NON-ARCHIMEDEAN DYNAMICS IN DIMENSION ONE: COURSE OUTLINE

ROBERT L. BENEDETTO

Overview: In complex dynamics, given a rational function $\phi(z) \in \mathbb{C}(z)$, one considers the action of the family of iterates $\{\phi^n\}_{n\geq 0}$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, where $\phi^n := \phi \circ \phi \circ \cdots \circ \phi$. (That is, $\phi^0(z) = z$, and $\phi^{n+1}(z) = \phi \circ \phi^n(z)$.) The most obvious dynamical features are *fixed points*, i.e., points $P \in \mathbb{P}^1(\mathbb{C})$ such that $\phi(P) = P$. More generally, P is *periodic* if $\phi^n(P) = P$ for some $n \geq 1$; and even more generally, P is *preperiodic* if $\phi^n(P) = \phi^m(P)$ for some $n > m \geq 0$. In addition, the spherical metric on $\mathbb{P}^1(\mathbb{C})$ allows us to define and study the *Fatou set* and *Julia set* of ϕ.

Loosely speaking, the Fatou set $\mathcal{F} = \mathcal{F}_\phi$ is the set of all points $P \in \mathbb{P}^1(\mathbb{C})$ for which, given any point $Q \in \mathbb{P}^1(\mathbb{C})$ close enough to P, $\phi^n(Q)$ remains close to $\phi^n(P)$ for all $n \geq 1$. The Fatou set is open, and therefore its complement, the Julia set $\mathcal{J} = \mathcal{J}_\phi$, is closed. Both sets are invariant under ϕ; that is, $\phi(\mathcal{F}) = \phi^{-1}(\mathcal{F}) = \mathcal{F}$, and $\phi(\mathcal{J}) = \phi^{-1}(\mathcal{J}) = \mathcal{J}$.

The action of ϕ on the connected components of the Fatou set is well understood in the complex setting. All such components are preperiodic (this is Sullivan's deep "No Wandering Domains" theorem), and the dynamics on a periodic component must be one of a short list of types. Moreover, the number (but not the length) of periodic cycles of Fatou components is bounded in terms of $\deg \phi$, defined to be the maximum of the degrees of the numerator and denominator of ϕ. Meanwhile, the Julia set is the closure of the set of so-called repelling periodic points. In addition, one can construct a probability measure μ on the Riemann sphere, with support exactly on $\mathcal{J}$, that is invariant under ϕ; in particular, $\mu(\phi^{-1}(X)) = \mu(X)$ for any Borel set $X \subseteq \mathbb{P}^1(\mathbb{C})$.

The goal of much of the study of non-archimedean dynamics in the past fifteen years has been to extend the complex theory to a complete, algebraically closed non-archimedean field $\mathbb{C}_v$. (A standard example is $\mathbb{C}_v = \mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$, the field of p-adic rational numbers.) It turns out that we need to study not just $\mathbb{P}^1(\mathbb{C}_v)$, but the Berkovich projective line $\mathbb{P}^1_{\text{Ber}}$, which contains $\mathbb{P}^1(\mathbb{C}_v)$ but has many advantages, such as being compact and path-connected. Still, a lot can be done with $\mathbb{P}^1(\mathbb{C}_v)$ directly, and so we will consider both spaces in this course. Here is the plan:

Lecture 1: After a review of various examples of fields $\mathbb{C}_v$, we'll discuss the mapping properties of non-archimedean power series and rational functions on disks. We'll also learn a little rigid analysis, mainly to discuss *connected affinoid domains* in $\mathbb{C}_v$, which are more general than disks. We'll then use disks and affinoids to prove some basic facts about non-archimedean dynamics and about Fatou and Julia sets in $\mathbb{P}^1(\mathbb{C}_v)$.

Lecture 2: We'll introduce the Berkovich projective line $\mathbb{P}^1_{\text{Ber}}$ and see how a rational function $\phi \in K(z)$ acts on it. We'll introduce Berkovich versions $\mathcal{F}_{\text{Ber}}$ and $\mathcal{J}_{\text{Ber}}$ of the Fatou and Julia sets, revisit our earlier results from the Berkovich perspective, and prove some new results about the structure of periodic Fatou components.

Lecture 3: We'll prove some more advanced results about Fatou and Julia sets, especially regarding wandering domains and repelling density.

Lecture 4: We'll get a taste of some more advanced results, like the construction of an invariant measure with support exactly $\mathcal{J}_{\text{Ber}}$.