ARIZONA WINTER SCHOOL 2013

EXERCISES: WEAK MAASS FORMS, MOCK MODULAR FORMS, AND q-HYPERGEOMETRIC SERIES

AMANDA FOLSOM

The following 54 exercises are grouped by category, divided into the following 5 sections:

1. **Weak Maass forms**
2. **Mock Jacobi forms**
3. **q-hypergeometric series**
4. **Partition theory**
5. **Quantum modular forms**

Each problem is labeled to indicate difficulty level:

$$\star = \text{less difficult},$$
$$\star\star = \text{medium difficulty},$$
$$\star\star\star = \text{more difficult}.$$

Problems are not necessarily meant to be completed in the order presented, although it will be clear by context that some problems are sequential.

1. Weak Maass forms

Let $H_\kappa(\Gamma,\chi)$ (resp. $S_\kappa(\Gamma,\chi), M_\kappa(\Gamma,\chi), M_\kappa^!(\Gamma,\chi)$) denote the space of harmonic weak Maass forms (resp. cusp forms, holomorphic modular forms, weakly holomorphic modular forms) of weight κ on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ with character χ, and $q := e^{2\pi i\tau}, \tau \in \mathbb{H}$. Note we will typically write $H_\kappa(\Gamma) := H_\kappa(\Gamma, 1)$ (resp. $S_\kappa(\Gamma), M_\kappa(\Gamma), M_\kappa^!(\Gamma)$), and $(\Gamma_0(N),\chi) = (N,\chi)$.

Problem 1. ($\star\star$) Suppose $N \in \mathbb{N}$ and $f \in H_{2-k}(\Gamma_1(N))$, $1 < k \in \frac{1}{2}\mathbb{Z}$. Prove that f has Fourier expansion of the form

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y)q^n,$$

where $\tau = x+iy \in \mathbb{H}, x, y \in \mathbb{R}$, and $\Gamma(a,x)$ is the incomplete Γ-function.

Problem 2. ($\star\star$) Let $0 < a < c$ be integers. Consider the weak Maass form

$$D(a,c;\tau) := q^{4f_c^2 \frac{a}{c}\left(1-\frac{a}{c}\right)} H(a,c;4f_c^2\tau) + V(a,c;2f_c^2\tau),$$

where $f_c := 2c/\gcd(2c,4)$, and

$$V(a,c;\tau) := -\frac{1}{2}\int_{-\overline{\tau}}^{i\infty} \frac{(-iz)^{-3/2}T(a,c;-1/2z)}{\sqrt{-i(z+\tau)}}dz,$$

$$T(a,c;\tau) := i\sum_{n\in\mathbb{Z}}(n+1/4)\cosh(2\pi i(n+1/4)(2a/c-1))e^{2\pi i\tau\left(n+\frac{1}{4}\right)^2},$$

$$H(a;c;\tau) := \sum_{n\geq 0}\frac{q^{n(n+1)/2}(-q;q)_n}{(q^{a/c};q)_{n+1}(q^{1-a/c};q)_{n+1}},$$

where for $n\in\mathbb{N}_0$, $(\alpha;q)_n := (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})$.

(a) Prove that $D(a,c;\tau)$ has a Fourier expansion as in Problem 1.

(b) Prove that $D(a,c;\tau)$ is annihilated by the weight 1/2 Laplacian operator

$$\Delta_{\frac{1}{2}} := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{iy}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Problem 3. ($\star\star$) Recall that the ξ_k-operator is defined by

$$\xi_k := 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}}.$$

Let $1 < k \in \frac{1}{2}\mathbb{Z}$, and prove that for $f\in H_{2-k}(N,\chi)$ (with Fourier expansion as in Problem 1),

$$\xi_{2-k} : H_{2-k}(N,\chi) \to S_k(N,\overline{\chi}),$$

and

$$\xi_{2-k}(f) = -(4\pi)^{k-1}\sum_{n\geq 1}\overline{c_f^-(-n)}n^{k-1}q^n.$$

Problem 4. ($\star$) Let $R_k = -4\pi D + \frac{k}{y}$, where $D := \frac{1}{2\pi i}\frac{d}{d\tau}$. Prove Bol's identity, that

