Non-archimedean Dynamics in Dimension One: Lecture 3

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Still Sunday, π Day, 2010
Revisiting the Quadratic Polynomial Example

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- If \(|a| \leq 1\), then \(\phi(\zeta(0, 1)) = \zeta(0, 1)\), with \(\overline{\phi} = z^2 + \overline{a}z\), and hence with \(\deg_{\zeta(0, 1)} \phi = 2\).
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Each residue class \( \overline{x} \) is mapped to the residue class \( \overline{\phi}(\overline{x}) \).
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So \(F_{\phi, \Ber} = \mathbb{P}_\Ber^1 \setminus \{\zeta(0, 1)\}\), and \(J_{\phi, \Ber} = \{\zeta(0, 1)\}\).
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So \(\mathcal{F}_{\phi, \text{Ber}} = \mathbb{P}^1_{\text{Ber}} \setminus \{\zeta(0, 1)\}\), and \(\mathcal{J}_{\phi, \text{Ber}} = \{\zeta(0, 1)\}\).

- If \(|a| > 1\), then \(\mathcal{J}_{\phi, \text{Ber}} = \mathcal{J}_\phi \subseteq \mathbb{P}^1(\mathbb{C}_K)\) is the same Cantor set as before.

Then \(\mathcal{F}_{\phi, \text{Ber}} = \mathbb{P}^1_{\text{Ber}} \setminus \mathcal{J}_\phi\), all points of which are attracted to \(\infty\) under iteration.
Revisiting Hsia’s Cubic Polynomial Example

\[ \phi(z) = az^3 + z^2 + bz + c, \text{ where } 0 < |a| < 1, \text{ and } |b|, |c| \leq 1. \]
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Then \( \bar{\phi}(z) = z^2 + \bar{b}z + \bar{c} \), so that \( \phi \) maps \( \zeta(0, 1) \) to itself with multiplicity 2.
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So \( \zeta(0, 1) \in J_{\phi, Ber} \) is a (Type II) repelling fixed point.
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So \( \zeta(0, 1) \in \mathcal{J}_{\phi, \text{Ber}} \) is a (Type II) repelling fixed point.

\( \phi \) maps each residue class \( \overline{\subset} \) other than \( \overline{\infty} \) to its image under \( \phi \).
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Then \( \overline{\phi}(z) = z^2 + \overline{b}z + \overline{c} \), so that \( \phi \) maps \( \zeta(0, 1) \) to itself with multiplicity 2.

So \( \zeta(0, 1) \in J_{\phi, \text{Ber}} \) is a (Type II) repelling fixed point.

\( \phi \) maps each residue class except \( \overline{\infty} \) to its image under \( \overline{\phi} \). Since none of them ever hits \( \overline{\infty} \), they are all contained in \( \mathcal{F}_{\text{Ber}} \).
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\( \phi \) maps each residue class \( \overline{x} \) other than \( \overline{\infty} \) to its image under \( \overline{\phi} \).

Since none of them ever hits \( \overline{\infty} \), they are all contained in \( F_{\text{Ber}} \).

However, \( \phi \) maps the residue class \( \overline{\infty} \) onto all of \( \mathbb{P}^1_{\text{Ber}} \). The Julia set \( J_{\phi,\text{Ber}} \) is scattered through this residue class.
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Recall that the classical Julia set \( J_{\phi} \) was not compact; but of course the Berkovich Julia set \( J_{\phi, \text{Ber}} \) must be compact.
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However, \( \phi \) maps the residue class \( \overline{\infty} \) onto all of \( \mathbb{P}^1_{\text{Ber}}. \) The Julia set \( J_{\phi, \text{Ber}} \) is scattered through this residue class.

Recall that the classical Julia set \( J_{\phi} \) was not compact; but of course the Berkovich Julia set \( J_{\phi, \text{Ber}} \) must be compact.

In particular, that sequence \( \beta_1, \beta_2, \ldots \) (of preimages of the repelling fixed point \( \alpha \)) accumulates at \( \zeta(0, 1) \in J_{\phi, \text{Ber}}. \)
Components of the (Berkovich) Fatou Set

**Theorem**

Let \( \phi(z) \in \mathbb{C}_K(z) \) be a rational function of degree \( d \geq 2 \), with (Berkovich) Fatou set \( \mathcal{F}_{\phi,\text{Ber}} \) and Julia set \( \mathcal{J}_{\phi,\text{Ber}} \).
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Let $U \subseteq \mathcal{F}_{\phi,\text{Ber}}$ be a connected component of $\mathcal{F}_{\phi,\text{Ber}}$, and let $x \in U$. Then
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Let \( U \subseteq \mathcal{F}_{\phi,\text{Ber}} \) be a connected component of \( \mathcal{F}_{\phi,\text{Ber}} \), and let \( x \in U \). Then

