Periods and special values of L-functions

Course outline

Kartik Prasanna

December 18, 2010

1 Introduction

This course and project will be centered on *period relations* for modular forms, more precisely on a specific conjecture about periods of Hilbert modular forms. The romantic background for this is provided by the Hodge and Tate conjectures that predict in certain instances the existence of interesting cycles on algebraic varieties. If true, they imply nontrivial relations between periods of algebraic differential forms up to algebraic or even rational factors. In the absence of direct progress on the Hodge and Tate conjectures, (and for other reasons: eg. the study of special values of L-functions), it is of some interest to verify these period relations directly.

A natural family of examples to work with is provided by (products of) Shimura varieties, since Langlands' conjectures on functoriality often predict that one can find the same *motive* on Shimura varieties attached to different groups. What is truly remarkable is that in investigating this problem, one finds that something much stronger is likely to be true: namely, if everything is properly normalized, then there should be *integral* relations between periods that are not in any way predicted by the Hodge/Tate conjectures. Further the integers that appear seem to be of arithmetic significance.

2 Periods of Hilbert (and quaternionic) modular forms

Let $F \subset \mathbb{C}$ be a totally real number field, $[F : \mathbb{Q}] = d$, and f a holomorphic Hilbert modular form on F. For simplicity, we will assume that f has parallel weight $(2, 2, \cdots, 2)$, trivial central character and square-free conductor $\mathfrak{n}$. [1] Let B be a quaternion algebra over F whose discriminant D_B divides $\mathfrak{n} \cdot \infty_1 \infty_2 \cdots \infty_d$, where the ∞_i are the different infinite places of F. Then f admits a *Jacquet-Langlands* transfer f_B to the group $B^\times$, that may be chosen to be a *newvector*. This defines f_B up to scaling. To fix the scaling factor, one notices that the form f_B may be viewed in a natural way as a section of an automorphic vector bundle on the Shimura variety X_B, and uses the fact that this variety and the vector bundle on it admit *canonical models* over a suitable number

[1] More generally, one could allow any set of *motivic* weights, i.e. $(k_1, \cdots, k_d)$ with $k_1 \equiv k_2 \equiv \cdots \equiv k_d \mod 2$ and no restriction at the finite places.

field, and even canonical integral models outside finitely many bad primes. It will be convenient in what follows to fix an embedding $\lambda : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ such that all the X_B have good reduction at λ, so that f_B may be fixed up to a λ-adic unit in a (totally real) number field.

Remark 2.1. At bad primes it is not quite clear how to pick f_B. What one needs is a canonical integral structure on the de Rham cohomology of the X_B and of the automorphic vector bundles on them. There is work in progress of Bhatt and Snowden [BS] that defines such canonical integral structures for smooth varieties over number fields. In fact, using their work it should be possible to give a global definition of f_B instead of working locally, one prime at a time.

We would like to understand how the periods of f_B, in particular the Petersson inner products $\langle f_B, f_B\rangle$ vary as B varies. In the early 80's, Shimura made the following conjecture (see [Sh]).

Conjecture 2.2. *Let Σ_∞ denote the set of infinite places of F. Then there exists a function $c : \Sigma_\infty \to \mathbb{C}^\times$, $v \mapsto c_v$, such that*

$$\langle f_B, f_B\rangle \sim_{\bar{\mathbb{Q}}^\times} \prod_{\substack{v\in\Sigma_\infty \\ v\nmid D_B}} c_v. \tag{1}$$

Remark 2.3. Since there are only d different c_v but many more quaternion algebras B, the conjecture above should be viewed as giving a large number of period relations between the $\langle f_B, f_B\rangle$. One should certainly expect that the c_v are transcendental and algebraically independent (unless the form f is special in some way: eg. comes from a base change from a smaller field).

Shimura proved some partial results towards this conjecture. For example, he showed that if B_1 and B_2 have complementary ramification at infinity, then

$$\langle f_{B_1}, f_{B_1}\rangle \cdot \langle f_{B_2}, f_{B_2}\rangle \sim_{\bar{\mathbb{Q}}^\times} \langle f, f\rangle. \tag{2}$$

A few years later, the full conjecture was mostly proved by Michael Harris [Har]. In fact, Harris shows it to hold under the hypothesis that the automorphic representation $\pi = \otimes\pi_v$ associated to f is discrete series for at least one finite place v i.e. if $\mathfrak{n} \neq (1)$ in our setup. This condition was removed by later by Yoshida using a trick involving base change.

