Non-archimedean Dynamics in Dimension One: Lecture 2

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The Gauss Norm

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$$f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}_K[[z]] \qquad \text{such that } \lim_{n\to\infty} c_n = 0,$$

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Equivalently, for all $f \in \overline{\mathcal{A}}(0,1)$,

$$\begin{aligned} \|f\|_{\zeta(0,1)} &:= \sup\{|f(x)| : x \in \overline{D}(0,1)\} \\ &= \max\{|f(x)| : x \in \overline{D}(0,1)\} \end{aligned}$$

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By the way: we get $\|f+g\|_\zeta \leq \max\{\|f\|_\zeta, \|g\|_\zeta\}$ for free.

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Since $\| \cdot \|_{\overline{D}(a,r)} = \| \cdot \|_{D(a,r)}$, we can denote both by $\| \cdot \|_{\zeta(a,r)}$.

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This is the weakest topology such that for every $f \in \overline{\mathcal{A}}(0,1)$, the map $\overline{D}_{\mathrm{Ber}}(0,1) \to \mathbb{R}$ given by

$$\zeta \mapsto \|f\|_\zeta$$

is continuous.

Berkovich's Classification of Points

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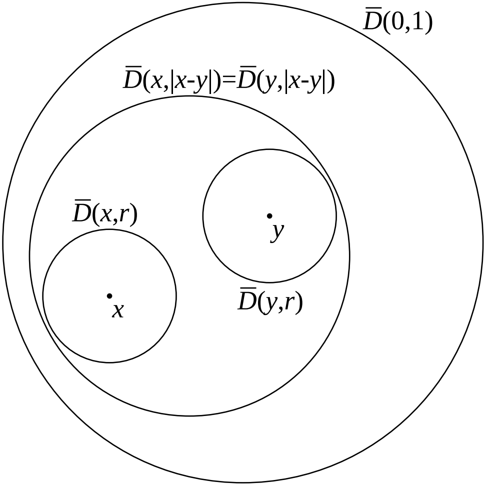
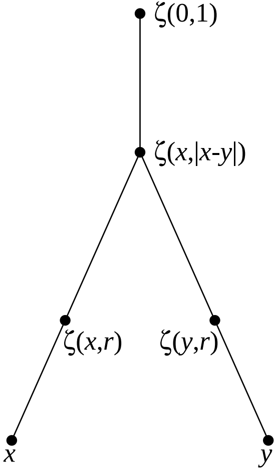
4. Type IV: norms $\|\cdot\|_{\zeta}$ corresponding to (equivalence classes of) decreasing chains $D_1 \supseteq D_2 \supseteq \cdots$ of disks with **empty intersection**.

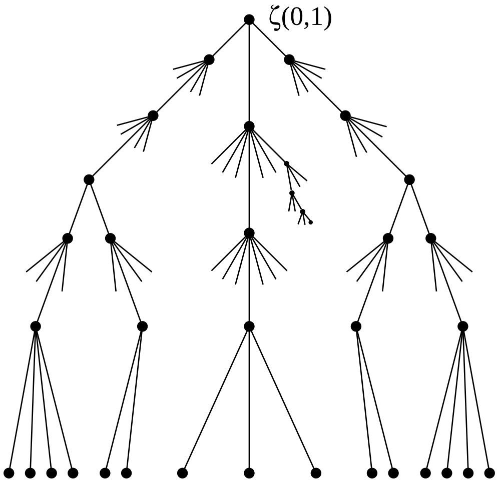
Chains of disks as in Type IV must have radius **bounded below**.

Path-connectedness, intuitively



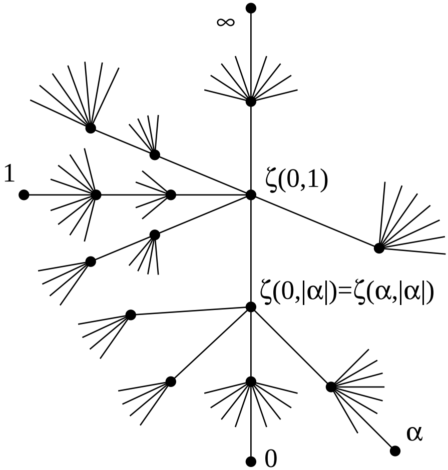


$\overline{D}_{\mathrm{Ber}}(0,1)$ as an $\mathbb{R}$-tree



The Berkovich Projective Line $\mathbb{P}^1_{\mathrm{Ber}}$

Glue two copies of $\overline{D}_{\mathrm{Ber}}(0,1)$ along $|z| = 1$ via $z \mapsto 1/z$.



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Let $a \in \mathbb{C}_K$ and $r > 0$.

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Fact:

$$D_{\mathrm{Ber}}(a, r) \quad \text{is open,} \quad \text{and} \quad \overline{D}_{\mathrm{Ber}}(a, r) \quad \text{is closed.}$$

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Moreover:

The open Berkovich disks and the complements of closed Berkovich disks together form a **subbasis** for the Gel'fand topology.