$$D^{k-1} = \frac{1}{(-4\pi)^{k-1}}R_{2-k}^{k-1}.$$

Problem 5. ($\star\star$) Let $2\leq k\in\mathbb{Z}$. Prove that if $f\in H_{2-k}(N)$ (with Fourier expansion as in Problem 1), then

$$D^{k-1}(f)\in M_k^!(N),$$

and

$$D^{k-1}f = \sum_{n\gg-\infty} c_f^+(n)n^{k-1}q^n.$$

Let $k \in \frac{1}{2}\mathbb{Z}$. For primes p, and $F(\tau) = \sum_{n \gg -\infty} a_F(n) q^n \in M_k^!(N,\chi)$, the $T_k(p)$ Hecke operator is defined by

$$F|T_k(p) := \sum_{n \gg -\infty} \left(a_F(pn) + \chi(p) p^{k-1} a_F(n/p) \right) q^n, \quad \text{if } k \in \mathbb{Z},$$

$$:= \sum_{n \gg -\infty} \left(a_F(p^2 n) + \chi(p) \left(\frac{(-1)^\lambda n}{p} \right) p^{\lambda-1} a_F(n) + \chi(p^2) p^{2\lambda-1} a_F(n/p^2) \right) q^n, \quad \text{if } k = \lambda + \frac{1}{2}, \lambda \in \mathbb{Z}.$$

A Hecke action on weak Maass forms is defined analogously.

Problem 6. ($\star\star$) Let $f \in H_{2-k}(N,\chi)$ and $p \nmid N$ a prime for which $\xi_{2-k}(f) \in S_k(N, \overline{\chi})$ is an eigenform of $T_k(p)$ with eigenvalue $\lambda(p)$. Prove that

$$f|T_{2-k}(p) - p^{h(k)} \lambda(p) f \in M_{2-k}^!(N,\chi),$$

where $h(k) := 2 - 2k$ if $k \in \frac{1}{2} + \mathbb{Z}$, and $h(k) := 1 - k$ if $k \in \mathbb{Z}$.

Problem 7. ($\star\star$) Fill in the details of the proof of Theorem 4.5 of the notes, which pertains to periods and weak Maass forms.

Let ρ_L denote the Weil representation associated to L'/L, where $L \subseteq V$ is an even lattice and L' its dual, and let $M_{k,\rho_L}^!$ denote the space of $\mathbb{C}[L'/L]$-valued, weight k, weakly holomorphic functions of type ρ_L for $\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$. (Other spaces $M_{k,\rho}, H_{k,\rho}$ etc. are defined analogously.) For $g \in M_{2-k,\overline{\rho}_L}$ and $f \in H_{k,\rho_L}$, define the bilinear pairing

$$\{g, f\} = (g, \xi_k(f))_{2-k} := \int_{\Gamma\backslash H} \langle g, \xi_k(f) \rangle y^{2-k} \frac{dxdy}{y^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson scalar product.

Problem 8. ($\star\star\star$) Prove that $\{g, f\}$ depends only on the principal part of f.

Problem 9. ($\star\star$) Prove that the Hecke operator $T_k(\ell)$ is up to scalar self adjoint with respect to the pairing $\{\cdot,\cdot\}$. That is, show that

$$\{g, f|T_k(\ell)\} = \ell^{2k-2} \{g|T_{2-k}(\ell), f\}$$

for any $g \in S_{2-k,\overline{\rho}}$ and $f \in H_{k,\rho}$.

Problem 10. ($\star\star$) Let $g \in S_{2-k,\overline{\rho}}$, $f \in H_{k,\rho}$, and suppose $\{g, f\} = 1$ and $\{g', f\} = 0$ for all $g' \in S_{2-k,\overline{\rho}}$ orthogonal to g. Show that $\xi_k(f) = \|g\|^{-2} g$, where $\|\cdot\|$ denotes the Petersson norm.