- \( U \) is the union of all connected Berkovich affinoids containing \( x \) and contained in \( \mathcal{F}_{\phi,\text{Ber}} \).
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- $U$ is the union of all connected Berkovich affinoids containing $x$ and contained in $\mathcal{F}_{\phi,\text{Ber}}$.
- $\phi(U)$ is a connected component of $\mathcal{F}_{\phi,\text{Ber}}$. 
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- \( U \) is the union of all connected Berkovich affinoids containing \( x \) and contained in \( \mathcal{F}_{\phi,\text{Ber}} \).
- \( \phi(U) \) is a connected component of \( \mathcal{F}_{\phi,\text{Ber}} \).
- \( \phi^{-1}(U) \) is a disjoint union of at most \( d \) connected components \( V_1, \ldots, V_\ell \) of \( \mathcal{F}_{\phi,\text{Ber}} \).
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Each \( V_i \) maps \( d_i \)-to-1 onto \( U \), for some \( d_i \geq 1 \), and \( d_1 + \cdots + d_\ell = d \).
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- We say $U$ is an \textit{indifferent component} if the mapping $\phi^m : U \to U$ is one-to-one.
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- We say $U$ is an *attracting component* if there is an attracting periodic point $x \in U$ of period $m$, and if $\lim_{n \to \infty} \phi^{mn}(\zeta) = x$ for all $\zeta \in U$. 

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A connected component of $\mathcal{F}_{\phi,\text{Ber}}$ that is not preperiodic is called a wandering domain.
Theorem (Rivera-Letelier, 2000)

Let \( \phi \in \mathbb{C}_K(z) \) be a rational function of degree \( d \geq 2 \) with Fatou set \( \mathcal{F}_{\phi,\text{Ber}} \).

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2. Some iterate \( \phi^n(U) \) is an attracting periodic component.
3. \( U \) is a wandering domain.
Example: Good Reduction

Recall that \( \phi \) has **good reduction** if when we write \( \phi(z) = \frac{f(z)}{g(z)} \) where \( f, g \in \mathcal{O}[z] \) satisfy

- \((f, g) = 1\),
- at least one coefficient of \( f \) and/or \( g \) is a unit (i.e., \(|a| = 1\)).
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In that case, $\mathcal{J}_{\phi, \text{Ber}} = \{ \zeta(0, 1) \}$, and each residue class is a Fatou component.
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The components map to each other as dictated by $\bar{\phi} := \bar{f}/\bar{g}$ acting on $\mathbb{P}^1(k).$
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The components map to each other as dictated by $\overline{\phi} := \overline{f}/\overline{g}$ acting on $\mathbb{P}^1(k)$.

An $n$-periodic residue class $D_{\text{Ber}}(a, 1)$ is attracting if and only if $\overline{\phi}$ has a critical point among $\{\overline{a}, \overline{\phi(\overline{a})}, \ldots, \overline{\phi}^{n-1}(\overline{a})\}$. 
Example: Polynomials

If \( \phi(z) \in \mathbb{C}[[z]] \) with \( \deg \phi \geq 2 \) is a polynomial, then the Fatou component \( W \) containing \( \infty \) is fixed and attracting.
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If $\phi(z) \in \mathbb{C}_v[z]$ with $\deg \phi \geq 2$ is a polynomial, then the Fatou component $W$ containing $\infty$ is fixed and attracting.

If $\phi$ is not of potentially good reduction,
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If \( \phi \) is not of potentially good reduction, then \( W \) is not a disk. Instead, it is of Cantor type.
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That is, let \( V_0 = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}_{\text{Ber}}(a, r) \subseteq \mathcal{F}_\phi \) be the largest open \( \mathbb{P}^1_{\text{Ber}} \)-disk containing \( \infty \).
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\( V_1 := \phi^{-1}(V_0) \supset V_0 \) is a non-disk open affinoid, with at least two ends outside (the unique) end of \( V_0 \).
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That is, let $V_0 = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}_{\text{Ber}}(a, r) \subseteq \mathcal{F}_\phi$ be the largest open $\mathbb{P}^1_{\text{Ber}}$-disk containing $\infty$.

$V_1 := \phi^{-1}(V_0) \supset V_0$ is a non-disk open affinoid, with at least two ends outside (the unique) end of $V_0$.

$V_2 := \phi^{-1}(V_1) \supset V_1$ is a non-disk open affinoid; with at least two ends outside each end of $V_1$. 
Example: Polynomials

If \( \phi(z) \in \mathbb{C}_v[z] \) with \( \deg \phi \geq 2 \) is a polynomial, then the Fatou component \( W \) containing \( \infty \) is fixed and attracting.

