# Some aspects of the algebraic theory of quadratic forms

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March 14 – March 18, 2009

(Notes for lectures at AWS 2009)

There are many good references for this material including [EKM], [L], [Pf] and [S].

## 1 Quadratic forms

Let $k$ be a field with $\operatorname{char} k \neq 2$.

**Definition 1.1.** A **quadratic form** $q\colon V \to k$ on a finite-dimensional vector space $V$ over $k$ is a map satisfying:

1. $q(\lambda v) = \lambda^2 q(v)$ for $v \in V$, $\lambda \in k$.
2. The map $b_q\colon V \times V \to k$, defined by
$$b_q(v,w) = \frac{1}{2}[q(v+w) - q(v) - q(w)]$$
is bilinear.

We denote a quadratic form by $(V,q)$, or simply as $q$.

The bilinear form $b_q$ is symmetric; $q$ determines $b_q$ and for all $v \in V$, $q(v) = b_q(v,v)$.

For a choice of basis $\{e_1, \ldots, e_n\}$ of $V$, $b_q$ is represented by a symmetric matrix $A(q) = (a_{ij})$ with $a_{ij} = b_q(e_i, e_j)$. If $v = \sum_{1 \le i \le n} X_i e_i \in V$, $X_i \in k$, then

$$q(v) = \sum_{1 \le i,j \le n} a_{ij} X_i X_j = \sum_{1 \le i \le n} a_{ii} X_i^2 + 2 \sum_{i<j} a_{ij} X_i X_j.$$

Thus $q$ is represented by a homogeneous polynomial of degree 2. Clearly, every homogeneous polynomial of degree 2 corresponds to a quadratic form on $V$ with respect to the chosen basis. We define the *dimension* of $q$ to be the dimension of the underlying vector space $V$ and denote it by $\dim(q)$.

**Definition 1.2.** Two quadratic forms $(V_1, q_1)$, $(V_2, q_2)$ are **isometric** if there is an isomorphism $\phi\colon V_1 \xrightarrow{\sim} V_2$ such that $q_2(\phi(v)) = q_1(v)$, $\forall v \in V_1$.

If $A(q_1)$, $A(q_2)$ are the matrices representing $q_1$ and $q_2$ with respect to bases $B_1$ and $B_2$ of $V_1$ and $V_2$ respectively, $\phi$ yields a matrix $T \in M_n(k)$, $n = \dim V$, such that

$$T A(q_2)\, T^t = A(q_1).$$

In other words, the symmetric matrices $A(q_1)$ and $A(q_2)$ are congruent. Thus isometry classes of quadratic forms yield congruence classes of symmetric matrices.

**Definition 1.3.** The form $q\colon V \to k$ is said to be **regular** if $b_q\colon V \times V \to k$ is nondegenerate.

Thus $q$ is regular if and only if the map $V \to V^* = \operatorname{Hom}(V, k)$, defined by $v \mapsto (w \mapsto b_q(v, w))$, is an isomorphism. This is the case if $A(q)$ is invertible.

Henceforth, we shall only be concerned with regular quadratic forms.

**Definition 1.4.** Let $W$ be a subspace of $V$ and $q\colon V \to k$ be a quadratic form. The **orthogonal complement** of $W$ denoted $W^\perp$ is the subspace

$$W^\perp = \{v \in V : b_q(v, w) = 0 \ \forall\ w \in W\}.$$

**Exercise 1.5.** Let $(V, q)$ be a regular quadratic form and $W$ a subspace of $V$. Then

1. $\dim(W) + \dim(W^\perp) = \dim(V)$.
2. $(W^\perp)^\perp = W$.

## 1.1 Orthogonal sums

Let $(V_1, q_1)$, $(V_2, q_2)$ be quadratic forms. The form

$$(V_1, q_1) \perp (V_2, q_2) = (V_1 \oplus V_2, q_1 \perp q_2),$$

with $q_1 \perp q_2$ defined by

$$(q_1 \perp q_2)(v_1, v_2) = q_1(v_1) + q_2(v_2),\ v_1 \in V_1,\ v_2 \in V_2$$

is called the *orthogonal sum* of $(V_1, q_1)$ and $(V_2, q_2)$.

## 1.2 Diagonalization

Let $(V, q)$ be a quadratic form. There exists a basis $\{e_1, \ldots, e_n\}$ of $V$ such that $b_q(e_i, e_j) = 0$ for $i \neq j$. Such a basis is called an *orthogonal basis* for $q$ and, with respect to an orthogonal basis, $b_q$ is represented by a diagonal matrix.

If $\{e_1, \ldots, e_n\}$ is an orthogonal basis of $q$ and $q(e_i) = d_i$, we write $q = \langle d_1, \ldots, d_n \rangle$. In this case, $V = ke_1 \oplus \cdots \oplus ke_n$ is an orthogonal sum and $q|ke_i$ is represented by $\langle d_i \rangle$. Thus every quadratic form is diagonalizable.

## 1.3 Hyperbolic forms

**Definition 1.6.** A quadratic form $(V, q)$ is said to be **isotropic** if there is a nonzero $v \in V$ such that $q(v) = 0$. It is **anisotropic** if $q$ is not isotropic. A quadratic form $(V, q)$ is said to be **universal** if it represents every nonzero element of $k$.

**Example 1.7.** The quadratic form $X^2 - Y^2$ is isotropic over $k$. Suppose $(V, q)$ is a regular form which is isotropic. Let $v \in V$ be such that $q(v) = 0$, $v \neq 0$. Since $q$ is regular, there exists $w \in V$ such that $b_q(v, w) \neq 0$. After scaling we may assume $b_q(v, w) = 1$. If $q(w) \neq 0$, we may replace $w$ by $w + \lambda v$, $\lambda = -\frac{1}{2}q(w)$, and assume that $q(w) = 0$. Thus $W = kv \oplus kw$ is a 2-dimensional subspace of $V$ and $q|W$ is represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with respect to $\{v, w\}$.

**Definition 1.8.** A binary quadratic form isometric to $(k^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ is called a **hyperbolic plane**. A quadratic form $(V, q)$ is **hyperbolic** if it is isometric to an orthogonal sum of hyperbolic planes. A subspace $W$ of $V$ such that $q$ restricts to zero on $W$ and $\dim W = \frac{1}{2} \dim V$ is called a **Lagrangian**.

Every regular quadratic form which admits a Lagrangian can easily be seen to be hyperbolic.

**Exercise 1.9.** Let $(V, q)$ be a regular quadratic form and $(W, q|W)$ a regular form on the subspace $W$. Then $(V, q) \xrightarrow{\sim} (W, q|W) \perp (W^\perp, q|W^\perp)$.

Let $(V, q)$ be a quadratic form. Then

$$V_0 = \{v \in V \,:\, b_q(v, w) = 0 \;\forall\; w \in V\}$$

is called the **radical** of $V$. If $V_1$ is any complementary subspace of $V_0$ in $V$, then $q|V_1$ is regular and $(V, q) = (V_0, 0) \perp (V_1, q|V_1)$. Note that $V$ is regular if and only if the radical of $V$ is zero. If $(V, q)$ is any quadratic form, we define the **rank** of $q$ to be the dimension of $V/V^\perp$. Of course if $(V, q)$ is regular, then $\operatorname{rank}(q) = \dim(q)$.

**Theorem 1.10** (Witt's Cancellation Theorem)**.** *Let $(V_1, q_1)$, $(V_2, q_2)$, $(V, q)$ be quadratic forms over $k$. Suppose*

$$(V_1, q_1) \perp (V, q) \cong (V_2, q_2) \perp (V, q).$$

*Then $(V_1, q_1) \cong (V_2, q_2)$.*

The key ingredient of Witt's cancellation theorem is the following.

**Proposition 1.11.** *Let $(V, q)$ be a quadratic form and $v, w \in V$ with $q(v) = q(w) \neq 0$. Then there is an isometry $\tau \colon (V, q) \xrightarrow{\sim} (V, q)$ such that $\tau(v) = w$.*

*Proof.* Let $q(v) = q(w) = d \neq 0$. Then

$$q(v + w) + q(v - w) = 2q(v) + 2q(w) = 4d \neq 0.$$

Thus $q(v + w) \neq 0$ or $q(v - w) \neq 0$. For any vector $u \in V$ with $q(u) \neq 0$, define $\tau_u \colon V \to V$ by

$$\tau_u(z) = z - \frac{2b_q(z, u)u}{q(u)}.$$

Then $\tau_u$ is an isometry called the *reflection with respect to $u$*.

Suppose $q(v - w) \neq 0$. Then $\tau_{v-w} \colon V \to V$ is an isometry of $V$ which sends $v$ to $w$. Suppose $q(v + w) \neq 0$. Then $\tau_w \circ \tau_{v+w}$ sends $v$ to $w$. $\square$

**Remark 1.12.** The orthogonal group of $(V, q)$ denoted by $O(q)$ is the set of isometries of $V$ onto itself. This group is generated by reflections. This is seen by an inductive argument on $\dim(q)$, using the above proposition.

