Non-archimedean Dynamics in Dimension One: Lecture 2

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Problems with $\mathbb{P}^1(\mathbb{C}_K)$

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- is path-connected.
The Gauss Norm

\[ \overline{A}(0, 1) = \mathbb{C}_K \langle \langle z \rangle \rangle \] is the ring of all power series

\[ f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}_K[[z]] \] such that \( \lim_{n \to \infty} c_n = 0 \),

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The Gauss norm on \( \overline{A}(0, 1) \) is \( \| \cdot \|_{\zeta(0,1)} : \overline{A}(0, 1) \to [0, \infty) \), by

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Equivalently, for all \( f \in \overline{A}(0,1) \),

\[
\| f \|_{\zeta(0,1)} := \sup \{ |f(x)| : x \in \overline{D}(0,1) \} = \max \{ |f(x)| : x \in \overline{D}(0,1) \}.
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**Definition**

A **bounded multiplicative seminorm** on $\mathcal{A}(0, 1)$ is a function $\zeta = \| \cdot \|_\zeta : \mathcal{A}(0, 1) \rightarrow [0, \infty)$ such that
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- $\|0\|_\zeta = 0$ and $\|1\|_\zeta = 1$,
- $\|fg\|_\zeta = \|f\|_\zeta \cdot \|g\|_\zeta$ for all $f, g \in \overline{\mathbb{A}}(0, 1)$,
- $\|f + g\|_\zeta \leq \|f\|_\zeta + \|g\|_\zeta$ for all $f, g \in \overline{\mathbb{A}}(0, 1)$, and
- $\|f\|_\zeta \leq \|f\|_{\zeta(0,1)}$ for all $f \in \overline{\mathbb{A}}(0, 1)$. 
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**Note:** We do not require that $\|f\|_{\zeta} = 0$ implies $f = 0$. 
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4. $\|f\|_\zeta \leq \|f\|_{\zeta(0,1)}$ for all $f \in \overline{A}(0, 1)$.

Note: We do not require that $\|f\|_\zeta = 0$ implies $f = 0$.

By the way: we get $\|f + g\|_\zeta \leq \max\{\|f\|_\zeta, \|g\|_\zeta\}$ for free.
Examples of Bounded Multiplicative Seminorms

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If $D = \overline{D}(a, r)$ or $D = D(a, r)$, and $f(z) = \sum c_n(z - a)^n$, then

$$\| f \|_D = \max \{|c_n|r^n : n \geq 0\}.$$ 

If $D$ is rational closed, then $\| f \|_D = \max \{|f(x)| : x \in D\}$. 
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Since $\| \cdot \|_{\overline{D}(a,r)} = \| \cdot \|_{D(a,r)}$, we can denote both by $\| \cdot \|_{\zeta(a,r)}$. 
The Berkovich Disk

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As a topological space, $\overline{D}_{\text{Ber}}(0,1)$ is equipped with the Gel’fand topology.
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As a topological space, $\overline{D}_{\text{Ber}}(0, 1)$ is equipped with the **Gel'fand topology**.

This is the weakest topology such that for every $f \in \overline{A}(0, 1)$, the map $\overline{D}_{\text{Ber}}(0, 1) \to \mathbb{R}$ given by

$$\zeta \mapsto \|f\|_\zeta$$

is continuous.
Berkovich’s Classification of Points

There are four kinds of points in $\overline{D}_{\text{Ber}}(0, 1)$.

1. Type I: seminorms $\| \cdot \|_x$ corresponding to (classical) points $x \in \overline{D}(0, 1)$. 
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4. Type IV: norms $\| \cdot \|_{\zeta}$ corresponding to (equivalence classes of) decreasing chains $D_1 \supseteq D_2 \supseteq \cdots$ of disks with empty intersection.

Chains of disks as in Type IV must have radius bounded below.
Path-connectedness, intuitively

$\overline{D}(x,|x-y|) = \overline{D}(y,|x-y|)$

$\overline{D}(x,r) \cap \overline{D}(y,r)$

$\zeta(x,r)$

$\zeta(y,r)$

$\zeta(0,1)$

$x$

$y$
$\overline{D}_{\text{Ber}}(0, 1)$ as an $\mathbb{R}$-tree
Glue two copies of $\overline{D}_{\text{Ber}}(0, 1)$ along $|z| = 1$ via $z \mapsto 1/z$. 
Berkovich Disks

Definition
Let $a \in \mathbb{C}_K$ and $r > 0$.

