Group actions on curves and the lifting problem
Arizona winter school 2012: project description

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This document contains a more detailed description of the project for the Arizona Winter School. It may also be regarded as a leitfaden for preparing for the talks.

The problem we are concerned with in our lectures and which we shall refer to as the *lifting problem* was originally formulated by Frans Oort in [1]. To state it, we fix an algebraically closed field κ of positive characteristic p. Let $W(\kappa)$ be the ring of Witt vectors over κ. Throughout our notes, $\mathfrak{o}$ will denote a finite local ring extension of $W(\kappa)$ and $k = \mathrm{Frac}(\mathfrak{o})$ the fraction field of $\mathfrak{o}$. Note that $\mathfrak{o}$ is a complete discrete valuation ring of characteristic zero with residue field κ.

Definition 1 Let C be a smooth proper curve over κ. Let $G \subset \mathrm{Aut}_\kappa(C)$ be a finite group of automorphisms of C. We say that the pair (C, G) *lifts to characteristic zero* if there exists a finite local extension $\mathfrak{o}/W(\kappa)$, a smooth projective $\mathfrak{o}$-curve $\mathcal{C}$ and an $\mathfrak{o}$-linear action of G on $\mathcal{C}$ such that

(a) $\mathcal{C}$ is a lift of C, i.e. there exists an isomorphism $\lambda : \mathcal{C} \otimes_{\mathfrak{o}} \kappa \cong C$, and

(b) the G-action on $\mathcal{C}$ restricts, via the isomorphism λ, to the given G-action on C.

Problem 2 (The lifting problem) Which pairs (C, G) as in Definition 1 can be lifted to characteristic zero?

Rather than considering pairs (C, G), one may consider the corresponding Galois cover $C \to D := C/G$. A variant of Problem 2 is to ask for which groups G all pairs (C, G) lift to characteristic zero. This is known classically for so-called tame actions, i.e. in the case that the characteristic p of κ does not divide the ramification indices of the associated cover $C \to D$. Therefore, we may restrict to wild actions, i.e. the case that p divides some of the ramification indices of $C \to D$. Here it is known that all G-covers with $G = \mathbb{Z}/p\mathbb{Z}$ or $G = D_p$ may be lifted to characteristic zero. On the other hand, it is known that for many groups G there exist pairs (C, G) that do not lift. Interesting examples discussed in the notes are for $G = Q_8$ (in the case that $p = 2$) and $G = (\mathbb{Z}/p\mathbb{Z})^n$ (for p odd and $n > 1$).

It is known that to solve the Lifting Problem (Problem 2), one may restrict to lifting local actions $(\hat{\mathcal{O}}_{C,y} \simeq \kappa[[z]], G_y)$ (see § 1.2 of the notes). With such a local action, we may associate a so-called Katz–Gabber cover $f_0 : C_0 \to D_0 \simeq \mathbb{P}^1_\kappa$ (§ 4.1), which is Galois with Galois group the decomposition group G_y. Locally, in one point y this cover is isomorphic to the given local action. The cover is uniquely determined by this and the requirement that the genus of C_0 is as small as possible. (In particular, the cover f_0 is tame outside y.) To solve the lifting problem, we may therefore restrict to such Katz–Gabber covers.

The main goal of the project is to prove the following result.

Conjecture 3 Let G be the alternating group A_4 and κ an algebraically closed field of characteristic 2. Then every G-action (C, G) over κ lifts to characteristic zero.

The notes prove an analogous statement for Galois covers in characteristic p with Galois group $\mathbb{F}_p \rtimes \mathbb{Z}/m\mathbb{Z}$. In this case the statement is slightly different: not all covers lift, but one can exactly characterize those that do.