**Theorem 1.13** (Witt's decomposition)**.** *Let $(V, q)$ be a quadratic form. Then there is a decomposition*

$$(V, q) = (V_0, 0) \perp (V_1, q_1) \perp (V_2, q_2)$$

*where $V_0$ is the radical of $q$, $q_1 = q|V_1$ is anisotropic and $q_2 = q|V_2$ is hyperbolic. If $(V, q) = (V_0, 0) \perp (W_1, f_1) \perp (W_2, f_2)$ with $f_1$ anisotropic and $f_2$ hyperbolic, then*

$$(V_1, q_1) \cong (W_1, f_1),\ (V_2, q_2) \cong (W_2, f_2).$$

**Remark 1.14.** A hyperbolic form $(W, f)$ is determined by $\dim(W)$; for if $\dim(W) = 2n$, $(W, f) \cong nH$, where $H = (k^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ is the hyperbolic plane.

From now on, we shall assume $(V, q)$ is a regular quadratic form. We denote by $q_{an}$ the quadratic form $(V_1, q_1)$ in Witt's decomposition which is determined by $q$ up to isometry. We call $\frac{1}{2}\dim(V_2)$ the *Witt index* of $q$. Thus any regular quadratic form $q$ admits a decomposition $q \cong q_{an} \perp (nH)$, with $q_{an}$ anisotropic and $H$ denoting the hyperbolic plane. We also sometime denote by $H^n$ the sum of $n$ hyperbolic planes.

# 2 Witt group of forms

## 2.1 Witt groups

We set

$$W(k) = \{\text{isomorphism classes of regular quadratic forms over } k\}/\sim$$

where the Witt equivalence $\sim$ is given by:

$$(V_1, q_1) \sim (V_2, q_2) \qquad \Longleftrightarrow \qquad \begin{array}{l} \text{there exist } r,\, s \in \mathbb{Z} \text{ such that} \\ (V_1, q_1) \perp H^r \cong (V_2, q_2) \perp H^s \end{array} .$$

$W(k)$ is a group under orthogonal sum:

$$[(V_1, q_1)] \perp [(V_2, q_2)] = [(V_1, q_1) \perp (V_2, q_2)].$$

The zero element in $W(k)$ is represented by the class of hyperbolic forms. For a regular quadratic form $(V, q)$, $(V, q) \perp (V, -q)$ has Lagrangian

$$W = \{(v, v) : v \in V\}$$

so that $(V, q) \perp (V, -q) \cong H^n$, $n = \dim(V)$. Thus, $[(V, -q)] = -[(V, q)]$ in $W(k)$.

It follows from Witt's decomposition theorem that every element in $W(k)$ is represented by a unique anisotropic quadratic form up to isometry. Thus $W(k)$ may be thought of as a group made out of isometry classes of anisotropic quadratic forms over $k$.

The abelian group $W(k)$ admits a ring structure induced by tensor product on the associated bilinear forms. For example, if $q_1 \cong \langle a_1, \dots, a_n \rangle$ and $q_2$ is a quadratic form, then $q_1 \otimes q_2 \cong a_1 q_2 \perp a_2 q_2 \perp \cdots \perp a_n q_2$.

**Definition 2.1.** Let $I(k)$ denote the ideal of classes of even-dimensional quadratic forms in $W(k)$. The ideal $I(k)$ is called the **fundamental ideal**. $I^n(k)$ stands for the $n^{\text{th}}$ power of the ideal $I(k)$.

**Definition 2.2.** Let $P_n(k)$ denote the set of isomorphism classes of forms of the type

$$\langle\langle a_1, \dots, a_n \rangle\rangle := \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle.$$

Elements in $P_n(k)$ are called **$n$-fold Pfister forms**.

The ideal $I(k)$ is generated by the forms $\langle 1, a \rangle$, $a \in k^*$. Moreover, the ideal $I^n(k)$ is generated additively by $n$-fold Pfister forms. For instance, for $n = 2$, the generators of $I^2(k)$ are of the form

$$\langle a, b \rangle \otimes \langle c, d \rangle \cong \langle 1, ac, ad, cd \rangle - \langle 1, cd, -bc, -bd \rangle = \langle\langle ac, ad \rangle\rangle - \langle\langle cd, -bc \rangle\rangle$$

**Example 2.3.** If $k = \mathbb{C}$, every 2-dimensional quadratic form over $k$ is isotropic.

$$W(k) \cong \mathbb{Z}/2\mathbb{Z}$$

$$[(V, q)] \mapsto \dim(V) \pmod 2$$

is an isomorphism.

**Example 2.4.** If $k = \mathbb{R}$, every quadratic form $q$ is represented by

$$\langle 1, \ldots, 1, -1, \ldots, -1 \rangle$$

with respect to an orthogonal basis. The number $r$ of $+1$'s and the number $s$ of $-1$'s in the diagonalization above are uniquely determined by the isomorphism class of $q$. The *signature* of $q$ is defined as $r - s$. The signature yields a homomorphism $\mathrm{sgn}\colon W(\mathbb{R}) \to \mathbb{Z}$ which is an isomorphism.

## 2.2 Quadratic forms over $p$-adic fields

Let $k$ be a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers. We call $k$ a non-dyadic $p$-adic field if $p \neq 2$. The field $k$ has a discrete valuation $v$ extending the $p$-adic valuation on $\mathbb{Q}_p$. Let $\pi$ be a uniformizing parameter for $v$ and $\kappa$ the residue field for $v$. The field $\kappa$ is a finite field of characteristic $p \neq 2$. Let $u$ be a unit in $k^*$ such that $\overline{u} \in \kappa$ is not a square. Then

$$k^*/k^{*2} = \{1, u, \pi, u\pi\}.$$

Since $\kappa$ is finite, every 3-dimensional quadratic form over $\kappa$ is isotropic. By Hensel's lemma, every 3-dimensional form $\langle u_1, u_2, u_3 \rangle$ over $k$, with $u_i$ units in $k$ is isotropic. Since every form $q$ in $k$ has a diagonal representation

$$\langle u_1, \ldots, u_r \rangle \perp \pi \langle v_1, \ldots, v_s \rangle,$$

if $r$ or $s$ exceeds 3, $q$ is isotropic. In particular every 5-dimensional quadratic form over $k$ is isotropic. Further, up to isometry, there is a unique quadratic form in dimension 4 which is anisotropic, namely,

$$\langle 1, -u, -\pi, u\pi \rangle.$$

This is the norm form of the unique quaternion division algebra $H(u, \pi)$ over $k$ (cf. section 2.3).

## 2.3 Central simple algebras and the Brauer group

Recall that a finite-dimensional algebra $A$ over a field $k$ is a *central simple algebra* over $k$ if $A$ is simple (has no two-sided ideals) and the center of $A$ is $k$. Recall also that for a field $k$,

$$\mathrm{Br}(k) = \{\text{Isomorphism classes of central simple algebras over } k\} \,/\sim$$

where the Brauer equivalence $\sim$ is given by: $A \sim B$ if and only if $M_n(A) \cong M_m(B)$ for some integers $m, n$. The pair $(\mathrm{Br}(k), \otimes)$ is a group. The inverse of $[A]$ is $[A^{\mathrm{op}}]$ where $A^{\mathrm{op}}$ is the *opposite algebra* of $A$: the multiplication structure, $*$, on $A^{\mathrm{op}}$ is given by $a * b = ba$. We have a $k$-algebra isomorphism $\phi\colon A \otimes A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_k(A)$ induced by $\phi(a \otimes b)(c) = acb$. The identity element in $\mathrm{Br}(k)$ is given by $[k]$. By Wedderburn's theorem on central simple algebras, the elements of $\mathrm{Br}(k)$ parametrize the isomorphism classes of finite-dimensional central division algebras over $k$.

For elements $a, b \in k^*$, we define the **quaternion algebra** $H(a, b)$ to be the 4-dimensional central simple algebra over $k$ generated by $\{i, j\}$ with the relations $i^2 = a$, $j^2 = b$, $ij = -ji$. This is a generalization of the standard Hamiltonian quaternion algebra $H(-1, -1)$. The algebra $H(a, b)$ admits a canonical involution $^-\colon H(a, b) \to H(a, b)$ given by

$$\overline{\alpha + i\beta + j\gamma + ij\delta} = \alpha - i\beta - j\gamma - ij\delta$$

This involution gives an isomorphism $H(a, b) \cong H(a, b)^{\mathrm{op}}$; in particular, $H(a, b)$ has order 2 in ${}_2\mathrm{Br}(k)$, where ${}_2\mathrm{Br}(k)$ denotes the 2-torsion subgroup of the Brauer group of $k$. The norm form for this algebra is given by $N(x) = x\overline{x}$, which is a quadratic form on $H(a, b)$ represented with respect to the orthogonal basis $\{1, i, j, ij\}$ by $\langle 1, -a, -b, ab\rangle = \langle\langle -a, -b\rangle\rangle$.