If \( \phi \) is not of potentially good reduction, then \( W \) is not a disk. Instead, it is of **Cantor type**.

That is, let \( V_0 = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}\text{Ber}(a, r) \subseteq \mathcal{F}_\phi \) be the largest open \( \mathbb{P}^1_{\text{Ber}} \)-disk containing \( \infty \).

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\[ V_2 := \phi^{-1}(V_1) \supsetneq V_1 \] is a non-disk open affinoid; with at least two ends outside each end of \( V_1 \).

etc. In the end, \( W = \bigcup_{n \geq 0} V_n \).
Rivera-Letelier’s Classification: Continued

Theorem (Rivera-Letelier, 2000)

Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_{\phi,\text{Ber}}$, and let $U \subseteq \mathcal{F}_{\phi,\text{Ber}}$ be a periodic connected component of the Fatou set.
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1. If $U$ is indifferent, then $U$ is a rational open connected affinoid, and $\phi$ permutes the (finitely many) boundary points of $U$. 
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1. If $U$ is indifferent, then $U$ is a rational open connected affinoid, and $\phi$ permutes the (finitely many) boundary points of $U$. The boundary points are all type II periodic Julia points.
2. If $U$ is attracting, then $U$ is either a rational open disk or a domain of Cantor type.
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Let \( \phi \in \mathbb{C}_K(z) \) be a rational function of degree \( d \geq 2 \) with Fatou set \( \mathcal{F}_{\phi, \text{Ber}} \), and let \( U \subseteq \mathcal{F}_{\phi, \text{Ber}} \) be a **periodic** connected component of the Fatou set.

1. **If** \( U \) **is indifferent**, then \( U \) is a rational open connected affinoid, and \( \phi \) permutes the (finitely many) boundary points of \( U \). The boundary points are all type II periodic Julia points.

2. **If** \( U \) **is attracting**, then \( U \) is either a rational open disk or a domain of Cantor type.
   
   For an open disk, the unique boundary point is a type II repelling periodic (Julia) point.
Rivera-Letelier’s Classification: Continued

Theorem (Rivera-Letelier, 2000)

Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_{\phi,\text{Ber}}$, and let $U \subseteq \mathcal{F}_{\phi,\text{Ber}}$ be a periodic connected component of the Fatou set.

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Example: A Non-disk Indifferent Component

\[ \phi(z) = \frac{1}{1 - z} + \pi^2 z = \frac{\pi^2 z^2 - \pi^2 z - 1}{z - 1} \in \mathbb{C}_K[z], \text{ with } 0 < |\pi| < 1. \]
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Note that \( \overline{\phi}^3(z) = z \), with \( \infty \mapsto 0 \mapsto 1 \mapsto \infty \).
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It is not hard to check that

\[ \phi \text{ maps } \begin{cases} \zeta(0, |\pi|^{-1}) \mapsto \zeta(0, |\pi|) & \text{with multiplicity } 2, \\ \zeta(0, |\pi|) \mapsto \zeta(1, |\pi|) & \text{with multiplicity } 1, \\ \zeta(1, |\pi|) \mapsto \zeta(0, |\pi|^{-1}) & \text{with multiplicity } 1, \end{cases} \]

so these three type II points form a repelling cycle of period 3.
Example: A Non-disk Indifferent Component

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so these three type II points form a repelling cycle of period 3.

It’s also easy to check that \( \phi \) maps the open connected affinoid

\[ U := D_{\text{Ber}}(0, |\pi|^{-1}) \setminus (\overline{D}_{\text{Ber}}(0, |\pi|) \cup \overline{D}_{\text{Ber}}(1, |\pi|)) \]

bijectively onto itself.
No Wandering Domains

Definition

Let $\phi \in \mathbb{C}_K(z)$ be a rational function, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$.

- $x$ is **recurrent** under $\phi$ if $x \in \bigcup_{n \geq 1} \phi^n(x)$. 
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- $x$ is recurrent under $\phi$ if $x \in \bigcup_{n \geq 1} \phi^n(x)$.
- $x$ is a wild critical point of $\phi$ if the multiplicity $\deg_x \phi$ of $\phi$ at $x$ is divisible by the residue characteristic of $\mathbb{C}_K$. 
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Theorem (RB, 1998)

*Let \( K \) be a locally compact non-archimedean field, with \( \mathbb{C}_K \) the completion of an algebraic closure of \( K \).*
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Then $F_{\phi, \text{Ber}}$ has no wandering domains.
A Power Series Lemma

Lemma

Let $a \in \mathbb{C}_K^\times$, set $r = |a|$, and let $0 < s < r$.