Problem 11. ($\star\star$) Let $F \subset \mathbb{C}$ be a subfield, and $g \in S_{2-k,\overline{\rho}}(F)$ a newform. (Here, $S_{k,\rho}(F)$ denotes those forms with Fourier coefficients in the field F.) Show that there is some $f \in H_{k,\rho}(F)$ such that

$$\xi_k(f) = \|g\|^{-2} g.$$

Problem 12. ($\star\star$)

(a) Let $f(\tau) := \sum_{n=h}^{\infty} a_f(n) q^n$ be meromorphic in a neighborhood of $q = 0$, and suppose $a_f(h) = 1$. Prove there exist unique numbers $c(n)$ such that

$$f(\tau) = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$.

(b) Prove that

$$\frac{\Theta(f)}{f} = h - \sum_{n=1}^{\infty} \sum_{d|n} c(d) d q^n,$$

where the Ramanujan Θ-operator is defined by

$$\Theta\left(\sum_{n=m}^{\infty} b(n) q^n\right) = \sum_{n=m}^{\infty} n b(n) q^n.$$

(Equivalently, $\Theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$.)

Problem 13. ($\star\star\star$) The Eisenstein series $E_4(\tau)$ is defined by

$$E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

where $\sigma_m(n) := \sum_{d|n} d^m$. Without using the previous exercise, prove that $E_4(\tau)$ satisfies

$$E_4(\tau) = \prod_{n=1}^{\infty} (1 - q^n)^{c(n^2)},$$

where

$$g(\tau) = \sum_{n \geq -3} c(n) q^n = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + 1707264q^8 \dots.$$

Investigate this property with respect to $E_6(\tau)$ and $E_{12}(\tau)$ as well.

The next two problems concern Borcherds products and mock theta functions. Consider one of Ramanujan's mock theta functions,

$$\omega(q) = \sum_{n=0}^{\infty} a_\omega(n) q^n := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n}{(q;q^2)_{n+1}},$$

where the last equality follows from a q-hypergeometric identity of Fine. Note that by Problem 48, we have a combinatorial interpretation of the coefficients $a_\omega(n)$ of the mock theta function $\omega(q)$. Consider the functions

$$L_\omega(q) := \sum_{n\geq 1} \widehat{\sigma}_\omega(n) q^n, \qquad \widetilde{L}_\omega(q) := \sum_{\substack{n\geq 1 \\ \gcd(n,6)=1}} \widehat{\sigma}_\omega(n) q^n,$$

where the divisor-like function $\widehat{\sigma}_\omega$ is defined on $\mathbb{N}$ by

$$\widehat{\sigma}_\omega(n) := \sum_{1\leq d|n} \left(\frac{d}{3}\right) \chi\left(\frac{n}{d}\right) d\cdot a_\omega\left(\frac{2d^2-2}{3}\right),$$

and $\chi(m) := \left(\frac{-8}{m}\right)$ is defined by the Jacobi symbol.

Problem 14. ($\star\star$) Define the "Borcherds product"

$$B_\omega(\tau) := \prod_{m=1}^{\infty} \left(\frac{1+\sqrt{-2}q^m - q^{2m}}{1-\sqrt{-2}q^m - q^{2m}}\right)^{-4\left(\frac{m}{3}\right)a_\omega\left(\frac{2m^2-2}{3}\right)}$$

from the coefficients $a_\omega(n)$ of the mock theta function $\omega(q)$. Using results in [4], argue that $B_\omega(\tau)$ is a modular form of level 6 and weight 0.

The next exercises will establish that $L_\omega(q)$ and $\widetilde{L}_\omega(q)$ are in fact weight 2 modular forms.

Problem 15. ($\star\star$)

(a) Prove that

$$\frac{\Theta(B_\omega(\tau))}{B_\omega(\tau)} = -8\sqrt{-2}L_\omega(q),$$

where Θ is the operator defined previously within §1.

(b) Deduce that $L_\omega(q)$ is modular of weight 2.

(c) Using the operators U_ℓ and V_ℓ defined by

$$\sum b(n)q^n|U_\ell := \sum b(\ell n)q^n, \qquad \sum b(n)q^n|V_\ell := \sum b(n)q^{\ell n},$$

deduce that $\widetilde{L}_\omega(q)$ is a modular form of weight 2.