## 2.4 Classical invariants for quadratic forms

Let $(V, q)$ be a regular quadratic form. We define $\dim(q) = \dim(V)$ and $\dim_2(q) = \dim(V)$ modulo 2. We have a ring homomorphism $\dim_2\colon W(k) \to \mathbb{Z}/2\mathbb{Z}$. We note that $I(k)$ is the kernel of $\dim_2$. This gives an isomorphism

$$\dim_2\colon W(k)/I(k) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}.$$

Let $\mathrm{disc}(q) = (-1)^{n(n-1)/2}[\det(A(q))] \in k^*/k^{*2}$. Since $A(q)$ is determined up to congruence, $\det(A(q))$ is determined modulo squares. We have $\mathrm{disc}(H) = 1$, and $\mathrm{disc}(q)$ induces a group homomorphism

$$\mathrm{disc}\colon I(k) \to k^*/k^{*2}$$

which is clearly onto. It is easy to verify that $\ker(\mathrm{disc}) = I^2(k)$. Thus the discriminant homomorphism induces an isomorphism $I(k)/I^2(k) \to k^*/k^{*2}$.

The next invariant for quadratic forms is the Clifford invariant. To each quadratic form $(V, q)$ we wish to construct a central simple algebra containing $V$ whose multiplication on elements of $V$ satisfies $v \cdot v = q(v)$. The smallest such algebra (defined by a universal property) will be the Clifford algebra.

**Definition 2.5.** The **Clifford algebra** $C(q)$ of the quadratic form $(V, q)$ is $T(V)/I_q$, where $I_q$ is the two-sided ideal in the tensor algebra $T(V)$ generated by $\{v \otimes v - q(v), v \in V\}$.

The algebra $C(q)$ has a $\mathbb{Z}/2\mathbb{Z}$ gradation $C(q) = C_0(q) \oplus C_1(q)$ induced by the gradation $T(V) = T_0(V) \oplus T_1(V)$, where

$$T_0(V) = \bigoplus_{i \geq 0,\ i \text{ even}} V^{\otimes i} \qquad \text{and} \qquad T_1(V) = \bigoplus_{i \geq 1,\ i \text{ odd}} V^{\otimes i}.$$

If $\dim(q)$ is even, then $C(q)$ is a central simple algebra over $k$. If $\dim(q)$ is odd, $C_0(q)$ is a central simple algebra over $k$. The Clifford algebra $C(q)$ comes equipped with an involution $\tau$ defined by $\tau(v) = -v$, $v \in V$. Thus, if $\dim(q)$ is even, $C(q)$ determines a 2-torsion element in $\mathrm{Br}(k)$.

**Definition 2.6.** The **Clifford invariant** $c(q)$ of $(V, q)$ in $\mathrm{Br}(k)$ is defined as

$$c(q) = \begin{cases} [C(q)], & \text{if } \dim(q) \text{ is even} \\ [C_0(q)], & \text{if } \dim(q) \text{ is odd} \end{cases}$$

The Clifford invariant induces a homomorphism $c\colon I^2(k) \to {}_2\mathrm{Br}(k)$, ${}_2\mathrm{Br}(k)$ again denoting the 2-torsion in the Brauer group of $k$. The very first case of the Milnor conjecture (see section 3) states: $c$ is surjective and $\ker(c) = I^3(k)$.

**Theorem 2.7** (Merkurjev [M1])**.** *The map $c$ induces an isomorphism*

$$I^2(k)/I^3(k) \cong {}_2\mathrm{Br}(k)$$

**Example 2.8.** Let $q \cong \otimes_{i=1}^n \langle\langle -a_i, -b_i \rangle\rangle \in I^2(k)$. Then

$$c(q) \cong \otimes_{1 \leq i \leq n} H_i$$

where $H_i = H(a_i, b_i)$.

**Exercise 2.9.** Given $\bigotimes_{1\le i\le n} H_i$, a tensor product of $n$ quaternion algebras over $k$, show that there is a quadratic form $q$ over $k$ of dimension $2n+2$ such that $c(q) \cong \bigotimes_{1\le i\le n} H_i$.

Thus the image of $I^2(q)$ in ${}_2\mathrm{Br}(k)$ is spanned by quaternion algebras. It was a longstanding question whether ${}_2\mathrm{Br}(k)$ is spanned by quaternion algebras. Merkurjev's theorem answers this question in the affirmative; further, it gives precise relations between quaternion algebras in ${}_2\mathrm{Br}(k)$.

# 3 Galois cohomology and the Milnor conjecture

Let $\Gamma_k = \mathrm{Gal}(\bar{k}|k)$, $\bar{k}$ denoting the separable closure of $k$, be the absolute Galois group of $k$. The group

$$\Gamma_k = \varprojlim_{L\subset\bar{k},\, L|k \text{ finite Galois}} \mathrm{Gal}(L|k)$$

is a profinite group. A *discrete $\Gamma_k$-module $M$* is a continuous $\Gamma_k$-module for the discrete topology on $M$ and the profinite topology on $\Gamma_k$. For a discrete $\Gamma_k$-module $M$, we define $H^n(k, M)$ as the direct limit of the cohomology of the finite quotients

$$H^n(k, M) = \varinjlim_{L\subset\bar{k},\, L|k \text{ finite Galois}} H^n(\mathrm{Gal}(L|k), M^{\Gamma_L}).$$

Suppose $\mathrm{char}(k) \neq 2$ and $M = \mu_2$. The module $\mu_2$ has trivial $\Gamma_k$ action. We denote this module by $\mathbb{Z}/2\mathbb{Z}$. We have

$$H^0(k, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

$$H^1(k, \mathbb{Z}/2\mathbb{Z}) \cong k^*/k^{*2}$$

$$H^2(k, \mathbb{Z}/2\mathbb{Z}) \cong {}_2\mathrm{Br}(k)$$

These can be seen from the Kummer exact sequence of $\Gamma_k$-modules:

$$0 \longrightarrow \mu_2 \longrightarrow \bar{k}^* \xrightarrow{\cdot 2} \bar{k}^* \longrightarrow 0$$

and noting that $H^1(\Gamma_k, \bar{k}^*) = 0$ (Hilbert's Theorem 90) and $H^2(\Gamma_k, \bar{k}^*) = \mathrm{Br}(k)$.

For an element $a \in k^*$, we denote by $(a)$ its class in $H^1(k, \mathbb{Z}/2\mathbb{Z})$ and for $a_1, \ldots, a_n \in k^*$, the cup product $(a_1) \cup \cdots \cup (a_n) \in H^n(k, \mathbb{Z}/2\mathbb{Z})$ is denoted by $(a_1). \cdots .(a_n)$.

For $a, b \in k^*$, the element $(a).(b)$ represents the class of $H(a, b)$ in ${}_2\mathrm{Br}(k)$. The map

$$c\colon I^2(k) \to H^2(k, \mathbb{Z}/2\mathbb{Z})$$

sends $\langle 1, -a, -b, ab \rangle$ to the class of $H(a, b)$ in $H^2(k, \mathbb{Z}/2\mathbb{Z})$. The forms $\langle 1, -a, -b, ab \rangle$ additively generate $I^2(k)$. Merkurjev's theorem asserts that $H^2(k, \mathbb{Z}/2\mathbb{Z})$ is generated by $(a).(b)$, with $a, b \in k^*$. The Milnor conjecture (quadratic form version) proposes higher invariants $I^n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z})$ extending the classical invariants.

**Milnor Conjecture.** *The assignment*

$$\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle \mapsto (a_1). \cdots .(a_n)$$

*yields a map* $e_n\colon P_n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z})$. *This map extends to a homomorphism* $e_n\colon I^n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z})$ *which is onto and* $\ker(e_n) = I^{n+1}(k)$.

The maps *dimension mod* 2, *discriminant* and *Clifford invariant* coincide with $e_0$, $e_1$ and $e_2$. Unlike these classical invariants, which are defined on all quadratic forms, conjecturally $e_n$, $n \geq 3$, are defined only on elements in $I^n(k)$ on which the invariants $e_i$, $i \leq n-1$, vanish. In 1975, Arason [Ar] proved that $e_3\colon I^3(k) \to H^3(k, \mathbb{Z}/2\mathbb{Z})$ is well defined and is one-one on $P_3(k)$. As we mentioned earlier, the first nontrivial case of the Milnor conjecture was proved by Merkurjev for $n = 2$. The Milnor conjecture (quadratic form version) is now a theorem due to Orlov–Vishik–Voevodsky [OVV].

The Milnor conjecture gives a classification of quadratic forms by their Galois cohomology invariants: Given anisotropic quadratic forms $q_1$ and $q_2$, suppose $e_i(q_1 \perp -q_2) = 0$ for $i \geq 0$. Then $q_1 = q_2$ in $W(k)$. We need only to verify $e_i(q_1 \perp -q_2) = 0$ for $i \leq N$ where $N \leq 2^n$ and $\dim(q_1 \perp -q_2) \leq 2^n$, by the following theorem of Arason and Pfister.

**Theorem 3.1** (Arason–Pfister Hauptsatz)**.** *Let $k$ be a field. The dimension of an anisotropic quadratic form in $I^n(k)$ is at least $2^n$.*

# 4 Pfister forms

The theory of Pfister forms (or multiplicative forms, as Pfister called them) evolved from questions on classification of quadratic forms whose nonzero values form a group (hereditarily).

