Arithmetic Quantum Unique Ergodicity

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First Goal: Theorem (microlocal lift).

Let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$ be a lattice, and let $M = \Gamma \backslash \mathbb{H}$. Suppose that (ϕ_i) is an L^2-normalized sequence of eigenfunctions of Δ in $C^\infty(M) \cap L^2(M)$, with the corresponding eigenvalues λ_i satisfying $|\lambda_i| \to \infty$ as $i \to \infty$, and assume that the weak*-limit μ of $|\phi_i|^2 \,\mathrm{dvol}_M$ exists. If $\widetilde{\phi}_i$ denotes the sequence of lifted functions defined later, then (possibly after choosing a subsequence to achieve convergence) the weak*-limit $\widetilde{\mu}$ of $|\widetilde{\phi}_i|^2 \,\mathrm{d}m_X$ has the following properties:

[L] Projecting $\widetilde{\mu}$ on $X = \Gamma \backslash G$ to $M = \Gamma \backslash G / K$ gives μ.

[I] $\widetilde{\mu}$ is invariant under the right action of A.

The measure $\widetilde{\mu}$ is called a *microlocal lift* of μ, or a *quantum limit* of (ϕ_i).

Fourier series – the action of K

For $f \in C^\infty(X)$

$$f_n(x) = f *_K e_n(x) = \int_K f(xk_\theta)e_n(-k_\theta)\,\mathrm{d}m_K(k_\theta),$$

which is an eigenfunction under K in the sense that

$$f_n(xk_\psi) = f_n(x)e_n(k_\psi)$$

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We have $f = \sum_n f_n$.

K-finite functions

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K-finite functions

$$\mathcal{A}_n = \{f \mid f(xk_\theta) = e_n(k_\theta)f(x)\} = \{f : W * f = inf\}$$

f is K-finite if $f \in \sum_{n=-N}^{N} \mathcal{A}_n$ for some n.

Proposition.

For $m, w \in \mathfrak{sl}_2(\mathbb{R})$ we have

$$m \circ w - w \circ m = [m, w]$$

where $[m, w] = mw - wm$ is the Lie bracket, defined by the difference of the matrix products. More concretely, this means that

$$m * (w * f) - w * (m * f) = ([m, w]) * f$$

for any $f \in C^\infty(X)$.

Proposition.

Let

$$\mathcal{H} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \mathcal{U}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{U}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$\mathcal{W} = \mathcal{U}^+ - \mathcal{U}^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the degree-2 element – called the *Casimir element* (or *Casimir operator*) – defined by

$$\begin{aligned} \Omega_c &= \mathcal{H} \circ \mathcal{H} + \tfrac{1}{2} (\mathcal{U}^+ \circ \mathcal{U}^- + \mathcal{U}^- \circ \mathcal{U}^+) \\ &= \mathcal{H} \circ \mathcal{H} + \tfrac{1}{4} (\mathcal{U}^+ + \mathcal{U}^-) \circ (\mathcal{U}^+ + \mathcal{U}^-) - \tfrac{1}{4} \mathcal{W} \circ \mathcal{W}, \end{aligned}$$

is fixed under the action of $\mathrm{SL}_2(\mathbb{R})$ (equivalently, under the derived action of $\mathfrak{sl}_2(\mathbb{R})$).

For $f \in C^\infty(\Gamma\backslash\mathbb{H}) \subseteq C^\infty(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$, we have

$$\Omega_c * f = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = \Delta f. \tag{1}$$

The *raising operator* is the element of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$\mathcal{E}^+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \mathcal{H} + \frac{i}{2}(\mathcal{U}^+ + \mathcal{U}^-), \tag{2}$$

and the *lowering operator* is the element of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$\mathcal{E}^- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = \mathcal{H} - \frac{i}{2}(\mathcal{U}^+ + \mathcal{U}^-). \tag{3}$$

The operators $\mathcal{E}^+$ and $\mathcal{E}^-$ raise and lower in the sense that

$$\begin{aligned} \mathcal{E}^+ : \mathcal{A}_n &\rightarrow \mathcal{A}_{n+2}, \\ \mathcal{E}^- : \mathcal{A}_n &\rightarrow \mathcal{A}_{n-2} \end{aligned}$$

for all $n \in \mathbb{Z}$.

