Non-archimedean Dynamics in Dimension One: Lecture 1

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(All Cauchy sequences converge).

Fun Fact: Let K be a complete non-archimedean field, and let $\{a_n\}_{n\geq 0}$ be a sequence in K. Then

$$\sum_{n\geq 0} a_n \text{ converges} \qquad \text{if and only if} \qquad \lim_{n\to\infty} a_n = 0.$$

The Residue Field and Value Group

Let K be a non-archimedean field.
The ring of integers and (unique) maximal ideal of K are

$$\mathcal{O}_K = \{x \in K : |x| \leq 1\} \quad \text{and} \quad \mathcal{M}_K = \{x \in K : |x| < 1\}.$$

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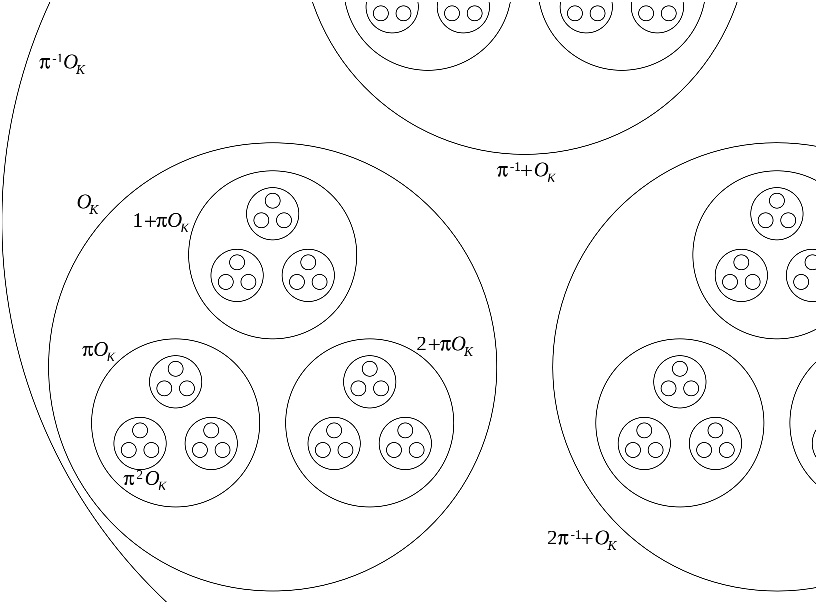
The *value group* of K is

$$|K^\times| \subseteq (0, \infty).$$

A Sketch of a Non-archimedean Field with $k \cong \mathbb{F}_3$



Extension Fields

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The new value group $|L^\times|$ contains $|K^\times|$ as a subgroup.

The algebraic closure $\overline{K}$ of K may **not** be complete.

But its completion $\mathbb{C}_K$ is both complete and algebraically closed.

Example: p-adic numbers

Fix $p \geq 2$ prime. The p-adic absolute value on $\mathbb{Q}$ is given by

$$\left|\frac{r}{s}p^n\right|_p = p^{-n} \qquad \text{for } r, s \in \mathbb{Z} \text{ not divisible by } p.$$

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$$\mathbb{Q}_p := \left\{ \sum_{n \geq n_0} a_n p^n : n_0 \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \right\}$$

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maximal ideal $\mathcal{M}_{\mathbb{Q}_p} := p\mathbb{Z}_p$, value group $|\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}}$, and residue field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

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The completion $\mathbb{C}_p$ of an algebraic closure $\overline{\mathbb{Q}}_p$ has residue field $\overline{\mathbb{F}}_p$ and value group $|\mathbb{C}_p^\times| = p^{\mathbb{Q}}$.

Example: Laurent and Puiseux Series

Fix $\mathbb{F}$ a field. The field of formal Laurent series

$$\mathbb{F}((t)) := \left\{ \sum_{n \geq n_0} a_n t^n : n_0 \in \mathbb{Z}, a_n \in \mathbb{F} \right\}$$

has a non-archimedean absolute value

$$|f| := \varepsilon^{\mathrm{ord}_{t=0} f},$$

where $0 < \varepsilon < 1$ is any (fixed) thing you want.