2. Mock Jacobi forms

Problem 16. ($\star$) Let $e(z) := e^{2\pi i z}$. For $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, define the Mordell integral

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e(\tau x^2/2)e^{-2\pi z x}}{\cosh(\pi x)} dx.$$

Prove that

$$h(z;\tau) + e(-z)q^{-1/2}h(z+\tau;\tau) = 2e(-z/2)q^{-1/8}.$$

Problem 17. ($\star\star$) Prove that

$$h(z;\tau) + h(z+1;\tau) = \frac{2}{\sqrt{-i\tau}} e^{\pi i (z+1/2)^2/\tau}.$$

Problem 18. ($\star\star$) Prove that $h(z;\tau)$ is the unique holomorphic function (in z) satisfying the properties from the previous two problems.

For $\tau \in \mathbb{H}$, and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, define

$$\mu(u,v;\tau) := \frac{e(u/2)}{\vartheta(v;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} e(nv)}{1 - q^n e(u)},$$

where the Jacobi ϑ-function is defined by

$$\vartheta(z;\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e\left(\left(n+1/2\right)\left(z+1/2\right)\right).$$

Problem 19. ($\star$) Prove that $\mu(u,v;\tau) + e(v-u)q^{-1/2}\mu(u+\tau,v) = -ie((v-u)/2)q^{-1/8}$.

Problem 20. ($\star$) Prove that $\mu(u,v;\tau)$ is a meromorphic function in the variable u, with simple poles for $u \in \mathbb{Z}\tau + \mathbb{Z}$, and residue $-1/(2\pi i \vartheta(v;\tau))$ at $u = 0$.

Problem 21. ($\star\star$) Prove that

$$\frac{1}{\sqrt{-i\tau}} e^{\pi i (u-v)^2/\tau} \mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) + \mu(u,v;\tau) = \frac{1}{2i} h(u-v;\tau).$$

Let

$$R(u;\tau) := \sum_{n\in\mathbb{Z}} \left\{ \operatorname{sgn}\left(n+\frac{1}{2}\right) - E\left(\left(n + \frac{\operatorname{Im}(u)}{\operatorname{Im}(\tau)} + \frac{1}{2}\right)\sqrt{2y}\right) \right\} (-1)^n q^{-\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e\left(-\left(n+\frac{1}{2}\right)u\right),$$

where

$$E(z) := 2\int_0^z e^{-\pi t^2} dt, \quad z \in \mathbb{C}.$$

Problem 22. ($\star$) Prove that

$$R(u;\tau) + e(-u)q^{-\frac{1}{2}}R(u+\tau;\tau) = 2e\left(-u/2\right)q^{-\frac{1}{8}}.$$

Problem 23. ($\star\star$) Prove that

$$\frac{1}{\sqrt{-i\tau}}e^{\pi i u^2/\tau}R\left(\frac{u}{\tau};-\frac{1}{\tau}\right) + R(u,\tau) = h(u;\tau).$$

Problem 24. ($\star$) Problem 21 establishes a key property of the function $\mu(u,v;\tau)$, namely, it shows precisely how $\mu(u,v;\tau)$ falls short of transforming like a Jacobi form (see [5]). Use the function $R(w;\tau)$ to construct a new function $\widetilde{\mu}(u,v;\tau)$ from $\mu(u,v;\tau)$ that corrects the "error to modularity" exhibited by $\mu(u,v;\tau)$ in Problem 21. Discuss the analytic properties of the new function $\widetilde{\mu}(u,v;\tau)$.

Problem 25. ($\star\star$) Prove that under suitable specializations of parameters, the Mordell integral can be expressed in a different manner, i.e. show that for $u = 0$,

$$-h(0;\tau) = \int_0^{i\infty} \frac{\theta(u)}{\sqrt{-i(u+\tau)}}du,$$

where the modular theta function

$$\theta(\tau) := \sum_{v\in\frac{1}{2}+\mathbb{Z}} vq^{v^2/2}e(v/2).$$

Problem 26. ($\star\star$) Similarly, prove that

$$R\left(\frac{\tau}{4};\tau\right) = -\zeta_4 q^{\frac{1}{32}}\int_{-\overline{\tau}}^{i\infty} \frac{\sum_{n\in\mathbb{Z}}(-1)^n\left(n+\frac{3}{4}\right)e\left(\frac{1}{2}\left(n+\frac{3}{4}\right)^2 z\right)}{\sqrt{-i(z+\tau)}}dz,$$

where $\zeta_m := e^{2\pi i/m}$.