**Definition 4.1.** A regular quadratic form $q$ over $k$ is called **multiplicative** if the nonzero values of $q$ over any extension field $L$ over $k$ form a group.

We have the following examples of quadratic forms which are multiplicative.

**Example 4.2.** $\langle 1 \rangle$: nonzero squares are multiplicatively closed in $k^*$.

**Example 4.3.** $\langle 1, -a \rangle$: $x^2 - ay^2$, $a \in k^*$ is the norm from the quadratic algebra $k[t]/(t^2 - a)$ over $k$ and the norm is multiplicative.

**Example 4.4.** $\langle 1, -a \rangle \otimes \langle 1, -b \rangle$: $x^2 - ay^2 - bz^2 + abt^2$ is a norm form from the quaternion algebra $H(a, b)$: $N(\alpha + i\beta + j\gamma + ij\delta) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2$. The norm once again is multiplicative.

**Example 4.5.** $\langle 1, -a \rangle \otimes \langle 1, -b \rangle \otimes \langle 1, -c \rangle$: $(x^2 - ay^2 - bz^2 + abt^2) - c(u^2 - av^2 - bw^2 + abs^2)$ is the norm form from an octonion algebra associated to the triple $(a, b, c)$; it is a non-associative algebra obtained from the quaternion algebra $H(a, b)$ by a doubling process. The norm is once again multiplicative.

**Theorem 4.6** (Pfister)**.** *An anisotropic quadratic form $q$ over $k$ is multiplicative if and only if $q$ is isomorphic to a Pfister form.*

We shall sketch a proof of this theorem. The main ingredients are

**Theorem 4.7** (Cassels–Pfister)**.** *Let $q = \langle a_1, \ldots, a_n \rangle$ be a regular quadratic form over $k$ and $f(X) \in k[X]$, a polynomial over $k$ which is a value of $q$ over $k(X)$. Then there exist polynomials $g_1, \ldots, g_n \in k[X]$ such that $f(X) = a_1 g_1^2(X) + \cdots + a_n g_n^2(X)$.*

**Corollary 4.8** (Specialization Lemma)**.** *Let $q = \langle a_1, \dots, a_n \rangle$ be a quadratic form over $k$, $X = \{X_1, \dots, X_n\}$, $p(X) \in k(X)$ a rational function represented by $q$ over $k(X)$. Then for any $v \in k^n$ where $p(v)$ is defined, $p(v)$ is represented by $q$ over $k$.*

*Proof.* We may assume, by multiplying $p(X)$ by a square, that $p(X) \in k[X]$. Let $p(X) = p_1(X_n)$, where $p_1$ is a polynomial in $X_n$ with coefficients in $k[X_1, \dots, X_{n-1}]$. By Cassels–Pfister theorem, $p_1(X_n)$ is represented by $q$ over $k(X_1, \dots, X_{n-1})[X_n]$. Let $v = (v_1, \dots, v_n)$. Then specializing $X_n$ to $v_n$, we have $p_1(v_n) \in k[X_1, \dots, X_{n-1}]$ is represented by $q$ over $k(X_1, \dots, X_{n-1})$. By an induction argument, one concludes that $p(v_1, \dots, v_n)$ is a value of $q$ over $k$. □

**Corollary 4.9.** *Let $q$ be an anisotropic quadratic form over $k$ of dimension $n$. Then $q$ is multiplicative if and only if for indeterminates $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$, $q(X)\, q(Y)$ is a value of $q$ over $k(X_1, \dots, X_n, Y_1, \dots, Y_n)$.*

*Proof.* The only non-obvious part is "only if". Suppose $L|k$ is a field extension and $v, w \in L^n$. Let $q(v) = c$ and $q(w) = d$. Since $q(X)\, q(Y)$ is a value of $q$ over $k(X, Y)$, by Specialization Lemma, $q(X)\, q(w)$ is a value of $q$ over $L(X)$ and by the same lemma, $q(v)\, q(w)$ is a value of $q$ over $L$. □

**Theorem 4.10** (Subform Theorem)**.** *Let $q = \langle a_1, \dots, a_n \rangle$, $\gamma = \langle b_1, \dots, b_m \rangle$ be anisotropic quadratic forms over $k$. Then $\gamma$ is a subform of $q$ (i.e., $q \cong \gamma \perp \gamma'$ for some form $\gamma'$ over $k$) if and only if $b_1 X_1^2 + \cdots + b_m X_m^2$ is a value of $q$ over $k(X_1, \dots, X_m)$.*

**Corollary 4.11.** *Let $q$ be an anisotropic quadratic form over $k$ of dimension $n$. Let $X = \{X_1, \dots, X_n\}$ be a set of $n$ indeterminates. Then $q$ is multiplicative if and only if $q \cong q(X)\, q$ over $k(X)$.*

*Proof.* Suppose $q \cong q(X)\, q$ over $k(X)$. Let $A$ be the matrix representing $q$ over $k$. There exists $W \in \mathrm{GL}_n(k(X))$ such that $q(X)A = WAW^t$. Let $Y = \{Y_1, \dots, Y_n\}$ be a set of $n$ indeterminates. Over $k(X, Y)$,

$$q(X)\, q(Y) = Y(q(X)A)Y^t = (YW)A(YW)^t = q(Z)$$

where $Z = YW$. Thus $q(X)\, q(Y)$ is a value of $q$ over $k(X, Y)$ and by Corollary 4.9, $q$ is multiplicative. Suppose conversely that $q$ is multiplicative. Then

$q(X)\,q(Y)$ is a value of $q$ over $k(X,Y)$. By the subform theorem, $q(X)\,q$ is a subform of $q$. A dimension count yields $q \cong q(X)\,q$. □

*Proof of Pfister's theorem 4.6.* Let $q = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ be an anisotropic quadratic form over $k$. Over any field extension $L|k$, either $q$ is an anisotropic Pfister form or isotropic in which case it is universal. Thus it suffices to show that the nonzero values of $q$ form a subgroup of $k^*$ for any anisotropic $n$-fold Pfister form $q$. The proof is by induction on $n$; for $n = 1$, $q$ is the norm form from a quadratic extension of $k$ (see Example 4.3). Let $n \geq 2$. We have $q \cong q_1 \perp a_n q_1$, where $q_1 = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_{n-1}\rangle$ is an anisotropic $(n-1)$-fold Pfister form. Let $X = \{X_1, \ldots, X_{2^{n-1}}\}$, $Y = \{Y_1, \ldots, Y_{2^{n-1}}\}$ be two sets of $2^{n-1}$ indeterminates. Since $q_1$ is multiplicative, by Corollary 4.11, $q_1(X)\,q_1 \cong q_1$ over $k(X)$ and $q_1(Y)\,q_1 \cong q_1$ over $k(Y)$. We have, over $k(X,Y)$,

$$q \cong q_1(X)\,q_1 \perp a_n q_1(Y)\,q_1 \cong \langle q_1(X), a_n q_1(Y)\rangle \otimes q_1.$$

Since $q(X,Y) = q_1(X) + a_n q_1(Y)$, $\langle q_1(X), a_n q_1(Y)\rangle$ represents $q(X,Y)$. Therefore, by a comparison of discriminants,

$$\begin{aligned} \langle q_1(X), a_n q_1(Y)\rangle &\cong \langle q(X,Y), a_n q(X,Y) q_1(X) q_1(Y)\rangle \\ &\cong q(X,Y)(1 \perp a_n q_1(X) q_1(Y)) \end{aligned}$$

In particular,

$$\begin{aligned} q &\cong q(X,Y)\langle 1, a_n q_1(X) q_1(Y)\rangle \otimes q_1 \\ &\cong q(X,Y)(q_1 \perp a_n q_1) \\ &\cong q(X,Y)\,q \end{aligned}$$

Thus by Corollary 4.11, $q$ is multiplicative.

Conversely, let $q$ be an anisotropic quadratic form over $k$ which is multiplicative. Let $n$ be the largest such that $q$ contains an $n$-fold Pfister form $q_1 = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ as a subform. Suppose $q \cong q_1 \perp \gamma$, $\gamma = \langle b_1, \ldots, b_m\rangle$, with $m \geq 1$. Let $Z = \{Z_1, \ldots, Z_{2^n}\}$. Over $k(Z)$,

$$q \cong q(Z,0)\,q \cong q_1(Z)(q_1 \perp \gamma) \cong q_1(Z)\,q_1 \perp q_1(Z)\,\gamma \cong q_1 \perp q_1(Z)\,\gamma.$$

By Witt's cancellation, $\gamma \cong q_1(Z)\,\gamma$ over $k(Z)$. Thus $\gamma$ represents $b_1 q_1(Z)$ over $k(Z)$ and by the subform theorem, $\gamma \cong b_1\,q_1 \perp \gamma_1$. Then $q \cong q_1 \perp$

$b_1 q_1 \perp \gamma_1 \cong \langle 1, b_1 \rangle \otimes q_1 \perp \gamma_1$ contains an $(n+1)$-fold Pfister form $\langle 1, b_1 \rangle \otimes q_1$, leading to a contradiction to maximality of $n$. Thus $q \cong q_1$. $\square$

An important property of Pfister forms is stated in the following.