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For any $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$, the complement $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta\}$ consists of

1. one component if ζ is type I or type IV,
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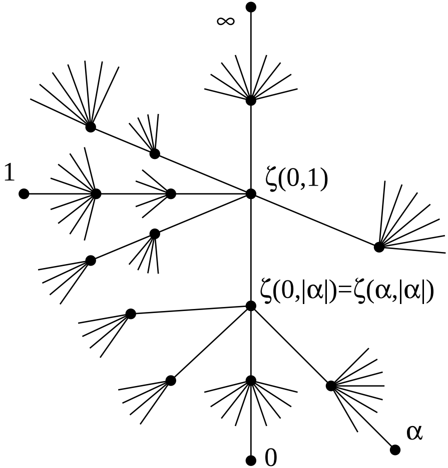
The components of $\mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \{\zeta\}$ are called the **directions** at ζ.

Recall: The Berkovich Projective Line $\mathbb{P}^1_{\mathrm{Ber}}$



Rational Functions Acting on $\mathbb{P}^1_{\text{Ber}}$

Let $\phi(z) \in \mathbb{C}_K(z)$. Then for each point $\zeta \in \mathbb{P}^1_{\text{Ber}}$, there is a unique point $\phi(\zeta) \in \mathbb{P}^1_{\text{Ber}}$ such that

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Then $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ is the unique continuous extension of $\phi : \mathbb{P}^1(\mathbb{C}_K) \to \mathbb{P}^1(\mathbb{C}_K)$.

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- $\phi(z) = 1/z$ maps $\zeta(a, r)$ to $\begin{cases} \zeta(0, 1/r) & \text{if } 0 \in \overline{D}(a, r), \\ \zeta(1/a, r/|a|^2) & \text{if } 0 \notin \overline{D}(a, r). \end{cases}$

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- So for any $\phi \in \mathrm{PGL}(2, \mathbb{C}_K)$, i.e., $\phi(z) = \dfrac{az + b}{cz + d}$ with $ad - bc \neq 0$, you can figure out what $\phi(\zeta)$ is for any $\zeta \in \mathbb{P}^1_{\text{Ber}}$.

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- Given $\phi \in \mathrm{PGL}(2, \mathbb{C}_K)$, then

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with $a_i, b_i \in \mathcal{O}$ and **some** $|a_i| = 1$ and/or **some** $|b_j| = 1$.

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Then $\overline{\phi}(z) := \dfrac{\overline{a}_d z^d + \cdots + \overline{a}_1 z + \overline{a}_0}{\overline{b}_d z^d + \cdots + \overline{b}_1 z + \overline{b}_0} \in \overline{k}(z)$.

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But we might have cancellation in $\overline{\phi}$.

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For more general $\phi \in \mathbb{C}_K(z)$, when does $\phi(\zeta(0,1)) = \zeta(0,1)$?

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Fact: $\phi(\zeta(0,1)) = \zeta(0,1)$ if and only if ϕ has nonconstant reduction.

Understanding $\phi \in \mathbb{C}_K(z)$ at type II points

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- $\eta, \theta \in \mathrm{PGL}(2, \mathbb{C}_K)$ are not unique,
  but the cosets $\mathrm{PGL}(2, \mathcal{O})\eta$ and $\mathrm{PGL}(2, \mathcal{O})\theta$ are unique.

Example

$\mathbb{C}_K = \mathbb{C}_p$, $\zeta = \zeta(0, |p|_p)$, and $\phi(z) = \dfrac{z^3 - z^2 + z + p^2}{z}$.

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So $\phi(\zeta) = \theta^{-1}(\zeta(0,1)) = \zeta(1, |p|_p)$.

Dynamics on $\mathbb{P}^1_{\text{Ber}}$: Classifying Periodic Points

Definition

If ζ and ξ are type II points and $\phi(\zeta) = \xi$, then the **local degree** or **multiplicity** of ϕ at ζ is

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Note: Periodic type III and IV points are always indifferent.

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An open set $U \subseteq \mathbb{P}^1_{\text{Ber}}$ is **dynamically stable** under $\phi \in \mathbb{C}_K(z)$ if $\bigcup_{n \geq 0} \phi^n(U)$ omits infinitely many points of $\mathbb{P}^1_{\text{Ber}}$.

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The **(Berkovich) Julia set of** ϕ is the set

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  In general, if $\zeta(0,1)$ is fixed by ϕ,
  and if $\overline{\phi}^m(z) = z$ for some $m \geq 1$,
  then $\zeta(0,1)$ is Fatou.

$\mathbb{P}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C}_K)$, and $\mathbb{P}^1_{\text{Ber}}$

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Some indifferent points are Fatou, and some are Julia	**All** indifferent points are Fatou	**Most** indifferent points are Fatou.

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$\mathcal{J}$ is the closure of the set of repelling periodic points	??? (see Project #1)	$\mathcal{J}_{\text{Ber}}$ is the closure of the set of repelling periodic (Type I & II) points