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We have $\overline{\mathcal{E}^+} = \mathcal{E}^-$, $\mathcal{E}^+ + \mathcal{E}^- = 4\mathcal{H}$, and

$$\Omega_c = \mathcal{E}^- \circ \mathcal{E}^+ - \frac{1}{4}\mathcal{W} \circ \mathcal{W} - \frac{i}{2}\mathcal{W} = \mathcal{E}^+ \circ \mathcal{E}^- - \frac{1}{4}\mathcal{W} \circ \mathcal{W} + \frac{i}{2}\mathcal{W}.$$

Adjoints of elements of $\mathfrak{sl}_2(\mathbb{C})$

Let $m \in \mathfrak{sl}_2(\mathbb{C})$, and consider $m*$ as an unbounded operator on $L^2(\mathrm{SL}_2(\mathbb{R}))$ with domain $C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. Then

$$(m*)^* f = -\overline{m} * f$$

for $f \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$.
By definition, this means that for $f_1, f_2 \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$ we have

$$\langle m * f_1, f_2 \rangle = -\langle f_1, \overline{m} * f_2 \rangle .$$

If $f \in \mathcal{C}$, then

$$\begin{aligned} (\Omega_c *)^* f &= \Omega_c * f, \text{ and} \\ (\mathcal{E}^{\pm} *)^* f &= -\mathcal{E}^{\mp} * f. \end{aligned}$$

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If $f \in \mathcal{C} \cap \mathcal{A}_n$ satisfies $\Omega_c \cdot f = \lambda f$ with $\lambda = -(\frac{1}{4} + r^2)$ then

$$\begin{aligned} \|\mathcal{E}^{+} * f\|_2 &= |ir + \tfrac{1}{2} + \tfrac{1}{2}n| \|f\|_2, \text{ and} \\ \|\mathcal{E}^{-} * f\|_2 &= |ir + \tfrac{1}{2} - \tfrac{1}{2}n| \|f\|_2. \end{aligned}$$

From now on ϕ has weight 0 and Δ-eigenvalue $\lambda = -(\frac{1}{4} + r^2)$

Inductively define functions by

$$\phi_0(x) = \phi(xK) \in \mathcal{A}_0,$$

and

$$\phi_{2n+2} = \frac{1}{ir + \frac{1}{2} + n}\mathcal{E}^+ * \phi_{2n} \in \mathcal{A}_{2n+2} \text{ for } n \geqslant 0, \tag{4}$$

$$\phi_{2n-2} = \frac{1}{ir + \frac{1}{2} - n}\mathcal{E}^- * \phi_{2n} \in \mathcal{A}_{2n-2} \text{ for } n \leqslant 0. \tag{5}$$

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$$\phi_{2n-2} = \frac{1}{ir + \frac{1}{2} - n}\mathcal{E}^- * \phi_{2n} \in \mathcal{A}_{2n-2} \text{ for } n \leqslant 0. \qquad (5)$$

These formulas hold for all n and $\|\phi_{2n}\| = 1$.

The lift

Define, for $N = N(\lambda)$ to be chosen later,

$$\widetilde{\phi} = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \phi_{2n},$$

Theorem

Let $\phi \in L^2(M)$ be an eigenfunction of Δ with corresponding eigenvalue $\lambda = -(\frac{1}{2} + r^2)$. If $f \in C_c^\infty(M)$, then

$$\begin{aligned}\int f|\widetilde{\phi}|^2 \, \mathrm{d}m_X &= \langle f\phi, \phi\rangle_{L^2(M)} + \mathrm{O}\left(Nr^{-1}\right) \\ &= \int f|\phi|^2 \, \mathrm{dvol}_M + \mathrm{O}\left(Nr^{-1}\right).\end{aligned}$$

More generally, if f is a K-finite function in $C_c^\infty(M)$, then

$$\int f|\widetilde{\phi}|^2 \, \mathrm{d}m_X = \left\langle f \sum_{n=-N}^{N} \phi_{2n}, \phi \right\rangle_{L^2(X)} + \mathrm{O}_f\left(\max\{N^{-1}, Nr^{-1}\}\right)$$

Corollary

Suppose that $N = N(\lambda)$ is a function of λ chosen so that $Nr^{-1} = \mathrm{O}(N|\lambda|^{-1/2}) \to 0$ as $|\lambda| \to \infty$. Assume also that (ϕ_i) is a sequence of Maass cusp forms, with corresponding eigenvalues $|\lambda_i| \to \infty$ as $i \to \infty$, and that $|\phi_i|^2 \, \mathrm{dvol}_M$ converges weak*. Then any weak*-limit of $|\widetilde{\phi}_i|^2 \, \mathrm{d}m_X$ projects to the weak*-limit of $|\phi_i|^2 \, \mathrm{dvol}_M$.