Let $f(q)$ denote one of Ramanujan's mock theta functions, defined by

$$f(q) := \sum_{n\geq 0}\frac{q^{n^2}}{(-q;q)_n^2}.$$

(For the next 5 problems, see also §3.)

Problem 27. ($\star\star$)

(a) Prove that

$$f(q)=\frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n^2+n}{2}}}{1+q^n}.$$

(b) Prove that

$$\frac{iq^{-1/24}}{2}f(q)=\frac{\eta^3(3\tau)}{\eta(\tau)\vartheta(3/2;3\tau)}-\mu(3/2;-\tau;3\tau)-\mu(3/2,\tau;3\tau),$$

where $\eta(\tau)$ is Dedekind's η-function, and $\vartheta(z;\tau)$ is defined previously within §2.

Problem 28. ($\star\star$) Prove that $q^{-1/24}f(q)$ is a weight 1/2 mock modular form with shadow proportional to

$$\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)n\cdot e(n/4)\cdot q^{n^2/24}.$$

Consider the "universal" mock theta functions of Gordon and McIntosh

$$g_2(w;q):=\sum_{n\geq 0}\frac{(-q;q)_n q^{n(n+1)/2}}{(w;q)_{n+1}(q/w;q)_{n+1}},$$

$$g_3(w;q):=\sum_{n\geq 0}\frac{q^{n(n+1)}}{(w;q)_{n+1}(q/w;q)_{n+1}}.$$

Problem 29. ($\star\star\star$)

(a) For $\alpha\in\mathbb{C}$, $\alpha\notin\mathbb{Z}\tau+\frac{1}{2}\mathbb{Z}$, prove that

$$e(\alpha)g_2(w;q)=\frac{\eta^4(2\tau)}{i\eta^2(\tau)\vartheta(2\alpha;2\tau)}+e(\alpha)q^{-1/4}\mu(2\alpha,\tau;2\tau).$$

(b) Prove that for $\zeta\neq 1$ a root of unity, $\zeta g_2(\zeta;q)+1/2$ is a mock modular form of weight 1/2 with shadow proportional to

$$\sum_{n\in\mathbb{Z}}(-1)^n n\zeta^{-2n}q^{n^2}.$$

Consider another of Ramanujan's mock theta functions

$$\psi(q) := \sum_{n\geq 1} \frac{q^{n^2}}{(q;q^2)_n}.$$

Problem 30. ($\star\star$) Prove that

$$f(-q) + 4\psi(q) = \frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3 (q^4;q^4)_\infty^3} =: c(q),$$

and deduce that $q^{-1/24}(f(-q)+4\psi(q))$ is a modular form of weight 1/2 on a congruence subgroup.

Problem 31. ($\star\star$)

(a) Find 3 different mock modular forms of weight 1/2 with the same shadow as $f(q)$.

(b) Use these mock modular forms to create non-trivial, and different, modular forms.

3. q-HYPERGEOMETRIC SERIES

Let $(a_1, a_2, \ldots, a_r; q)_n := \prod_{j=1}^{r} (a_j;q)_n$. The q-hypergeometric series are defined by

$${}_r\phi_s \left(\begin{matrix} a_1, & a_1, & \ldots & a_r \\ b_1, & b_2, & \ldots & b_s \end{matrix} \; q; z \right) := \sum_{n\geq 0} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s, q; q)_n} ((-1)^n q^{\frac{n(n-1)}{2}})^{1+s-r} z^n$$

where $r, s \in \mathbb{N}_0$, $|z| < 1, |q| < 1, b_j \neq q^{-m}$ for any $m \in \mathbb{N}_0$. The celebrated Watson-Whipple transformation is given by

$$\begin{aligned} &{}_8\phi_7 \left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N} \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1} & \end{matrix} ; q; \frac{a^2 q^{N+2}}{bcde} \right) \\ &= \frac{(aq;q)_N (aq/de;q)_N}{(aq/d;q)_N (aq/e;q)_N} {}_4\phi_3 \left(\begin{matrix} aq/bc, & d, & e, & q^{-N} \\ deq^{-N}/a, & aq/b, & aq/c & \end{matrix} ; q; q \right) \end{aligned}$$