**Proposition 4.12.** *Let $\phi$ be an $n$-fold Pfister form. If $\phi$ is isotropic then $\phi$ is hyperbolic.*

*Proof.* Let $\phi = r \langle 1, -1 \rangle \perp \phi_0$, with $\phi_0$ anisotropic, $\dim(\phi_0) \geq 1$ and $r \geq 1$. Let $\dim(\phi) = m$ and $X = \{X_1, \ldots, X_m\}$ be a set of $m$ indeterminates. Over $k(X_1, \ldots, X_m)$

$$\phi = r \langle 1, -1 \rangle \perp \phi_0 \cong \phi(X_1, \ldots, X_m)\, \phi \cong r \langle 1, -1 \rangle \perp \phi(X_1, \ldots, X_m)\, \phi_0.$$

By Witt's cancellation theorem

$$\phi_0 \cong \phi(X_1, \ldots, X_m)\, \phi_0.$$

If $b$ is a value of $\phi_0$, $b\phi(X_1, \ldots, X_m)$ is a value of $\phi_0$ and by the subform theorem, $b\phi$ is a subform of $\phi_0$ contradicting $\dim(\phi_0) < \dim(\phi)$. Thus $\phi \cong r \langle 1, -1 \rangle$ is hyperbolic. $\square$

**Corollary 4.13.** *The only integers $n$ such that a product of sums of $n$ squares is again a sum of $n$ squares over any field of characteristic zero are $n = 2^m$ for all $m \geq 0$.*

# 5 Level of a field

**Definition 5.1.** The **level** of a field $k$ is the least positive integer $n$ such that $-1$ is a sum of $n$ squares in $k$. We denote the level of $k$ by $s(k)$.

If the field is formally real (i.e., $-1$ is not a sum of squares), then the level is infinite. It was a longstanding open question whether the level of a field, if finite, is always a power of 2. Pfister's theory of quadratic forms leads to an affirmative answer to this question.

**Theorem 5.2** (Pfister)**.** *The level of a field is a power of 2 if it is finite.*

*Proof.* Let $n = s(k)$. We choose an integer $m$ such that $2^m \leq n < 2^{m+1}$. Suppose

$$-1 = (u_1^2 + u_2^2 + \cdots + u_{2^m}^2) + (u_{2^m+1}^2 + \cdots + u_n^2) \tag{5.3}$$

The element $u_1^2 + u_2^2 + \cdots + u_{2^m}^2 \neq 0$ since $s(k) \geq 2^m$. Every ratio of sums of $2^m$ squares is again a sum of $2^m$ squares since $\langle 1, 1\rangle^{\otimes m}$ is a multiplicative form. Thus, from (5.3) we see that

$$\begin{aligned} 0 &= 1 + \frac{u_{2^m+1}^2 + \cdots + u_n^2 + 1}{u_1^2 + \cdots + u_{2^m}^2} \\ &= 1 + (v_1^2 + \cdots + v_{2^m}^2) \end{aligned}$$

Therefore, $-1 = v_1^2 + \cdots + v_{2^m}^2$ and $s(k) = 2^m$. □

**Remark 5.4.** There exist fields with level $2^n$ for any $n \geq 1$. For instance, $\mathbb{R}(X_1, \ldots, X_{2^n})(\sqrt{-(X_1^2 + \cdots + X_{2^n}^2)}\,)$ is a field of level $2^n$.

**Exercise 5.5.** Let $k$ be a $p$-adic field with $p \neq 2$ and with residue field $\mathbb{F}_q$. Prove the following:

1. $s(k) = 1$ if $q \equiv 1 \pmod 4$.
2. $s(k) = 2$ if $q \equiv -1 \pmod 4$.

# 6 The $u$-invariant

**Definition 6.1.** The ***u*-invariant** of a field $k$, denoted by $u(k)$, is defined to be the largest integer $n$ such that every $(n+1)$-dimensional quadratic form over $k$ is isotropic and there is an anisotropic form in dimension $n$ over $k$.

$$u(k) = \max\{\dim(q) \,:\, q \text{ anisotropic form over } k\}.$$

If $k$ admits an ordering, then sums of nonzero squares are never zero and there is a refined $u$-invariant for fields with orderings, due to Elman–Lam [EL].

**Example 6.2.** 1. $u(\mathbb{F}_q) = 2$.

2. $u(k(X)) = 2$, if $k$ is algebraically closed and $X$ is an integral curve over $k$ (Tsen's theorem).

3. $u(k) = 4$ for $k$ a $p$-adic field.

4. $u(k) = 4$ for $k$ a totally imaginary number field. This follows from the Hasse–Minkowski theorem.

5. Suppose $u(k) = n < \infty$. Let $k((t))$ denote the field of Laurent series over $k$. Then $u(k((t))) = 2n$. In fact, the square classes in $k((t))^*$ are $\{u_\alpha, tu_\alpha\}_{\alpha\in I}$ where $\{u_\alpha\}_{\alpha\in I}$ are the square classes in $k^*$. As in the $p$-adic field case, every form over $k((t))$ is isometric to $\langle u_1, \dots, u_r\rangle \perp t\langle v_1, \dots, v_s\rangle$, $u_i, v_i \in k^*$ and this form is anisotropic if and only if $\langle u_1, \dots, u_r\rangle$ and $\langle v_1, \dots, v_s\rangle$ are anisotropic.

6. More generally, if $K$ is a complete discrete valuated field with residue field $\kappa$ of $u$-invariant $n$, then $u(K) = 2n$.

**Definition 6.3.** A field $k$ is $\boldsymbol{C_i}$ if every homogeneous polynomial in $N$ variables of degree $d$ with $N > d^i$ has a nontrivial zero.

**Example 6.4.** Finite fields and function fields in one variable over algebraically closed fields are $C_1$.

If $k$ is a $C_i$ field, $u(k) \leq 2^i$. Further, the property $C_i$ behaves well with respect to function field extensions. If $l|k$ is finite and $k$ is $C_i$ then $l$ is $C_i$; further, if $t_1, \dots, t_n$ are indeterminates, $k(t_1, \dots, t_n)$ is $C_{i+n}$.

**Example 6.5.** The $u$-invariant of transcendental extensions:

1. $u(k(t_1, \dots, t_n)) = 2^n$ if $k$ is algebraically closed. In fact,
$$u(k(t_1, \dots, t_n)) \leq 2^n$$
since $k(t_1, \dots, t_n)$ is a $C_n$ field. Further, the form
$$\langle\langle t_1, \dots, t_n\rangle\rangle = \langle 1, t_1\rangle \otimes \cdots \otimes \langle 1, t_n\rangle$$
is anisotropic over $k((t_1))((t_2))\dots((t_n))$ and hence also over $k(t_1, \dots, t_n)$.

2. $u(\mathbb{F}_q(t_1, \dots, t_n)) = 2^{n+1}$.

All fields of known $u$-invariant in the 1950's happened to have $u$-invariant a power of 2. Kaplansky raised the question whether the $u$-invariant of a field is always a power of 2.

**Proposition 6.6.** *The $u$-invariant does not take the values* $3, 5, 7$.

*Proof.* Let $q$ be an anisotropic form of dimension 3. By scaling, we may assume that $q \cong \langle 1, a, b \rangle$. Then the form $\langle 1, a, b, ab \rangle$ is anisotropic; if $\langle 1, a, b, ab \rangle$ is isotropic, since discriminant is one, it is hyperbolic and Witt's cancellation yields $\langle a, b, ab \rangle \cong \langle 1, -1, -1 \rangle$ is isotropic and $q \cong a\langle a, b, ab \rangle$ is isotropic leading to a contradiction. Thus $u(k) \neq 3$.

Let $u(k) < 8$. Every 3-fold Pfister form (which has dimension 8) is isotropic and hence hyperbolic. Thus $I^3(k)$ which is generated by 3-fold Pfister forms is zero. Let $q \in I^2(k)$ be any quadratic form. For any $c \in k^*$, $\langle 1, -c \rangle\, q \in I^3(k)$ is zero and $cq$ is Witt equivalent to $q$, hence isometric to $q$ by Witt's cancellation. We conclude that every quadratic form whose class is in $I^2(k)$ is universal.

Suppose $u(k) = 5$ or 7. Let $q$ be an anisotropic form of dimension $u(k)$. Since every form in dimension $u(k) + 1$ is isotropic, if $\mathrm{disc}(q) = d$, $q \perp -d$ is isotropic and therefore $q$ represents $d$. We may write $q \cong q_1 \perp \langle d \rangle$ where $q_1$ is even-dimensional with trivial discriminant. Hence $[q_1] \in I^2(k)$ so that $q_1$ is universal. This in turn implies that $q_1 \perp \langle d \rangle \cong q$ is isotropic, leading to a contradiction. □

In the 1990's Merkurjev [M2] constructed examples of fields $k$ with $u(k) = 2n$ for any $n \geq 1$, $n = 3$ being the first open case, answering Kaplansky's question in the negative. Since then, it has been shown that the $u$-invariant could be odd. In [I], Izhboldin proves there exist fields $k$ with $u(k) = 9$ and in [V] Vishik has shown that there exist fields $k$ with $u(k) = 2^r + 1$ for all $r \geq 3$.