Our interest is in formulating an integral version of this conjecture. Note that the conjecture above can equivalently be written as:

$$\langle f_B, f_B\rangle \sim_{\bar{\mathbb{Q}}^\times} \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_\infty \\ v\mid D_B}} c_v}. \tag{3}$$

Here is then the integral version of the above conjecture.

Conjecture 2.4. *([Pr2]) Suppose λ is not an Eisenstein prime for f i.e. the mod-λ representation $\bar{\rho}_{f,\lambda}$ associated to f is absolutely irreducible. Let $\Sigma(f)$ denote the set of places v of F such that π_v is discrete series (i.e. $\Sigma(f) = \Sigma_\infty \sqcup \{v : v \text{ divides } \mathfrak{n}\}$ in our setup.) Then there exists a function $c : \Sigma(f) \to \mathbb{C}^\times$, $v \mapsto c_v$, such that*

$$\langle f_B, f_B\rangle \sim_{\lambda-\text{units}} \frac{\langle f, f\rangle}{\prod_{v\mid D_B} c_v}. \tag{4}$$

Further for v finite, c_v is in $\bar{\mathbb{Q}}$ and is a λ-adic integer.

The main evidence for this is the following result (see [Pr1], §2.2.1), whose most significant input is a theorem of Ribet and Takahashi [RT] comparing degrees of modular parametrizations of elliptic curves over $\mathbb{Q}$ by different Shimura curves. This in turn uses all of the techniques that go into proving Ribet's famous level-lowering result [R].

Theorem 2.5. *Suppose $F = \mathbb{Q}$, $\mathfrak{n} = (N)$ with N a square-free positive integer so that there is an elliptic curve $E/\mathbb{Q}$ such that f corresponds to the isogeny class of E. Let B be an indefinite quaternion algebra over $\mathbb{Q}$ with $D_B \mid N$. Suppose p is not Eisenstein for f. Then*

$$\langle f_B, f_B \rangle \sim_{p-\text{units}} \frac{\langle f, f \rangle}{\prod_{q|D_B} c_q}, \tag{5}$$

where c_q is the order of the component group of the Néron model of E at q. (Since we are assuming E is semistable, we have $c_q = \operatorname{ord}_q(\Delta_E) = -\operatorname{ord}_q(j_E)$, where Δ_E is the minimal discrminant and j_E is the j-invariant of E.)

3 The lectures

We will roughly cover the following themes:

- Modular forms on $\mathbf{GL}(2)$ and quaternion algebras, congruences (raising and lowering the level), relation to adjoint L-value.
- Modular parametrizations, sketch of proof of Thm. 2.5 above.
- A sketch of partial result (from [Pr1]) in the same direction for forms of higher weight. This uses Waldspurger's formula [W] relating period integrals to L-values and some results from Iwasawa theory.
- Theta correspondence and see-saw pairs, applications of CM points to computing on Shimura curves.
- The conjecture over totally real fields up to algebraicity. Sketch of Harris's proof of algebraicity. This relies on understanding the arithmetic of theta correspondences for unitary groups.
- An outline of work in progress, joint with Ichino [IP], that gives a new approach in the totally real case. This gives a somewhat different proof of the algebraicity results without the technical hypothesis that the representation is discrete series at a finite place. It also seems promising for the study of integrality questions. There are connections here with Waldspurger's formula as well as triple product L-values. (There should also be connections therefore to the lectures of Darmon and Rotger.)

4 The project(s)

The main goal of the student projects will be to experimentally verify Conj. 2.4 in as many cases as possible. Along the way, one would like to develop new techniques to computationally study modular forms on quaternion algebras over totally real fields. We will go in the following order:

1. Definite quaternion algebras over $\mathbb{Q}$.

2. Totally definite quaternion algebras over totally real fields.

   In these first two cases, modular forms are just functions on certain finite double coset spaces, which makes them computationally easier to deal with. The next case already presents vexing computational problems.

3. Indefinite quaternion algebras over $\mathbb{Q}$. In this case, the conjecture is known to be true (Thm. 2.5 above). Nevertheless this is still a good case to look at since, firstly we can look at higher weight forms (where a similar conjecture should be true but is not known) and secondly, even in the weight 2 case, we can at least verify that our computational methods give the right answer. The difficulty here is that Shimura curves do not have cusps, so there are no q-expansions and it is hard to write down forms explicitly. We will try to deal with this using expansions at CM points.

4. If we can get past the previous stage, we will look at partially definite quaternion algebras over totally real fields.