The Watson-Whipple q-hypergeometric transformation formula leads to the following identity

$$\begin{aligned} &\sum_{n\geq 0} \frac{(\alpha,\beta,\gamma,\delta,\epsilon;q)_n (1-\alpha q^{2n}) q^{n(n+3)/2}}{(\alpha q/\beta, \alpha q/\gamma, \alpha q/\delta, \alpha q/\epsilon, q;q)_n (1-\alpha)} \left(-\frac{\alpha^2}{\beta\gamma\delta\epsilon} \right)^n . \\ &= \frac{(\alpha q, \alpha q/(\delta\epsilon);q)_\infty}{(\alpha q/\delta, \alpha q/\epsilon;q)_\infty} \sum_{n\geq 0} \frac{(\delta, \epsilon, \alpha q/(\beta\gamma);q)_n}{(\alpha q/\beta, \alpha q/\gamma, q;q)_n} \left(\frac{\alpha q}{\delta\epsilon} \right)^n . \end{aligned}$$

Problem 32. ($\star\star$) Prove that

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2}}{1-wq^n} = \frac{(q;q)_\infty^2}{(w;q)_\infty (q/w;q)_\infty}.$$

For the next two problems, see also §2.

Problem 33. ($\star\star$) Prove that the q-hypergeometric "universal" mock theta functions defined in §2 satisfy

$$g_2(w;q) = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(n+1)}}{1-wq^n},$$
$$g_3(w;q) = \frac{1}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{3n(n+1)/2}}{1-wq^n}.$$

Problem 34. ($\star\star$) Let $\alpha \in \mathbb{C} \setminus (\mathbb{Z}\tau + \frac{1}{2}\mathbb{Z})$. Prove that

$$e(\alpha)(g_2(e(\alpha);q) + g_2(-e(\alpha);q)) = 2\frac{\eta^4(2\tau)}{i\eta^2(\tau)\vartheta(2\alpha;2\tau)},$$

where $\vartheta(z;\tau)$ is the Jacobi ϑ-function defined in §2, and $e(z) := e^{2\pi i z}$.

Problem 35. ($\star\star\star$) Prove Ramanujan's ${}_1\psi_1$ summation formula

$${}_1\psi_1(\alpha,\beta;q;z) := \sum_{n\in\mathbb{Z}} \frac{(\alpha;q)_n}{(\beta;q)_n} z^n = \frac{(\beta/\alpha, \alpha z, q/(\alpha z), q; q)_\infty}{(q/\alpha, \beta/(\alpha z), \beta, z; q)_\infty}$$

for $|\beta/\alpha| < |z| < 1$.

Problem 36. ($\star\star$) Define

$${}_2\psi_2\left(\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix}\; q,\; z\right) := \sum_{n\in\mathbb{Z}} \frac{(a_1;q)_n (a_2;q)_n}{(b_1;q)_n (b_2;q)_n} z^n.$$

Prove Bailey's ${}_2\psi_2$ summation formula

$${}_2\psi_2\left(\begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix}\; q,\; z\right) = \frac{(\frac{b_2 q}{a_1 a_2 z};q)_\infty (\frac{b_1}{a_2};q)_\infty (a_1 z;q)_\infty (\frac{b_2}{a_1};q)_\infty}{(\frac{q}{a_2};q)_\infty (\frac{b_1 b_2}{a_1 a_2 z};q)_\infty (b_2;q)_\infty (z;q)_\infty} \cdot {}_2\psi_2\left(\begin{matrix} \frac{a_1 a_2 z}{b_2} & a_1 \\ a_1 z & b_1 \end{matrix}\; q,\; \frac{b_2}{a_1}\right).$$

4. Partition theory

Problem 37. ($\star$) Let S be a set of positive integers.

(a) Show that
$$P_S(q) := \sum_{n\geq 0} p_S(n)q^n = \prod_{n\in S} \frac{1}{1-q^n}$$
for $|q|<1$, where $p_S(n) :=$ number of partitions of n with parts in S.