Step I : Higher ramification groups The first step of the project is to describe A_4-covers $f : C \to \mathbb{P}^1_\kappa$ in characteristic 2 of Katz–Gabber type. (In particular, they are only wildly branched at one point ∞). Rather than describing all such covers by equations, it suffices for our purposes to study the wild ramification. An important invariant measuring how "wild" the ramification of a given cover f is, is the filtration of higher ramification groups. Associated with this are the Swan and Artin conductors. In the notes, one finds a short recapitulation of these concepts in § 2 of the notes (Situation A: the classical case). A more detailed description can be found in [2, Chapter 4].

The notes contain several sets of exercises on higher ramification groups. The case $G = \mathbb{Z}/p^n\mathbb{Z}$ in characteristic p is worked out in Exercises 2.11, 2.12 and 2.21. Exercise 2.23 describes filtrations of higher ramification groups of certain Q_8-Galois covers in characteristic 2. These exercises are a good preparation for the lectures, as higher ramification groups will be considered known in our talks.

(Project a) Classify all local A_4-action $A_4 \subset \mathrm{Aut}_\kappa(\kappa[[z]])$ over an algebraically closed field κ of characteristic 2 in terms of filtration of higher ramification groups (or equivalently, in terms of the Artin conductor). This part of the project is described in Exercises 2.13, 2.22, 3.19, and 3.20.

Step II: Hurwitz trees Let $f : C \to \mathbb{P}^1_\kappa$ be a wildly branched G-Galois cover. Suppose that f lifts to characteristic zero. Let e_P be the ramification index of a branch point P. The Riemann-Hurwitz formula implies that the contribution of a wild branch point P to $g(C)$ is $h_P \cdot (e_P - 1)/2$ for some $h_P \geq 2$, where h_P may be computed in terms of the higher ramification groups of P. This is in contrast to the situation in characteristic zero, where this contribution is $(e_P - 1)/2$. This implies that to lift f one has to lift the branch point P to h_P distinct points in characteristic zero. To find a lift of the cover f, these lifts of P need to have exactly the "right" p-adic distance. This makes it hard to lift f by explicitly writing down equations. In our project, we will therefore take a different route, namely we will use so-called *Hurwitz trees*.

The motivation for the definition of the Hurwitz trees comes from the theory of stable reduction. The definition of the stable reduction of a Galois cover can be found in § 3.1 of the notes. The notes contain two worked out examples. Example 3.8 computes the stable reduction of a p-cyclic cover branched at three points, where p is an odd prime. In Section 3.2, we discuss the reduction of a Q_8-cover to characteristic 2. The idea behind a Hurwitz tree is very important in the project. Therefore studying these examples is highly recommended as a preparation for the lectures and working on the project.

Let $f_0 : C \to D$ be a G-Galois cover in characteristic p, and suppose that f lifts to a cover $f_k : Y \to X \simeq \mathbb{P}^1_k$ over a field k of characteristic zero. We denote the stable reduction of f_k to characteristic p by $\bar{f} : \bar{Y} \to \bar{X}$. Excluding trivial cases, the curve $\bar{Y}$ will contain C as an irreducible component, but $\bar{Y}$ will also have other irreducible components. Let $\bar{Y}_i \neq C$ be one of these, and $\bar{X}_i$ the corresponding irreducible component of $\bar{X}$. Then the induced cover $\bar{Y}_i \to \bar{X}_i$ will be inseparable. With such an inseparable cover, we associate the differential data. These are combinatorial data which may be described purely in characteristic p.

Let L (resp. K) by the completion of the function field of Y (resp. X) at the generic points of $\bar{Y}_i$ (resp. $\bar{X}_i$). We may restrict to the case that the residue field extension of L/K is purely inseparable. In this case there exists a theory of higher ramification groups and Swan conductors analogous to that in the classical situation considered in Step I. This is described in § 2 of the notes (Situation B: the case of residual dimension one.) Especially important here is the definition and properties of the differential Swan conductor in § 2.4 of the notes.

Best understood is the case that $[L : K] = p$. In this case the differential data consists of a differential forms. This case is described more detailedly in § 2.5 of the notes. Concrete examples of differential data associated with the stable reduction of a Galois cover can be found in Exercises 3.16, 3.18 and 3.19.

