; z)$. We do not use this notation to avoid confusion.

as $t \to 0^+$. The numbers $L(-r, C)$ are given explicitly by

$$L(-r, C) = -\frac{M^r}{r+1} \sum_{n=1}^{M} C(n) B_{r+1}\left(\frac{n}{M}\right) \quad (r = 0, 1, \ldots)$$

where $B_k(x)$ denotes the kth Bernoulli polynomial and M is any period of the function $C(n)$.

Sketch of the proof of Theorem 2.11. We first prove (1). Let $\rho := q^{-1}$, where $q = e^{2\pi i z}$ and $z \in \mathbb{H}$. We compute

(14.4)

$$-\rho^{\frac{a}{b}-\frac{a^2}{b^2}} G(a, b; -z) = -\rho^{\frac{a}{b}} \sum_{n=0}^{\infty} \frac{(-\rho^{\frac{a}{b}}; \rho)_n}{(\rho^{\frac{a}{b}}; \rho)_{n+1}} (-\rho^{\frac{a}{b}})^n \;=\; \sum_{n=0}^{\infty} \frac{(-q^{\frac{a}{b}}; q)_n}{(q^{\frac{a}{b}}; q)_{n+1}} q^{n\left(1-\frac{a}{b}\right)} \;=: G^*(a, b; z),$$

where we use the fact that

$$(a; \rho)_n = (-a)^n q^{-\frac{n(n-1)}{2}} (a^{-1}; q)_n. \tag{14.5}$$

Thus from (2.19) and (14.4) we see that the series defining $G(a, b; z)$ is defined for $z \in \mathbb{H} \cup \mathbb{H}^-$. Similarly, using (14.5), it is not difficult to show that

$$q^{\frac{1}{8}} H(a, b; -z) = F\left(q^{-1}\zeta_b^a, \zeta_b^a, \zeta_b^{-a}; q^2\right), \tag{14.6}$$

and hence that $H(a, b; z)$ is defined for $z \in \mathbb{H} \cup \mathbb{H}^-$. We justify the convergence of the series $F\left(q^{-1}\zeta_b^a, \zeta_b^a, \zeta_b^{-a}; q^2\right)$ in (14.6) as follows. By considering the Rogers-Fine series $F(\alpha, \beta, t; q)$ purely formally, we have the functional equation

$$F\left(q^{-1}\zeta_b^a, \zeta_b^a, \zeta_b^{-a}; q^2\right) = -\zeta_b^a \left(1 - \frac{\zeta_b^a - q}{\zeta_b^a - 1} F\left(q^{-1}\zeta_b^a, \zeta_b^a, \zeta_b^{-a} q^2; q^2\right)\right).$$

(See also (2.4) of [114].) Iterating this recurrence relationship, it follows that $F(\alpha, \beta, t; q)$ converges for all $t \neq q^{-n}, n \in \mathbb{N}_0$. (See also [114] page 2.) This proves (1).

We now prove (2). We relate $G^*(a, b; z)$ to the mock modular form $g(a, b; z)$. The identity

$$(1-t)F(\alpha, \beta; t) = \sum_{n=0}^{\infty} \frac{\left(\frac{\beta}{\alpha}; q\right)_n}{(\beta q; q)_n (tq; q)_n} \cdot (-\alpha t)^n q^{\frac{n(n+1)}{2}} \tag{14.7}$$