(b) Find sets S for which $P_S(q)$ is modular (when $q = e(\tau)$).

Problem 38. ($\star$) Prove for $|z|<1$, that
$$1 + \sum_{n\geq 1} \frac{z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m\geq 0}(1-zq^m)^{-1}.$$

Problem 39. ($\star\star$) Show that
$$\sum_{n\geq 0} p(n)q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n\geq 0}\frac{q^n}{(q;q)_n},$$
where $p(n) := \#\{\text{number of partitions of } n\}$.

Problem 40. ($\star\star$)

(a) If $p_m(n) :=$ number of partitions of n with at most m parts, show that $p_m(n) \leq (n+1)^m$ for each $m>0$.

(b) Show that $\lim\limits_{n\to\infty} p(n)^{1/n} = 1$.

(c) Deduce that $\prod\limits_{k\geq 1}\frac{1}{1-q^k}$ converges for $|q|<1$.

(d) Prove that $\sum\limits_{n\geq 0} p(n)q^n = \prod\limits_{k\geq 1}\frac{1}{1-q^k}$.

Problem 41. ($\star$) Let S be a subset of $\mathbb{N}$. Prove that
$$\sum_{n\geq 0} p(n \mid \text{distinct parts in } S)q^n \;=\; \prod_{m\in S}(1+q^m).$$

Problem 42. ($\star$) Prove that

$$\begin{aligned} &p(n \mid \text{distinct parts congruent to 1,2,4} \mod 7)\\ &\quad = p(n \mid \text{parts congruent to 1,9,11} \mod 14). \end{aligned}$$

Problem 43. ($\star\star$) Prove that

$$1 + \sum_{n\geq 1}(p_e(n) - p_o(n))q^n = \prod_{m\geq 1}(1-q^m),$$

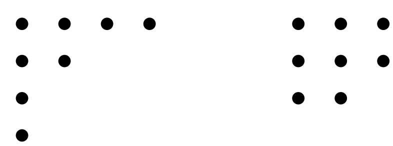
where
$p_e(n) := p(n \mid \text{even number of distinct parts})$,
$p_o(n) := p(n \mid \text{odd number of distinct parts})$.

Problem 44. ($\star$) Prove that

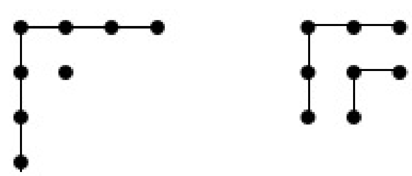
$$p(n \mid \text{all parts are odd}) \equiv 0 \mod 2$$

except when $n = j(3j \pm 1)/2,\ j \in \mathbb{Z}$.

Problem 45. ($\star$) A partition is self conjugate if it is equal to its conjugate. For example, the two self-conjugate partitions of 8 (4+2+1+1, and 3+3+2) are represented as:



By connecting dots lying on successive right angles, we obtain two new partitions of 8 (7 + 1, and 5 + 3) as follows:



Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

Problem 46. ($\star$) This problem is concerned with finding exact expressions for restricted partition numbers. Suppose $T = \{1, 2, 3\}$, and $\rho := e^{2\pi i/3}$.

(a) Verify the following generating function for $p_T(n) := p(n \mid \text{parts in T})$:

$$\sum_{n\geq 1} p_T(n)q^n = \frac{1}{6(1-q)^3} + \frac{1}{4(1-q)^2} + \frac{17}{72(1-q)} + \frac{1}{8(1+q)} + \frac{1}{9(1-\rho q)} + \frac{1}{9(1-\rho^2 q)}.$$

(b) Show that this implies

$$\begin{aligned} p_T(n) &= \frac{(n+2)(n+1)}{12} + \frac{n+1}{4} + \frac{17}{72} + \frac{(-1)^n}{8} + \frac{1}{9}(\rho^n + \rho^{2n}) \\ &= \frac{(n+3)^2}{12} + r(n), \end{aligned}$$

where $|r(n)| < \frac{1}{2}$.

(c) Deduce that $p_T(n)$ is the nearest integer to $\dfrac{(n+3)^2}{12}$.