- The **closed Berkovich disk** $\overline{D}_{\text{Ber}}(a, r)$ is the set of all $\zeta \in \mathbb{P}^1_{\text{Ber}}$ corresponding to a point/disk/chain of disks contained in $\overline{D}(a, r)$. 
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- The **open Berkovich disk** \( D_{\text{Ber}}(a, r) \) is the set of all \( \zeta \in \mathbb{P}^1_{\text{Ber}} \) corresponding to a point/disk/chain of disks contained in \( D(a, r) \), except \( \zeta(a, r) \) itself.
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**Fact:**

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D_{\text{Ber}}(a, r) \text{ is open, and } \overline{D}_{\text{Ber}}(a, r) \text{ is closed.}
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- The **open Berkovich disk** $D_{\text{Ber}}(a, r)$ is the set of all $\zeta \in \mathbb{P}^1_{\text{Ber}}$ corresponding to a point/disk/chain of disks contained in $D(a, r)$, except $\zeta(a, r)$ itself.

Fact:

$$D_{\text{Ber}}(a, r) \text{ is open, and } \overline{D}_{\text{Ber}}(a, r) \text{ is closed.}$$

Moreover:
The open Berkovich disks and the complements of closed Berkovich disks together form a **subbasis** for the Gel’fand topology.
More on the Gel’fand Topology

Definition

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- $\mathbb{P}^1_{\text{Ber}}$ is uniquely path-connected.
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Theorem

1. The open connected Berkovich affinoids form a basis for the Gel’fand topology.
2. \( \mathbb{P}^1_{\text{Ber}} \) is uniquely path-connected.

For any \( \zeta \in \mathbb{P}^1_{\text{Ber}} \), the complement \( \mathbb{P}^1_{\text{Ber}} \setminus \{\zeta\} \) consists of

1. one component if \( \zeta \) is type I or type IV,
2. infinitely many components if \( \zeta \) is type II,
3. two components if \( \zeta \) is type III.
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The components of $\mathbb{P}^1_{\text{Ber}} \setminus \{\zeta\}$ are called the directions at $\zeta$. 
Recall: The Berkovich Projective Line $\mathbb{P}^1_{\text{Ber}}$

\[ \zeta(0,1) = \zeta(\alpha,|\alpha|) \]
Rational Functions Acting on $\mathbb{P}^1_{\text{Ber}}$

Let $\phi(z) \in \mathbb{C}_K(z)$. Then for each point $\zeta \in \mathbb{P}^1_{\text{Ber}}$, there is a unique point $\phi(\zeta) \in \mathbb{P}^1_{\text{Ber}}$ such that

$$\|h\|_{\phi(\zeta)} = \|\phi \circ h\|_{\zeta}$$

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If $\zeta$ is type I, then $\phi(\zeta)$ is what you think.
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If $\zeta$ is type I, then $\phi(\zeta)$ is what you think.

Then $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$ is the unique continuous extension of $\phi : \mathbb{P}^1(\mathbb{C}_K) \to \mathbb{P}^1(\mathbb{C}_K)$. 
Understanding degree one maps on $\mathbb{P}^1_{\text{Ber}}$

- $\phi(z) = cz$ maps $\zeta(a, r)$ to $\zeta(ca, |c|r)$. 
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- $\phi(z) = cz$ maps $\zeta(a, r)$ to $\zeta(ca, |c|r)$.
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- $\phi(z) = 1/z$ maps $\zeta(a, r)$ to
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  \begin{cases}
  \zeta(0, 1/r) & \text{if } 0 \in \overline{D}(a, r), \\
  \zeta(1/a, r/|a|^2) & \text{if } 0 \not\in \overline{D}(a, r).
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So for any $\phi \in \text{PGL}(2, \mathbb{C}_K)$, i.e., $\phi(z) = \frac{az + b}{cz + d}$ with $ad - bc \neq 0$, you can figure out what $\phi(\zeta)$ is for any $\zeta \in \mathbb{P}^1_{\text{Ber}}$. 
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Given $\phi \in \text{PGL}(2, \mathbb{C}_K)$, then

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\phi(\zeta(0, 1)) = \zeta(0, 1) \quad \text{if and only if} \quad \phi \in \text{PGL}(2, \mathcal{O}),
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Understanding degree one maps on $\mathbb{P}^1_{\text{Ber}}$

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i.e., $\phi(z) = \frac{az + b}{cz + d}$ with $|a|, |b|, |c|, |d| \leq 1$ and $|ad - bc| = 1$. 
Reduction of $\phi \in \mathbb{C}_K(z)$

For more general $\phi \in \mathbb{C}_K(z)$, when does $\phi(\zeta(0, 1)) = \zeta(0, 1)$?
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For more general $\phi \in \mathbb{C}_K(z)$, when does $\phi(\zeta(0, 1)) = \zeta(0, 1)$?

Write $\phi(z) = \frac{a_d z^d + \cdots + a_1 z + a_0}{b_d z^d + \cdots + b_1 z + b_0}$,

with $a_i, b_i \in \mathcal{O}$ and some $|a_i| = 1$ and/or some $|b_j| = 1$. 
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Then $\bar{\phi}(z) := \frac{\bar{a}_d z^d + \cdots + \bar{a}_1 z + \bar{a}_0}{\bar{b}_d z^d + \cdots + \bar{b}_1 z + \bar{b}_0} \in \bar{k}(z)$. 
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But we might have cancellation in $\overline{\phi}$. 
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But we might have cancellation in $\bar{\phi}$.