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The ring of integers is the ring $\mathbb{F}[[t]]$ of power series, with maximal ideal $t\mathbb{F}[[t]]$, residue field

$$k = \mathbb{F}[[t]]/t\mathbb{F}[[t]] \cong \mathbb{F},$$

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and value group $|\mathbb{F}((t))^\times| = \varepsilon^{\mathbb{Z}}$.
The completion $\mathbb{L}$ of an algebraic closure $\overline{\mathbb{F}((t))}$ is the field of formal *Puiseux series* over $\mathbb{F}$, with residue field $\overline{\mathbb{F}}$ and value group $|\mathbb{L}^\times| = \varepsilon^{\mathbb{Q}}$.

Disks

Given $a \in \mathbb{C}_K$ and $r > 0$,

$$D(a, r) := \{x \in \mathbb{C}_K : |x - a| < r\} \quad \text{and}$$
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are the associated open disk and closed disk.

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Note:

- All disks are (topologically) **both** open and closed
- Any disk is **exactly one** of: rational open, rational closed, or irrational (as a disk).

More about Disks

- Any point of a disk is a center:
  $D(a, r) = D(b, r)$ (resp., $\overline{D}(a, r) = \overline{D}(b, r)$)
  for all $b \in D(a, r)$ (resp., $b \in \overline{D}(a, r)$)

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- Since our disks lie in $\mathbb{C}_K$, and $|\mathbb{C}_K^\times|$ is dense in $(0, \infty)$, the **radius** of a disk $D \subseteq \mathbb{C}_K$ is well-defined, and equal to the diameter $\sup\{|x - y| : x, y \in D\}$.

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- Two disks intersect if and only if one contains the other.
- All non-archimedean fields are totally disconnected.
  (I.e., the only connected nonempty subsets are singletons.)
- $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ are locally compact,
  but $\mathbb{C}_K$ is not locally compact.

(Power Series and) Polynomials on Disks

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Let $g(z) = c_0 + c_1(z-a) + \cdots + c_M(z-a)^M \in \mathbb{C}_K[z]$ *be a polynomial.*

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Let $s := \max_{n\geq 1}\{|c_n|r^n\}$, and

$$i := \text{ minimum } n \geq 1 \text{ for which } |c_n|r^n = s,$$
$$j := \text{ maximum } n \geq 1 \text{ for which } |c_n|r^n = s.$$

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Let $s := \max_{n\geq 1}\{|c_n|r^n\}$, *and*

$$\begin{aligned} i &:= \text{ minimum } n \geq 1 \text{ for which } |c_n|r^n = s, \\ j &:= \text{ maximum } n \geq 1 \text{ for which } |c_n|r^n = s. \end{aligned}$$

Then g *maps*

$$\begin{array}{lll} D(a,r) & i\text{-to-1 onto} & D(c_0,s), \qquad \text{and} \\ \overline{D}(a,r) & j\text{-to-1 onto} & \overline{D}(c_0,s), \end{array}$$

counting multiplicity.

Example

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[Note: $\overline{D}(p^3, s) = \overline{D}(0, s)$ for $s \geq |p|_p^3 = p^{-3}$.]

The mapping is 1-1 for $r < |p|_p$,
2-1 for $|p|_p \leq r < |p|_p^{-4/3}$,
5-1 for $r \geq |p|_p^{-4/3}$.

$\mathbb{P}^1(\mathbb{C}_K)$-Disks

Recall $\mathbb{P}^1(\mathbb{C}_K) = \mathbb{C}_K \cup \{\infty\}$.

Definition

A $\mathbb{P}^1(\mathbb{C}_K)$-*disk* is either

- a disk $D \subseteq \mathbb{C}_K$, or
- the complement $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus D$ of a disk $D \subseteq \mathbb{C}_K$.

We can attach the adjectives *rational open*, *rational closed*, or *irrational* in the obvious way.

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Theorem

Let $g(z) \in \mathbb{C}_K(z)$ be a non-constant rational function, and let $D \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a $\mathbb{P}^1(\mathbb{C}_K)$-disk. Then $g(D)$ is either

- *all of $\mathbb{P}^1(\mathbb{C}_K)$, or*
- *a $\mathbb{P}^1(\mathbb{C}_K)$-disk of the same type as D.*

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Let $g(z) \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 1$, and let $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a connected affinoid. Then

- *$g(U)$ is either $\mathbb{P}^1(\mathbb{C}_K)$ or a connected affinoid of the same type as U.*

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- *$g(U)$ is either $\mathbb{P}^1(\mathbb{C}_K)$ or a connected affinoid of the same type as U.*
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Connected Affinoids

Definition

A *connected affinoid* in $\mathbb{P}^1(\mathbb{C}_K)$ is a nonempty intersection of finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks. Equivalently, a connected affinoid is $\mathbb{P}^1(\mathbb{C}_K)$ with finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks removed.