Merkurjev's construction yields fields $k$ which are not of arithmetic type, i.e., not finitely generated over a number field or a $p$-adic field. It is still an interesting question whether $u(k)$ is a power of 2 if $k$ is of arithmetic type.

The behavior of the $u$-invariant is very little understood under rational function field extensions. For instance, it is an open question if $u(k) < \infty$ implies $u(k(t)) < \infty$ for the rational function field in one variable over $k$. This was unknown for $k = \mathbb{Q}_p$ until the late 1990's. Conjecturally, $u(\mathbb{Q}_p(t)) = 8$, in analogy with the positive characteristic local field case, $u(\mathbb{F}_p((X))(t)) = 8$.

We indicate some ways of bounding the $u$-invariant of a field $k$ once we know how efficiently the Galois cohomology groups $H^n(k, \mathbb{Z}/2\mathbb{Z})$ are generated by symbols for all $n$.

We set

$$H^n_{\rm dec}(k, \mathbb{Z}/2\mathbb{Z}) = \{(a_1). \cdots .(a_n),\ a_i \in k^*\}$$

and call elements in this set symbols. By Voevodsky's theorem on Milnor conjecture, $H^n(k, \mathbb{Z}/2\mathbb{Z})$ is additively generated by $H^n_{\rm dec}(k, \mathbb{Z}/2\mathbb{Z})$.

**Proposition 6.7.** *Let $k$ be a field such that $H^{n+1}(k, \mathbb{Z}/2\mathbb{Z}) = 0$ and for $2 \leq i \leq n$, there exist integers $N_i$ such that every element in $H^i(k, \mathbb{Z}/2\mathbb{Z})$ is a sum of $N_i$ symbols. Then $u(k)$ is finite.*

*Proof.* Let $q$ be a quadratic form over $k$ of dimension $m$ and discriminant $d$. Let $q_1 = \langle d \rangle$ if $m$ is odd and $\langle 1, -d \rangle$ if $m$ is even. Then $q \perp -q_1$ has even dimension and trivial discriminant. Hence $q \perp -q_1 \in I^2(k)$. Let $e_2(q \perp -q_1) = \sum_{j \leq N_2} \xi_{2j}$ where $\xi_{2j} \in H^2_{\rm dec}(k, \mathbb{Z}/2\mathbb{Z})$. Let $\phi_{2j}$ be 2-fold Pfister forms such that $e_2(\phi_{2j}) = \xi_{2j}$. Then $q_2 = \sum_{j \leq N_2} \phi_{2j}$ has dimension at most $4N_2$ and $e_2(q \perp -q_1 \perp -q_2) = 0$ and $q \perp -q_1 \perp -q_2 \in I^3(k)$, by Merkurjev's theorem. Repeating this process and using Milnor Conjecture, we get $q_i \in I^i(k)$ which is a sum of $N_i$ $i$-fold Pfister forms and $q - \sum_{1 \leq i \leq n} q_i \in I^{n+1}(k) = 0$, since $H^{n+1}(k, \mathbb{Z}/2\mathbb{Z}) = 0$. Thus $[q] = \sum_{1 \leq i \leq n} q_i$ and $\dim(q_{an}) \leq \sum_{1 \leq i \leq n} 2^i N_i$. Thus $u(k) \leq \sum_{1 \leq i \leq n} 2^i N_i$. □

**Definition 6.8.** A field $k$ is said to have **cohomological dimension at most $n$** (in symbols, $\operatorname{cd}(k) \leq n$) if $H^i(k, M) = 0$ for $i \geq n+1$ for all finite discrete $\Gamma_k$-modules $M$ (cf. [Se] §3).

**Example 6.9.** Finite fields and function fields in one variable over algebraically closed fields have cohomological dimension 1. Totally imaginary number fields and $p$-adic fields are of cohomological dimension 2. Thus if $k$ is a $p$-adic field, and $k(X)$ a function field in one variable over $k$, $\operatorname{cd}(k(X)) \leq 3$. In particular, $H^4(k(X), \mathbb{Z}/2\mathbb{Z}) = 0$.

**Theorem 6.10** (Saltman)**.** *Let $k$ be a non-dyadic $p$-adic field and $k(X)$ a function field in one variable over $k$. Every element in $H^2(k(X), \mathbb{Z}/2\mathbb{Z})$ is a sum of two symbols.*

**Theorem 6.11** (Parimala–Suresh)**.** *Let $k(X)$ be as in the previous theorem. Then every element in $H^3(k(X), \mathbb{Z}/2\mathbb{Z})$ is a symbol.*

**Corollary 6.12.** *For $k(X)$ as above, $u(k(X)) \leq 2 + 8 + 8 = 18$.*

It is not hard to show from the above theorems that $u(k(X)) \leq 12$. With some further work it was proved in [PS1] that $u(k(X)) \leq 10$. More recently in [PS2] the estimated value $u(k(X)) = 8$ was proved. For an alternate approach to $u(k(X)) = 8$, we refer to ([HH], [HHK], [CTPS]). More recently, Heath-Brown and Leep [HB] have proved the following spectacular theorem: If $k$ is *any* $p$-adic field and $k(X)$ the function field in $n$ variables over $k$, then $u(k(X)) = 2^{n+2}$.

# 7 Hilbert's seventeenth problem

An additional reference for sums of squares is available from H. Cohen at http://www.math.u-bordeaux1.fr/∼cohen/Cohensquares.pdf, which is a translation of the original paper [C].

**Definition 7.1.** An element $f \in \mathbb{R}(X_1, \ldots, X_n)$ is called **positive semi-definite** if $f(a) \geq 0$ for all $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ where $f$ is defined.

**Hilbert's seventeenth problem:**
Let $\mathbb{R}(X_1, \ldots, X_n)$ be the rational function field in $n$ variables over the field $\mathbb{R}$ of real numbers. Hilbert's seventeenth problem asks whether every positive semi-definite $f \in \mathbb{R}(X_1, \ldots, X_n)$ is a sum of squares in $\mathbb{R}(X_1, \ldots, X_n)$. E. Artin settled this question in the affirmative and Pfister gave an effective version of Artin's result (cf. [Pf], chapter 6).

**Theorem 7.2** (Artin, Pfister)**.** *Every positive semi-definite function $f \in \mathbb{R}(X_1, \ldots, X_n)$ can be written as a sum of $2^n$ squares in $\mathbb{R}(X_1, \ldots, X_n)$.*

For $n \leq 2$ the above was due to Hilbert himself. If one asks for expressions of positive definite polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ as sums of $2^n$ squares in $\mathbb{R}[X_1, \ldots, X_n]$, there are counterexamples for $n = 2$; the Motzkin polynomial

$$f(X_1, X_2) = 1 - 3X_1^2X_2^2 + X_1^4X_2^2 + X_1^2X_2^4$$

is positive semi-definite but not a sum of 4 squares in $\mathbb{R}[X_1, X_2]$. In fact, Pfister's result has the following precise formulation.

**Theorem 7.3** (Pfister). *Let $\mathbb{R}(X)$ be a function field in $n$ variables over $\mathbb{R}$. Then every $n$-fold Pfister form in $\mathbb{R}(X)$ represents every sum of squares in $\mathbb{R}(X)$.*

We sketch a proof of this theorem below.

**Definition 7.4.** Let $\phi$ be an $n$-fold Pfister form with $\phi = 1 \perp \phi'$. The form $\phi'$ is called the **pure subform of $\phi$.**

**Proposition 7.5** (Pure Subform Theorem). *Let $k$ be any field, $\phi$ an anisotropic $n$-fold Pfister form over $k$ and $\phi'$ its pure subform. If $b_1$ is any value of $\phi'$, then $\phi \cong \langle\langle b_1, \ldots, b_n \rangle\rangle$.*

*Proof.* The proof is by induction on $n$; for $n = 1$ the statement is clear. Let $n > 1$. We assume the statement holds for all $(n-1)$-fold Pfister forms. Let $\phi = \langle\langle a_1, \ldots, a_n \rangle\rangle$, $\psi = \langle\langle a_1, \ldots, a_{n-1} \rangle\rangle$, and let $\phi'$, $\psi'$ denote the pure subforms of $\phi$ and $\psi$ respectively. We have $\phi = \psi \perp a_n\psi$, $\phi' = \psi' \perp a_n\psi$. Let $b_1$ be a value of $\phi'$. We may write $b_1 = b_1' + a_n b$, with $b_1'$ a value of $\psi'$ and $b$ a value of $\psi$. The only nontrivial case to discuss is when $b \neq 0$ and $b_1' \neq 0$. By induction, $\psi \cong \langle\langle b_1', b_2, \ldots, b_{n-1} \rangle\rangle$ and $b\psi \cong \psi$. We thus have

$$\begin{aligned} \phi &\cong \langle\langle b_1', b_2, \ldots, b_{n-1}, a_n \rangle\rangle \cong \langle\langle b_1', b_2, \ldots, b_{n-1}, a_n b \rangle\rangle \\ &\cong \langle\langle b_1', a_n b \rangle\rangle \otimes \langle\langle b_2, \ldots, b_{n-1} \rangle\rangle \end{aligned}$$