(see (12.2) of [114]) with $\alpha = -q^{\frac{a}{b}-1}, \beta = q^{\frac{a}{b}}, t = q^{1-\frac{a}{b}}$ shows that

$$G^*(a, b; z) = g(a, b; z). \tag{14.8}$$

We use (14.7) again with $\alpha = q^{-1}\zeta_b^a, \beta = \zeta_b^a, t = \zeta_b^{-a}$, and $q \to q^2$, and find

$$h(a, b; z) = (1 - \zeta_b^{-a}) F\left(q^{-1}\zeta_b^a, \zeta_b^a, \zeta_b^{-a}; q^2\right) = (1 - \zeta_b^{-a}) q^{\frac{1}{8}} H(a, b; -z), \tag{14.9}$$

where the last equality in (14.9) follows from (14.6). The proof of part (2) now follows from part (1), (14.4), (14.6), (14.8), (14.9), and Theorem 14.2. While Theorem 14.2 gives a transformation law for $z \in \mathbb{H}$, we have continuation to $x \in Q_{a,b}$ by Lemma 14.1. As argued in [244] and [83], the integral appearing in (2) of Theorem 2.11 is real analytic.

We now prove part (3). The conclusion of part (3) follows from Proposition 14.3 once the hypotheses are confirmed for certain L-functions related to $G(a, b; z)$ and $H(a, b; z)$.

To this end, we let $c_G(n)$ and $c_H(n)$ be as defined in (14.1) and (14.2), respectively. Then using (2.21) and (2.22), we have

$$G\left(a, b; \frac{-h}{k} + \frac{it}{2\pi}\right) = \sum_{n=0}^{\infty} c_G(n) e^{-n^2 \frac{t}{b^2}},$$
$$H\left(a, b; \frac{k}{2h} + \frac{it}{2\pi}\right) = \sum_{n=0}^{\infty} c_H(n) e^{-n^2 \frac{t}{8}}.$$

It is clear that c_G is $2kb^2$ periodic, and c_H is $16h$ periodic. The arithmetic functions c_G and c_H have mean value zero. Proposition 14.3 gives part (3).

□

15. Research Projects

Here is a list of projects which arise naturally from the material discussed in these notes.

Project 15.1. *Theorem 10.7 is an example of a "class polynomial" for a nonholomorphic modular function obtained from the theory of harmonic Maass forms. Very little is known about the behavior of $p(n)$ modulo 2 and 3. It would be interesting to study the reductions modulo p of these polynomials when $p \nmid (24n-1)$. How are the polynomials distributed (e.g. factorizations modulo p, individual symmetric functions in the singular moduli)? In a related direction, one can ask the same questions for the "class polynomials" $H_n(\gamma; x)$ defined in* [80]*. Moreover, do any of these families of polynomials satisfy a Gross-Zagier type phenomenon concerning the prime factorizations of the constant terms?*

Project 15.2. *Theorem 11.3 gives Generalized Borcherds Products. These are often modular functions on some congruence subgroup. Prove theorems about the congruence properties of the coefficients of Ramanujan's mock theta functions by making use of the these products, the method of logarithmic differentiation, and the theory of modular forms modulo p and the Deligne-Serre theory of modular Galois representations.*

Project 15.3. *Here we only considered the function $U(w; q)$, defined by (2.13), when $w = 1$. Theorems 2.8, 2.9 and 2.17 related $U(1; q)$ to Kontsevich's "strange function", values of L-functions, and a certain weight 3/2 quantum modular form. Study this series for other roots of unity w. Investigate similar questions, and study the arithmetic properties of rank statistic for unimodal sequences when dissected by arithmetic progressions.*

Project 15.4. *Section 13 discusses recent work on Ramanujan's radial limits and his original vague notion of a mock theta function. In particular, Theorems 1.2 and 2.5 give finite basic hypergeometric-type formulas for the radial limits of some mock theta functions at roots of unity which are exponential singularities. Obtain more results along these lines and develop their connection with L-functions and quantum modular forms.*

Project 15.5. *In their 1988 Inventiones Mathematicae papers, Andrews, Dyson, and Hickerson* [26]*, and Cohen* [95] *related two basic hypergeometric series to a special Maass cusp form of Hecke-type (i.e. arising from a Hecke character for a real quadratic field). Study this example and obtain further examples where Fourier expansions of Maass cusp forms arise*

from basic hypergeometric series. Then relate these results to L-functions and quantum modular forms.

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