If $\deg \bar{\phi} = \deg \phi$, we say $\phi$ has good reduction.
If $\deg \bar{\phi} \geq 1$, we say $\phi$ has nonconstant reduction.
For more general $\phi \in \mathbb{C}_K(z)$, when does $\phi(\zeta(0,1)) = \zeta(0,1)$?

Write $\phi(z) = \frac{a_d z^d + \cdots + a_1 z + a_0}{b_d z^d + \cdots + b_1 z + b_0}$,

with $a_i, b_i \in \mathcal{O}$ and some $|a_i| = 1$ and/or some $|b_j| = 1$.

Then $\overline{\phi}(z) := \frac{\overline{a_d} z^d + \cdots + \overline{a_1} z + \overline{a_0}}{\overline{b_d} z^d + \cdots + \overline{b_1} z + \overline{b_0}} \in \overline{k}(z)$.

But we might have cancellation in $\overline{\phi}$.

If $\deg \overline{\phi} = \deg \phi$, we say $\phi$ has good reduction.
If $\deg \overline{\phi} \geq 1$, we say $\phi$ has nonconstant reduction.

Fact: $\phi(\zeta(0,1)) = \zeta(0,1)$ if and only if $\phi$ has nonconstant reduction.
For any type II point $\zeta \in \mathbb{P}^1_{\text{Ber}}$, there is some $\eta \in \text{PGL}(2, \mathbb{C}_K)$ such that $\eta(\zeta) = \zeta(0, 1)$.
Understanding $\phi \in \mathbb{C}_K(z)$ at type II points

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- Given $\phi \in \mathbb{C}_K(z)$ nonconstant and $\zeta \in \mathbb{P}^1_{\text{Ber}}$ of type II, choose $\eta \in \text{PGL}(2, \mathbb{C}_K)$ for $\zeta$ as above.
Understanding $\phi \in \mathbb{C}_K(z)$ at type II points

- For any type II point $\zeta \in \mathbb{P}_\text{Ber}^1$, there is some $\eta \in \text{PGL}(2, \mathbb{C}_K)$ such that $\eta(\zeta) = \zeta(0, 1)$.

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$$\theta \circ \phi \circ \eta^{-1}(z) \in \mathbb{C}_K(z)$$

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- Then $\phi(\zeta) = \theta^{-1}(\zeta(0, 1))$.

- $\eta, \theta \in \text{PGL}(2, \mathbb{C}_K)$ are not unique, but the cosets $\text{PGL}(2, \mathcal{O})\eta$ and $\text{PGL}(2, \mathcal{O})\theta$ are unique.
Example

\[ \mathbb{C}_K = \mathbb{C}_p, \ \zeta = \zeta(0, |p|_p), \text{ and } \phi(z) = \frac{z^3 - z^2 + z + p^2}{z}. \]

What is \( \phi(\zeta) \)?
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What is $\phi(\zeta)$?

$\eta(z) = z/p$ maps $\zeta$ to $\zeta(0, 1)$, and

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So \( \phi(\zeta) = \theta^{-1}(\zeta(0, 1)) = \zeta(1, |p|_p) \).
Dynamics on $\mathbb{P}^1_{\text{Ber}}$: Classifying Periodic Points

Definition

If $\zeta$ and $\xi$ are type II points and $\phi(\zeta) = \xi$, then the **local degree** or **multiplicity** of $\phi$ at $\zeta$ is

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**Note:** Periodic type III and IV points are always indifferent.
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An open set $U \subseteq \mathbb{P}^1_{\text{Ber}}$ is dynamically stable under $\phi \in \mathbb{C}_K(z)$ if
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The (Berkovich) \textbf{Fatou set of} \( \phi \) is the set \( \mathcal{F}_{\text{Ber}} = \mathcal{F}_{\phi, \text{Ber}} \) given by

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Berkovich Fatou and Julia Sets

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Basic Properties of Berkovich Fatou and Julia Sets

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- \( \mathcal{F} = \mathcal{F}_{\text{Ber}} \cap \mathbb{P}_{\mathbb{K}}^1 \), and \( \mathcal{J} = \mathcal{J}_{\text{Ber}} \cap \mathbb{P}_{\mathbb{K}}^1 \).

- All attracting periodic points are Fatou.

- All repelling periodic points are Julia.

- Indifferent periodic type II points are Fatou if the residue field is algebraic over a finite field, but they can be Julia otherwise.

In general, if \( \zeta(0, 1) \) is fixed by \( \phi \), and if \( \overline{\phi}^m (z) = z \) for some \( m \geq 1 \), then \( \zeta(0, 1) \) is Fatou.
$\mathbb{P}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C}_K)$, and $\mathbb{P}^1_{\text{Ber}}$

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