Problem 47. ($\star\star$) The rank of a partition is defined to be its largest part of the partition minus the number of its parts. Let $N(n,m) := \#\{\text{partitions of } n \text{ with rank } m\}$.

(a) Show that

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} N(n,m) z^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(zq;q)_n (q/z;q)_n}.$$

(b) Determine a combinatorial interpretation for the coefficients of the mock theta function $f(q)$ defined in §2.

Problem 48. ($\star$) Determine a combinatorial interpretation for the coefficients $a_\omega(n)$ of the mock theta function $\omega(q)$ as defined in §1. (Check your interpretation by explicitly looking at partitions of a few small integers.)

5. Quantum modular forms

Consider another of Ramanujan's mock theta functions

$$\phi(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}.$$

The functions $c(q)$ and $\psi(q)$ are defined in §2. Take note of the singularites of the functions ϕ and ψ. The exercises in this section establish the following proposition, studied by Robert C. Rhoades.

Proposition. *As $q \to \zeta$ radially from within the unit disk, where ζ is a primitive $4k$th root of unity, we have that*

$$\lim_{q\to\zeta}(\phi(q) - c(q)) = -2\sum_{n\geq 0} \zeta^{n+1}(-\zeta^2;\zeta^2)_n = -\psi(\zeta).$$

Moreover, as $q \to \rho$ radially from within the unit disk, where ρ is a primitive odd order root of unity, we have that and

$$\lim_{q\to\rho}(\psi(q) - c(q)/2) = -\frac{1}{2}\left(1 + \sum_{n\geq 0}(-1)^n \rho^{2n+1}\left(\rho;\rho^2\right)_n\right).$$

Problem 49. ($\star\star$) Prove that

$$\psi(q) = \sum_{n=0}^{\infty} q^{n+1}(-q^2;q^2)_n.$$

Problem 50. ($\star\star$) Prove that

$$\phi(q) + 2\psi(q) = c(q).$$

Problem 51. ($\star\star$) Prove for any primitive $4k$th root of unity ζ, we have

$$\lim_{q\to\zeta}\left(\phi(q) - c(q)\right) = -2\sum_{n\geq 0}\zeta^{n+1}(-\zeta^2;\zeta^2)_n.$$

Problem 52. ($\star\star$) Prove that

$$\phi(q) = 1 + \sum_{n\geq 0}(-1)^n q^{2n+1}(q;q^2)_n.$$

Problem 53. ($\star\star$) Prove for any primitive odd order root of unity ρ, we have

$$\lim_{q\to\rho}(\psi(q) - c(q)/2) = -\frac{1}{2}\left(1 + \sum_{n\geq 0}(-1)^n \rho^{2n+1}(\rho;\rho^2)_n\right).$$

Problem 54. ($\star\star$) What can you say about the series

$$\sum_{n\in\mathbb{Z}}(-q^2)^n(q;q^2)_n?$$

References

[1] G.E. Andrews and K. Eriksson, *Integer partitions,* Cambridge University Press, Cambridge, (2004).

[2] R.E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products,* Invent. Math. 120 (1995), 161-213.

[3] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. 125 (2004), 45-90.

[4] J.H. Bruinier and K. Ono, *Heegner divisors, L-functions, and Maass forms,* Ann. of Math., 172 (2010), 2135-2181.

[5] M. Eichler and D. Zagier, *The theory of Jacobi forms,* Progress in Mathematics, 55. Birkhäuser Boston, Boston, MA, (1985).

[6] B. Gordon and R.J. McIntosh, *A survey of the classical mock theta functions,* Dev. Math. 23 (2012), Springer, 95-144.

[7] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Proceedings of the 2008 Harvard-MIT Current Developments in Mathematics Conference, International Press, Somerville, MA, 2009, 347-454.

[8] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann),* Séminaire Bourbaki, Vol. 2007/2008. Astérisque No. 326 (2009), Exp. No. 986, vii-viii, 143-164 (2010).

[9] S. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, (2002).

Department of Mathematics, Yale University, P.O. Box 208283, New Haven, CT 06520-8283
E-mail address: amanda.folsom@yale.edu