Theorem (Zelditch)

If $f \in C_c^\infty(X)$ is a K-finite function, and N is sufficiently large (the lower bound depends on f), then

$$\left\langle [(r\mathcal{H} + \mathcal{V}) * f] \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle = 0$$

for some fixed degree-two differential operator $\mathcal{V}$. In particular,

$$\left\langle (\mathcal{H} * f) \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle = \mathrm{O}_f(r^{-1}).$$

Corollary

Suppose that N is defined as a function of λ so as to ensure that $Nr^{-1} \to 0$ and $N^{-1} \to 0$ as $|\lambda| \to \infty$. Assume that (ϕ_i) is a sequence of Maass cusp forms with corresponding eigenvalues $|\lambda_i| \to \infty$ as $i \to \infty$, and that $|\phi_i|^2 \,\mathrm{dvol}_M$ converges weak*. Then any weak*-limit of $|\widetilde{\phi}_i|^2 \,\mathrm{d}m_X$ is invariant under the geodesic flow.

Classical definition of Hecke operators

Let $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, let p be a prime, and let f be a function on M. Then the action of the Hecke operator T_p on f is defined by

$$(T_p(f))(z) = \frac{1}{p+1}\left[f(pz) + \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right)\right].$$

Second Defintion, using PGL_2

Let $X = \mathrm{PGL}_2(\mathbb{Z}) \backslash \mathrm{PGL}_2(\mathbb{R})$, let p be a prime, and let f be a function on X. Then the Hecke operator T_p is defined by sending f to the normalized sum

$$(T_p(f))([\Lambda]) = \frac{1}{p+1} \sum_{\substack{\Lambda' \subseteq \Lambda, \\ [\Lambda:\Lambda']=p}} f\left([\Lambda']\right)$$

of the values of f on all the sublattices Λ' of Λ with index p.

Proposition

If $\Lambda \subseteq \mathbb{R}^2$ is a lattice, then there are $p+1$ subgroups $\Lambda' \subseteq \Lambda$ with index p. The definition of $T_p(f)$ defines a function on the homothety classes $[\Lambda] \in \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$ of lattices $\Lambda \subseteq \mathbb{R}^2$.

Finally, for any $g \in \mathrm{PGL}_2(\mathbb{R})$, we have $g \centerdot T_p(f) = T_p(g \centerdot f)$.

Corollary

For any p the p-Hecke operator T_p commutes with any differential operator $m \in \mathfrak{sl}_2(\mathbb{C})$ (or even any element of the enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$).

In particular, if ϕ is a Maas cusp form that is an eigenfunction of T_p, then $\widetilde{\phi}$ is also an eigenfunction of T_p.

p-adic extension

Embed $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ as a subset of $\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_p)$ diagonally, by sending γ to (γ, γ). Then $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$ is a lattice in

$$\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_p),$$

and the double quotient

$$\mathrm{PGL}_2(\mathbb{Z}[\tfrac{1}{p}])\backslash \mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_p) / \mathrm{PGL}_2(\mathbb{Z}_p)$$

is naturally isomorphic to

$$\mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R}).$$

Lemma

$$K_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_p = K_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} \sqcup \bigsqcup_{j=0}^{p-1} K_p \begin{pmatrix} 1 & j \\ & p \end{pmatrix} = K_p \begin{pmatrix} 1 & \\ & p \end{pmatrix} K_p.$$

Lemma

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Choose a Haar measure on $\mathrm{PGL}_2(\mathbb{Q}_p)$ with $m(K_p) = 1$.

3-rd definition of T_p

For a function f on $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])\backslash \mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_p)$, we may obtain the p-Hecke operator by the convolution

$$T_p(f) = \frac{1}{p+1} f * \mathbf{1}_{K_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_p}.$$

For a function f on $\mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$, this agrees with T_p as defined before.

Corollary

T_p is a self-adjoint operator on $L^2(\mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R}))$ (and on $L^2(\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])\backslash \mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{Q}_p)))$.