For $G = A_4$, which we consider in the project, the differential data do not consist of a single differential form, but form a vector space of differential forms. While this case is slightly more difficult to understand than the case of degree p, it is still similar enough that one may adapt the methods. A concrete example of such a vector space of differential forms is constructed in Exercise 3.20.

Step III The Bertin Obstruction Suppose given a wildly branched G-Galois cover $f_0 : C \to D$ over κ. The Bertin Obstruction, described in § 4.1 of notes, is a first necessary condition for liftability of the cover. It gives a condition on the Artin representation corresponding to a wild ramification point of the cover. In Corollary 4.3 of the notes it is shown that the Bertin Obstruction vanishes for covers with Galois group $\mathbb{Z}/p^n\mathbb{Z}$ in characteristic $p > 0$. Further concrete instances of the Bertin Obstruction are discussed in Proposition 4.4 ($G = \mathbb{F}_p \rtimes \mathbb{Z}/m\mathbb{Z}$) and Exercise 4.7-4.8 ($G = Q_8$).

(Project b) Show that the Bertin Obstruction vanishes for every local A_4-action (Exercise 4.6).

Step IV Obstruction coming from the existence of a Hurwitz tree Suppose given a wildly branched G-Galois cover $f_0 : C \to D$ over κ. We have seen in Step II that if f_0 lifts to characteristic zero, then we may associate with f_0 a set of differential data in characteristic p. A Hurwitz tree consist of these differential data, together with additional combinatorial data, which satisfy certain compatibility conditions. The existence of such a Hurwitz tree for a given cover in characteristic p is a necessary condition for liftability. Explicitly constructing a Hurwitz tree is also a first important step in constructing a lift.

In the case that $G = \mathbb{Z}/p\mathbb{Z}$ one can find the definition of a Hurwitz tree in the literature. We do not give the full definition in the notes, as this is rather long and not very enlightening. However, in this case it is known that the existence of such a Hurwitz tree is also sufficient for liftability.

For more general groups, we only find partial definitions of a Hurwitz tree. Though a good candidate for the combinatorial structure is known, it is not known which compatibility conditions one has to require to ensure liftability. Defining Hurwitz trees for A_4-covers in characteristic 2, may be part of the project.

In § 5 of the notes we discuss methods for constructing differential data in positive characteristic. We mainly focus on the case that the Sylow p-subgroup of G has order p. This case is best understood. The main result of this part of described in § 5.3, where we take $G = \mathbb{F}_p \rtimes \mathbb{Z}/m\mathbb{Z}$. Namely in that section, we explicitly construct Hurwitz trees for G-Galois covers. It turns out that one of the main tools is using solutions of hypergeometric differential equations in positive characteristic.

(Project c) Construct Hurwitz trees for every local A_4-action (Section 5.4).

The main goal of this part of the project is to adapt techniques known in the case that $G = \mathbb{F}_p \rtimes \mathbb{Z}/m\mathbb{Z}$ to the case $G = A_4$.

Step V Lifting the cover to characteristic zero In the final step of the project, we prove the actual lifting result. Here we use in an essential way the Hurwitz trees constructed in Step IV. The Hurwitz tree seperate the points which eventually will be the reduction of the branch points of the characteristic-0 cover. In some sense, the Hurwitz tree already determines the p-adic distance between the branch points of the lifted cover in characteristic zero.

With this preparation, one may use techniques from formal patching to reduce the problem to a local problem around these distinguished points. This local problem may be solved "explicitly". This part will eventually be described in § 6 of the notes, which is currently not yet available.

(Project d) Show that every Hurwitz tree constructed in (c) can be lifted to an A_4-action in characteristic zero. (Section 6; this part is not yet written.)

References

[1] S.J. Edixhoven, B.J.J Moonen, and F. Oort. Open problems in algebraic geometry. *Bull. Sci. Math.*, 125:1–22, 2001.

[2] J.-P. Serre. *Corps locaux*. Hermann, 1968.