   Another related problem:

5. Let $E \supset F$ be two totally real fields. Let B be a totally definite quaternion algebra over F, set $B' = B \otimes_F E$. Compute the base change $f_{B'}$ of f_B to B'. How are the periods of f_B and $f_{B'}$ related ?

   In addition, here are some more theoretical problems to look at:

6. Suppose Nq_1q_2 is a square-free rational integer. Let B_1 and B_2 be the definite quaternion algebras with discriminant $q_1\infty$ and $q_2\infty$ respectively. Let E be an elliptic curve $\mathbb{Q}$ with conductor Nq_1q_2 and f the corresponding modular form of weight 2. Suppose p is not Eisenstein for f. The show (say, using the geometry of Shimura curves and modular curves) that
$$c_{q_1} \cdot \langle f_{B_1}, f_{B_1} \rangle \sim_{p-\text{units}} c_{q_2} \cdot \langle f_{B_2}, f_{B_2} \rangle, \tag{6}$$
   where c_{q_i} is the order of the component group of the Néron model of E at q_i.

7. Study the relation between Conj. 2.4 and the Bloch-Kato conjecture for $L(1, \mathrm{ad}^0 f)$, the adjoint L-function of f at $s = 1$. In particular, give an interpretation for the c_v at finite primes v in terms of Tamagawa numbers for the adjoint motive.

8. Should the conjecture continue to be true at bad primes (that are not Eisenstein) ? If not, make a prediction for how it needs to be modified.

5 Background reading and preparation

- On modular forms and Galois representations for $\mathbf{GL}(2)$, congruences and the adjoint L-value: [DDT].
- On level-lowering and modular degrees, as well as problem 6: [R], [RT].
- For problems 1-5: learn how to compute with quaternion algebras in SAGE/MAGMA. Also for problem 3 and 4, Waldspurger's article [W]. It might also be a good idea to look at Elkies article "Shimura Curve Computations" [E] for background.
- For problem 7, [BK] and [DFG].
- Warmup exercise: use SAGE to verify that (6) holds in as many cases as possible. If you think you've found a counterexample, email me promptly !

References

[BS] Bhatt, Bhargav, and Snowden, Andrew; *Integral structures on de Rham cohomology*, In preparation.

[BK] Bloch, Spencer; Kato, Kazuya, *L-functions and Tamagawa numbers of motives.* The Grothendieck Festschrift, Vol. I, 333400, Progr. Math., 86, Birkhuser Boston, Boston, MA, 1990.

[DDT] Darmon, Henri; Diamond, Fred; Taylor, Richard; *Fermat's last theorem. Elliptic curves, modular forms and Fermat's last theorem*, (Hong Kong, 1993), 2140, Int. Press, Cambridge, MA, 1997.

[DFG] Diamond, Fred; Flach, Matthias; Guo, Li, *The Tamagawa number conjecture of adjoint motives of modular forms.* Ann. Sci. cole Norm. Sup. (4) 37 (2004), no. 5, 663727

[E] Elkies, Noam, *Shimura curve computations*, Lecture Notes in Computer Science 1423 (proceedings of ANTS-3, 1998; J.P.Buhler, ed.), 1-47

[Har] Harris, Michael, *L-functions of 2 2 unitary groups and factorization of periods of Hilbert modular forms*, J. Amer. Math. Soc. 6 (1993), 637719.

[IP] Ichino, Atsushi, and Prasanna, Kartik; *Periods of quaternionic Shimura varieties*, in preparation.

[Pr1] Prasanna, Kartik, *Integrality of a ratio of Petersson norms and level-lowering congruences.* Ann. of Math. (2) 163 (2006), no. 3, 901967.

[Pr2] Prasanna, Kartik, *Arithmetic aspects of the theta correspondence and periods of modular forms.* Eisenstein series and applications, 251269, Progr. Math., 258, Birkhuser Boston, Boston, MA, 2008.

[R] Ribet, K. A., *On modular representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms.* Invent. Math. 100 (1990), no. 2, 431476.

[RT] K. A. Ribet and S. Takahashi, *Parametrizations of elliptic curves by Shimura curves and by classical modular curves*, in Elliptic Curves and Modular Forms , Proc. Natl. Acad. Sci. U.S.A. 94 (1997), 1111011114.

[Sh] Shimura, Goro, *Algebraic relations between critical values of zeta functions and inner products.* Amer. J. Math. 105 (1983), no. 1, 253285.

[W] Waldspurger, J.-L. , *Sur les valeurs de certaines fonctions L automorphes en leur centre de symtrie.* (French) [Values of certain automorphic L-functions at their center of symmetry] Compositio Math. 54 (1985), no. 2, 173242.