Since $b_1 = b_1' + a_n b$, $\langle b_1', a_n b \rangle \cong \langle b_1, b_1 b_1' a_n b \rangle$ and we have

$$\begin{aligned} \langle\langle b_1', a_n b \rangle\rangle &= \langle 1, b_1', a_n b, a_n b b_1' \rangle \\ &= \langle 1, b_1, b_1 b_1' a_n b, a_n b b_1' \rangle \\ &= \langle\langle b_1, c_1 \rangle\rangle, \end{aligned}$$

where $c_1 = b_1 b_1' a_n b$. Thus,

$$\phi \cong \langle\langle b_1, c_1, b_2, \cdots, b_{n-1} \rangle\rangle. \qquad \square$$

*Proof of Pfister's theorem.* Let $\phi$ be an anisotropic $n$-fold Pfister form over $K = \mathbb{R}(X)$. Let $b = b_1^2 + \cdots + b_m^2$, $b_i \in K^*$. We show that $\phi$ represents $b$ by induction on $m$. For $m = 1$, $b$ is a square and is represented by $\phi$. Suppose $m = 2$, $b = b_1^2 + b_2^2$, $b_1 \neq 0$, $b_2 \neq 0$. The field $K(\sqrt{-1})$ is a function field in $n$ variables over $\mathbb{C}$ and is $C_n$. Then $\phi$ is universal over $K(\sqrt{-1})$ and hence represents $\beta = b_1 + ib_2$. Let $v, w \in K^{2^n}$ such that $\phi_{K(\sqrt{-1})}(v + \beta w) = \beta$. Hence

$$\phi(v) + \beta^2\phi(w) + \beta(2\phi(v, w) - 1) = 0.$$

The irreducible polynomial of $\beta$ over $K$ is

$$\phi(w)X^2 + (2\phi(v, w) - 1)X + \phi(v)$$

and hence $N(\beta) = b = \frac{\phi(v)}{\phi(w)}$ is a value of $\phi$ since $\phi$ is multiplicative.

Suppose $m > 2$. We argue by induction on $m$. Suppose $\phi$ represents all sums of $m - 1$ squares. Let $b$ be a sum of $m$ squares. After scaling $b$ by a square, we may assume that $b = 1 + c$, $c = c_1^2 + \cdots + c_{m-1}^2$, $c \neq 0$. Let $\phi \cong 1 \perp \phi'$. By induction hypothesis, $\phi$ represents $c$. Let $c = c_0^2 + c'$, $c'$ a value of $\phi'$. Let $\psi = \phi \otimes \langle 1, -b\rangle$ and $\psi = 1 \perp \psi'$ with $\psi' = \langle -b\rangle \perp \phi' \perp -b\phi'$. The form $\psi'$ represents $c' - b = (c - c_0^2) - (1 + c) = -1 - c_0^2$. Thus, by the Pure Subform theorem,

$$\psi \cong \langle\langle -1 - c_0^2, d_1, \ldots, d_n\rangle\rangle = \langle 1, -1 - c_0^2\rangle \otimes \langle\langle d_1, \ldots, d_n\rangle\rangle.$$

By induction, the $n$-fold Pfister form $\langle\langle d_1, \ldots, d_n\rangle\rangle$ represents $1 + c_0^2$ which is a sum of 2 squares; thus $\psi$ is isotropic, hence hyperbolic. Thus $\phi \cong b\phi$ represents $b$. □

**Corollary 7.6.** *Let $K = \mathbb{R}(X)$ be a function field in $n$ variables over $\mathbb{R}$. Then every sum of squares in $K$ is a sum of $2^n$ squares.*

*Proof.* Set $\phi = \langle 1, 1\rangle^{\otimes n}$ in the above theorem. □

# 8 Pythagoras number

**Definition 8.1.** The **Pythagoras number** $p(k)$ of a field $k$ is the least positive integer $n$ such that every sum of squares in $k^*$ is a sum of at most $n$ squares.

**Example 8.2.** If $\mathbb{R}$ is the field of real numbers, $p(\mathbb{R}) = 1$.

**Example 8.3.** If $\mathbb{R}(X_1, \ldots, X_n)$ is a function field in $n$ variables over $\mathbb{R}$, by Pfister's theorem, $p(\mathbb{R}(X_1, \ldots, X_n)) \leq 2^n$.

## 8.1 Effectiveness of the bound $p(\mathbb{R}(X)) \leq 2^n$

Let
$$K = \mathbb{R}(X_1, \ldots, X_n)$$
be the rational function field in $n$ variables over $\mathbb{R}$. For $n = 1$ the bound is sharp. For $n = 2$ the Motzkin polynomial
$$f(X_1, X_2) = 1 - 3X_1^2X_2^2 + X_1^4X_2^2 + X_1^2X_2^4$$
is positive semi-definite; Cassels–Ellison–Pfister [CEP] show that this polynomial is not a sum of three squares in $\mathbb{R}(X_1, X_2)$ (see also [CT]). Therefore $p(\mathbb{R}(X_1, X_2)) = 4$.

**Lemma 8.4** (Key Lemma)**.** *Let $k$ be a field and $n = 2^m$. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n) \in k^n$ be such that $u \cdot v = \sum_{1 \leq i \leq n} u_i v_i = 0$. Then there exist $w_j \in k,\ 1 \leq j \leq n-1$ such that*
$$\sum_{1 \leq i \leq n} u_i^2 \sum_{1 \leq i \leq n} v_i^2 = \sum_{1 \leq j \leq n-1} w_j^2.$$

*Proof.* Let $\lambda = \sum_{1 \leq i \leq n} u_i^2$, $\mu = \sum_{1 \leq i \leq n} v_i^2$. We may assume without loss of generality that $u \neq 0$ and $v \neq 0$. The elements $\lambda$ and $\mu$ are values of $\phi_m = \langle 1, 1 \rangle^{\otimes m}$ and $\lambda\phi_m \cong \phi_m$, $\mu\phi_m \cong \phi_m$. We choose isometries $f \colon \lambda\phi_m \cong \phi_m$, $g \colon \mu\phi_m \cong \phi_m$ such that $f(1, 0, \ldots, 0) = u$ and $g(1, 0, \ldots, 0) = v$. If $U$ and $V$ are matrices representing $f$, $g$ respectively, we have
$$UU^t = \lambda^{-1}, \quad VV^t = \mu^{-1}, \quad \lambda^{-1}\mu^{-1} = \lambda^{-1}VV^t = (VU^t)(VU^t)^t.$$
The first row of $VU^t$ is of the form $(0, w_2, \ldots, w_n)$ since $u \cdot v = 0$. Thus $\lambda^{-1}\mu^{-1} = \sum_{2 \leq i \leq n} w_i^2$. □

**Corollary 8.5.** *Let $k$ be an ordered field with $p(k) = n$. Then $p(k(t)) \geq n+1$.*

*Proof.* Let $\lambda \in k^*$ be such that $\lambda$ is a sum of $n$ squares and not a sum of less than $n$ squares. Suppose $\lambda + t^2$ is a sum of $n$ squares in $k(t)$. By Cassels–Pfister theorem,
$$\lambda + t^2 = (\mu_1 + \nu_1 t)^2 + \cdots + (\mu_n + \nu_n t)^2$$

with $\mu_i, \nu_i \in k^*$. If $u = (\mu_1, \ldots, \mu_n)$, $v = (\nu_1, \ldots, \nu_n)$, then $u \cdot v = 0$, $\sum_{1\le i\le n} \mu_i^2 = \lambda$, $\sum_{1\le i\le n} \nu_i^2 = 1$. Thus $\lambda = (\sum_{1\le i\le n} \mu_i^2)(\sum_{1\le i\le n} \nu_i^2)$ is a sum of $n-1$ squares by the Key Lemma, 8.4, contradicting the choice of $\lambda$. □

**Corollary 8.6.** *$p(\mathbb{R}(X_1, \ldots, X_n)) \geq n+2$. Thus*

$$n + 2 \leq p(\mathbb{R}(X_1, \ldots, X_n)) \leq 2^n.$$

*Proof.* By [CEP], we know that $p(\mathbb{R}(X_1, X_2)) = 4$. The fact that $n + 2 \leq p(\mathbb{R}(X_1, \ldots, X_n))$ now follows by Corollary 8.5 and induction. □

**Remark 8.7.** It is open whether $p(\mathbb{R}(X_1, X_2, X_3)) = 5, 6, 7$ or $8$.

**Remark 8.8.** The possible values of the Pythagoras number of a field have all been listed ([H], [Pf, p. 97]).

**Proposition 8.9.** *If $k$ is a non-formally real field, $p(k) = s(k)$ or $s(k) + 1$.*

*Proof.* If $s(k) = n$, $-1$ is not a sum of less than $n$ squares, so that $p(k) \geq s(k)$. For $a \in k^*$,

$$a = \left(\frac{a+1}{2}\right)^2 + (-1)\left(\frac{a-1}{2}\right)^2$$

is a sum of $n+1$ squares if $-1$ is a sum of $n$ squares. Thus $p(k) \leq s(k)+1$. □

Let $k$ be a $p$-adic field and $K = k(X_1, \ldots, X_n)$ a rational function field in $n$ variables over $k$. Then $s(k) = 1, 2$ or $4$ so that $s(K) = 1, 2$, or $4$. Thus $p(K) \leq 5$ (in fact it is easy to see that if $s(k) = s$, $p(K) = s + 1$).