We can attach the adjectives *rational open*, *rational closed*, or *irrational* in the obvious way.

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$\mathbb{C}_K = \mathbb{C}_p$, and $g(z) = pz^3 - z^2 + z$. Then

- Let U be the rational closed annulus $\overline{D}(0,1) \setminus D(0,1)$. Then $g(U) = \overline{D}(0,1)$.

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- $g^{-1}(\overline{D}(0, |p|_p^{-3})) = \overline{D}(0, |p|_p^{-4/3})$, mapping 3-to-1.

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- $h^{-1}(\overline{D}(0,1))$ is the annulus $\overline{D}(0,1) \setminus D(0,1)$, which maps 2-to-1 onto $\overline{D}(0,1)$.

Dynamics on $\mathbb{P}^1(\mathbb{C}_K)$: Classifying Periodic Points

Fix a rational function $\phi(z) \in \mathbb{C}_K(z)$ of degree $d \geq 2$.

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Note:

- The multiplier is the the same for all points in the periodic cycle of x.
- The multiplier is coordinate-independent.

The Spherical Metric on $\mathbb{P}^1(\mathbb{C}_K)$

There is a spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$ analogous to that on $\mathbb{P}^1(\mathbb{C})$:

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More precisely, to allow the point at ∞,
in homogeneous coordinates we write:

$$\Delta([x_1, y_1], [x_2, y_2]) := \frac{|x_1 y_2 - x_2 y_1|}{\max\{|x_1|, |y_1|\}\max\{|x_2|, |y_2|\}}$$

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$$\begin{aligned}\mathcal{F} &= \left\{x \in \mathbb{P}^1 : \{\phi^n\}_{n\geq 0} \text{ is equicontinuous on a neighborhood of } x\right\} \\ &= \Big\{x \in \mathbb{P}^1 : \text{for all } n \geq 1 \text{ and } y \in \mathbb{P}^1(\mathbb{C}_K) \text{ s.t. } \Delta(x,y) \text{ is small,} \\ &\qquad\qquad \Delta(\phi^n(x), \phi^n(y)) \text{ is also small.}\Big\}\end{aligned}$$

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Idea:

- In the Fatou set, small errors stay small under iteration.
- In the Julia set, small errors may become large.

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An equivalent definition for $\mathbb{C}_K$:

Theorem

Let $\phi \in \mathbb{C}_K(z)$, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$. Then $x \in \mathcal{F}_\phi$ if and only if there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk $D \ni x$ such that

$$\#\mathbb{P}^1(\mathbb{C}_K) \setminus \left[\bigcup_{n \geq 0} \phi^n(D) \right] \geq 2.$$

A Quadratic Example

$\phi(z) = z^2 + az \in \mathbb{C}_K[z]$.

- If $|a| \leq 1$, then $\phi(\overline{D}(0,1)) \subseteq \overline{D}(0,1)$, and $\phi(\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(0,1)) \subseteq \mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(0,1)$.

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Similarly: Over $\mathbb{C}_p$, Smart and Woodcock showed
$\phi(z) = (z^p - z)/p$ has $\mathcal{J}_\phi = \mathbb{Z}_p$.

A Cubic Example (due to Hsia)

Assume the residue characteristic is not 2, and set

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Note: if we set $U_0 = \overline{D}(0, |a|^{-1})$, then

$$\phi(\mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0) \subseteq \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0$$

as before, and $U_n := \phi^{-n}(U_0)$ is a disjoint union of many disks.

In fact, $\mathcal{F}_\phi$ is the union of $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \bigcap_{n\geq 1} U_n$ and all preimages of $\overline{D}(0,1)$.

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$\mathcal{J}$ is the closure of the set of repelling periodic points	??? (see Project $\#$ 1)

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For example, $\mathbb{C}_p$ and the Puiseux series field $\mathbb{L}$ are **not** spherically complete.