Let $f(z) = c_0 + c_d z^d + \cdots \in \mathbb{C}_K[[z]]$ converge on $\overline{D}(0, r)$, and assume that $|c_n| r^n < |d c_d| r^d$ for all $n > d \geq 1$. 
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Let \( a \in \mathbb{C}_K^\times \), set \( r = |a| \), and let \( 0 < s < r \).
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So
\[
\text{diam} \left( f(\bar{D}(a, s)) \right) = |dc_d| r^{d-1} s, \quad \text{and} \quad \text{diam} \left( f(\bar{D}(0, r)) \right) = |c_d| r^d.
\]
Sketch of Proof of No Wandering Domains

Change coordinates so that $\mathcal{J}_\phi \subseteq \overline{D}(0, 1)$. Extend $K$ to include all critical points of $\phi$ and some point of a supposed wandering domain $U$. 
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For any $n \geq 0$, let $V_n \supset \phi^n(U)$ be a slightly larger disk.
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By the no wild (recurrent) Julia critical hypothesis, there is a
radius $R > 0$ so that $\phi^m(V_n)$ has to get up to radius at least $R$
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By the power series lemma, $\phi^{m+n}(U)$ has to have radius at least about $R$ (or $|p|^{M'} R$).
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So $U$ has infinitely many non-overlapping iterates of radius bounded below and intersecting the compact set $\mathcal{O}_K$,
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So $U$ has infinitely many non-overlapping iterates of radius bounded below and intersecting the compact set $\mathcal{O}_K$, a contradiction. QED
There **Can** be Wandering Domains

One of the hypotheses of the No Wandering Domains result is that \( \phi \) is defined over a locally compact subfield of \( \mathbb{C}_K \).
There **Can** be Wandering Domains

One of the hypotheses of the No Wandering Domains result is that $\phi$ is defined over a locally compact subfield of $\mathbb{C}_K$.

But if we relax that condition, we can find wandering domains.

**Theorem**

*Let $\mathbb{C}_K$ have residue field $\overline{k}$ that is **not** algebraic over a finite field.*
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**Theorem**

*Let \( \mathbb{C}_K \) have residue field \( \overline{k} \) that is **not** algebraic over a finite field.*

*Then any \( \phi(z) \in \mathbb{C}_K(z) \) with a type II Julia periodic point \( \zeta \) has wandering domains “in the basin of attraction” of \( \zeta \).*
There **Can** be Wandering Domains

One of the hypotheses of the No Wandering Domains result is that $\phi$ is defined over a locally compact subfield of $\mathbb{C}_K$.

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**Theorem**

*Let $\mathbb{C}_K$ have residue field $\overline{k}$ that is **not** algebraic over a finite field. Then any $\phi(z) \in \mathbb{C}_K(z)$ with a type II Julia periodic point $\zeta$ has wandering domains “in the basin of attraction” of $\zeta$.*

The wandering domains in question are just wandering residue classes of $\zeta$ whose iterates avoid “bad” residue classes.
Theorem (RB, 2005)

Let $K$ be a complete discretely valued non-archimedean field of residue characteristic zero, let $\mathbb{C}_K$ be the completion of an algebraic closure of $K$, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. 
Theorem (RB, 2005)

Let $K$ be a complete \textit{discretely valued} non-archimedean field of residue characteristic zero, let $\mathbb{C}_K$ be the completion of an algebraic closure of $K$, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

Then $\phi$ has no wandering domains besides those in attracting basins of periodic type II points.
No Other Wandering Domains

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Let $K$ be a complete **discretely valued** non-archimedean field of residue characteristic zero, let $\mathbb{C}_K$ be the completion of an algebraic closure of $K$, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

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Theorem (Trucco, 2009)

Let $\mathbb{L}$ be the field of Puiseux series over $\overline{\mathbb{Q}}$, and let $\phi \in \mathbb{L}[z]$ be a polynomial of degree $d \geq 2$. 
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Subtler Wandering Domains

Even for $\mathbb{C}_p$ and other fields with residue field $\overline{\mathbb{F}}_p$, there can be wandering domains not associated with type II periodic points.
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Even for $\mathbb{C}_p$ and other fields with residue field $\overline{\mathbb{F}}_p$, there can be wandering domains not associated with type II periodic points.

Theorem (RB, 2002)

Let $\mathbb{C}_K$ have residue characteristic $p > 0$. 


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$$\phi_a(z) := (1 - a)z^{p+1} + az^p$$

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(Idea of Proof: see Project #4)
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What if we stick to $K$ locally compact? It is easy to force a wild critical point into the Julia set.

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**Theorem (Rivera-Letelier, 2005)**

Let $K$ be a complete non-archimedean field of residue characteristic $p$. Then there are polynomials $\phi \in K[z]$ with wild recurrent Julia critical points.

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In both cases (char $K = p > 0$ with wild Julia critical points, or char $K = 0$ with wild recurrent Julia critical points), we don't know whether there can be wandering domains.