Thus we have bounds for $p(k(X_1, \ldots, X_n))$ if $k$ is the field of real or complex numbers or the field of $p$-adic numbers. The natural questions concern a number field $k$.

# 9 Function fields over number fields

Let $k$ be a number field and $F = k(t)$ the rational function field in one variable over $k$. In this case $p(k(t)) = 5$ is a theorem ([La]). The fact that $p(k(t)) \leq 8$ can be easily deduced from the following injectivity in the Witt groups ([CTCS], Prop. 1.1):

$$W(k(t)) \longrightarrow \prod_{w \in \Omega(k)} W(k_w(t)),$$

with $\Omega(k)$ denoting the set of places of $k$. In fact, if $f \in k(t)$ is a sum of squares, $f$ is a sum of at most two squares in $k_w(t)$ for a real place $w$, by Pfister's theorem (which in the case of function fields of curves goes back to Witt). Further, for a finite place $w$ of $k$ or a complex place, $\langle 1,1\rangle^{\otimes 3} = 0$ in $W(k_w)$. Thus $\langle 1,1\rangle^{\otimes 3} \otimes \langle 1,-f\rangle$ is hyperbolic over $k_w(t)$ for all $w \in \Omega(k)$.

By the above injectivity, this form is hyperbolic over $k(t)$, leading to the fact that $f$ is a sum of at most eight squares in $k(t)$.

We have the following conjecture due to Pfister for function fields over number fields.

**Conjecture** (Pfister)**.** *Let $k$ be a number field and $F = k(X)$ a function field in $d$ variables over $k$. Then*

1. *for $d = 1$, $p(F) \leq 5$.*
2. *for $d \geq 2$, $p(F) \leq 2^{d+1}$.*

For a general function field $k(X)$ in one variable over $k$, ($d = 1$), the best known result is due to F. Pop, $p(F) \leq 6$ [P]. We sketch some results and conjectures from the arithmetic side which could lead to a solution of the conjecture for $d \geq 2$ (see Colliot-Thélène, Jannsen [CTJ] for more details).

For any field $k$, by Voevodsky's theorem, we have an injection

$$e_n\colon P_n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z}).$$

In fact, for any field $k$, if $\phi_1$, $\phi_2 \in P_n(k)$ have the same image under $e_n$ then $\phi_1 \perp -\phi_2 \in \ker(e_n) = I^{n+1}(k)$. In $W(k)$, $\phi_1 \perp -\phi_2 = \phi_1' \perp -\phi_2'$ where $\phi_1'$ and $\phi_2'$ are the pure subforms of $\phi_1$ and $\phi_2$. Moreover, $\dim(\phi_1' \perp -\phi_2')_{\mathrm{an}} \leq 2^{n+1} - 2 < 2^{n+1}$. By the Arason–Pfister Hauptsatz, (Theorem 3.1), anisotropic forms in $I^{n+1}(k)$ must have dimension at least $2^{n+1}$. Therefore $\phi_1 = \phi_2$.

Let $k$ be a number field and $F = k(X)$ be a function field in $d$ variables over $k$. Let $f \in F$ be a function which is a sum of squares in $F$. One would like to show that $f$ is a sum of $2^{d+1}$ squares. Let $\phi_{d+1} = \langle 1,1\rangle^{\otimes(d+1)}$ and $q = \phi_{d+1} \otimes \langle 1,-f\rangle$. This is a $(d+2)$-fold Pfister form and $\phi_{d+1}$ represents $f$ if and only if $q$ is hyperbolic or equivalently, by the injectivity of $e_n$ above, $e_{d+2}(\phi_{d+1} \otimes \langle 1,-f\rangle) = 0$.

We look at this condition locally at all completions $k_v$ at places $v$ of $k$. Let $k_v(X)$ denote the function field of $X$ over $k_v$. (We may assume that $X$ is geometrically integral). Let $v$ be a complex place. The field $k_v(X)$ has cohomological dimension $d$ so that $H^m(k_v(X), \mathbb{Z}/2\mathbb{Z}) = 0$ for $m \geq d+1$. Hence $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$ over $k_v(X)$. Let $v$ be a real place. Over $k_v(X)$, $f$ is a sum of squares, hence a sum of at most $2^d$ squares (by Pfister's theorem 7.3) so that $\phi_{d+1} \otimes \langle 1, -f\rangle$ is hyperbolic over $k_v(X)$. Hence $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$.

Let $v$ be a non-dyadic $p$-adic place of $k$. Then $\phi_2$ is hyperbolic over $k_v$ so that $\phi_{d+1} \otimes \langle 1, -f\rangle = 0$ and $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$.

Let $v$ be a dyadic place of $k$. Over $k_v$, $\phi_3$ is hyperbolic so that $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$. Thus for all completions $v$ of $k$, $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle)$ is zero. The following conjecture of Kato implies Pfister's conjecture for $d \geq 2$.

**Conjecture** (Kato)**.** *Let $k$ be a number field, $X$ a geometrically integral variety over $k$ of dimension $d$. Then the map*

$$H^{d+2}(k(X), \mathbb{Z}/2\mathbb{Z}) \to \prod_{v \in \Omega_k} H^{d+2}(k_v(X), \mathbb{Z}/2\mathbb{Z})$$

*has trivial kernel.*

The above conjecture is the classical Hasse–Brauer–Noether theorem if the dimension of $X$ is zero, i.e., the injectivity of the Brauer group map:

$$\mathrm{Br}(k) \hookrightarrow \bigoplus_{v \in \Omega_k} \mathrm{Br}(k_v).$$

For $\dim X = 1$, the conjecture is a theorem of Kato ([K]). For $\dim X = 2$, Kato's conjecture was proved by Jannsen ([Ja]). Using Jannsens's theorem, Colliot-Thélène–Jannsen [CTJ] derived Pfister's conjecture: every sum of squares in $k(X)$, $X$ a surface over a number field, is a sum of at most 8 squares.

We explain how Kato's theorem was used by Colliot-Thélène to derive $p(k(X)) \leq 7$ for a curve $X$ over a number field.

Suppose $K = k(X)$ has no ordering. We claim that $s(K) \leq 4$. To show this it suffices to show that $\langle 1, 1\rangle^{\otimes 3}$ is zero over $k_v(X)$ for every place $v$ of $k$. At finite places $v$, $\langle 1, 1\rangle^{\otimes 3}$ is already zero in $k_v$. If $v$ is a real place of $k$, $k_v(X)$ is the function field of a real curve over the field of real numbers which has no orderings. By a theorem of Witt, $\mathrm{Br}(k_v(X)) = 0$ and every sum of squares is a sum of two squares in $k_v(X)$. Thus $-1$ is a sum of two squares in $k_v(X)$ and $\langle 1,1\rangle^{\otimes 3} = 0$ over $k_v(X)$. Since $H^3(k(X), \mathbb{Z}/2\mathbb{Z}) \to \prod_{v\in\Omega_k} H^3(k_v(X), \mathbb{Z}/2\mathbb{Z})$ is injective by Kato's theorem, $e_3(\langle 1,1\rangle^{\otimes 3}) = 0$ in $H^3(k(X), \mathbb{Z}/2\mathbb{Z})$. Since $e_3$ is injective on 3-fold Pfister forms, $\langle 1,1\rangle^{\otimes 3} = 0$ in $k(X)$. Thus $s(k(X)) \le 4$. In this case, $p(k(X)) \le 5$.

Suppose $K$ has an ordering. Let $f \in K^*$ be a sum of squares in $K$. Then $K(\sqrt{-f})$ has no orderings and hence $-1$ is a sum of 4 squares in $K(\sqrt{-f})$. Let $a_i, b_i \in K$ be such that

$$-1 = \sum_{1\le i\le 4} (a_i + b_i\sqrt{-f})^2, \quad a_i, b_i \in K.$$

Then

$$1 + \sum_{1\le i\le 4} a_i^2 = f\big(\sum_{1\le i\le 4} b_i^2\big), \quad \sum_{1\le i\le 4} a_i b_i = 0.$$

By the Key Lemma, 8.4, $(1 + \sum_{1\le i\le 4} a_i^2)\sum_{1\le i\le 4} b_i^2$ is a sum of at most 7 squares.

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