Transcendence in Positive Characteristic

Galois Group Examples and Applications

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Arizona Winter School 2008
March 18, 2008

Outline

Preliminaries

- Notation
- Transcendence degree theorem

Scalar quantities

Let p be a fixed prime; q a fixed power of p.

$$
\begin{array}{lcl}
A := \mathbb{F}_q[\theta] & \longleftrightarrow & \mathbb{Z} \\
k := \mathbb{F}_q(\theta) & \longleftrightarrow & \mathbb{Q} \\
\overline{k} & \longleftrightarrow & \overline{\mathbb{Q}} \\
k_\infty := \mathbb{F}_q((1/\theta)) & \longleftrightarrow & \mathbb{R} \\
\mathbb{C}_\infty := \widehat{\overline{k_\infty}} & \longleftrightarrow & \mathbb{C} \\
|f|_\infty = q^{\deg f} & \longleftrightarrow & |\cdot|
\end{array}
$$

Functions

- Rational functions:

$$\mathbb{F}_q(t), \quad \overline{k}(t), \quad \mathbb{C}_\infty(t).$$

- Analytic functions:

$$\mathbb{T} := \left\{ \sum_{i\geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; |a_i|_\infty \to 0 \right\}.$$

and

$$\mathbb{L} := \text{fraction field of } \mathbb{T}.$$

- Entire functions:

$$\mathbb{E} := \left\{ \sum_{i\geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; \begin{array}{c} \sqrt[i]{|a_i|_\infty} \to 0, \\ [k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty \end{array} \right\}.$$

Galois groups and transcendence degree

Theorem (P. 2008)

Let M be a t-motive, and let Γ_M be its associated group via Tannakian duality. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^\times$. Let Ψ be a rigid analytic trivialization of Φ in $\mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$. That is,

$$\Psi^{(-1)} = \Phi\Psi.$$

Finally let

$$L = \overline{k}(\Psi(\theta)) \subseteq \overline{k_\infty}.$$

Then

$$\mathrm{tr.\,deg}_{\overline{k}}\, L = \dim \Gamma_M \quad (= \dim \Gamma_\Psi).$$

Carlitz logarithms

- Difference equations for Carlitz logarithms
- Calculation of the Galois group
- Algebraic independence
- An explicit example: $\log_C(\zeta_\theta)$

Carlitz logarithms

- Recall the Carlitz exponential:

$$\exp_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})}.$$

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- Its formal inverse is the Carlitz logarithm,

$$\log_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^i})}.$$

- $\log_C(z)$ converges for $|z|_\infty < |\theta|^{q/(q-1)}$ and satisfies

$$\theta \log_C(z) = \log_C(\theta z) + \log_C(z^q).$$

The function $L_\alpha(t)$

- For $\alpha \in \overline{k}$, $|\alpha|_\infty < |\theta|^{q/(q-1)}$, we define

$$L_\alpha(t) = \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t-\theta^q)(t-\theta^{q^2})\cdots(t-\theta^{q^i})} \in \mathbb{T},$$

- Connection with Carlitz logarithms:

$$L_\alpha(\theta) = \log_C(\alpha).$$

- Functional equation:

$$L_\alpha^{(-1)} = \alpha^{(-1)} + \frac{L_\alpha}{t-\theta}.$$

Difference equations for Carlitz logarithms

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- If we set

$$\Phi = \begin{bmatrix} t-\theta & 0 & \cdots & 0 \\ \alpha_1^{(-1)}(t-\theta) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{(-1)}(t-\theta) & 0 & \cdots & 1 \end{bmatrix},$$

then Φ represents multiplication by σ on a t-motive M with

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then Φ represents multiplication by σ on a t-motive M with

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- We let

$$\Psi = \begin{bmatrix} \Omega & 0 & \cdots & 0 \\ \Omega L_{\alpha_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega L_{\alpha_r} & 0 & \cdots & 1 \end{bmatrix}.$$

Then

$$\Psi^{(-1)} = \Phi\Psi.$$

- Specialize Ψ at $t = \theta$ and find

$$\Psi(\theta) = \begin{bmatrix} -1/\pi_q & 0 & \cdots & 0 \\ -\log_C(\alpha_1)/\pi_q & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\log_C(\alpha_r)/\pi_q & 0 & \cdots & 1 \end{bmatrix}.$$

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- Thus we can determine

$$\mathrm{tr.\ deg}_{\overline{k}}\, \overline{k}(\pi_q, \log_C(\alpha_1), \ldots, \log_C(\alpha_r))$$

by calculating

$$\dim \Gamma_\Psi.$$

Calculating Γ_ψ

- Set $\Psi_1, \Psi_2 \in \mathrm{GL}_{r+1}(\mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L})$ so that
$$(\Psi_1)_{ij} = \Psi_{ij} \otimes 1, \quad (\Psi_2)_{ij} = 1 \otimes \Psi_{ij},$$
and set $\widetilde{\Psi} = \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_{r+1}(\mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L})$.

Calculating Γ_Ψ

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- Define an $\mathbb{F}_q(t)$-algebra map,
$$\mu = (X_{ij} \mapsto \widetilde{\Psi}_{ij}) : \mathbb{F}_q(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L},$$
which defines the $\mathbb{F}_q(t)$-subgroup scheme $\Gamma_\Psi \subseteq \mathrm{GL}_{r+1/\mathbb{F}_q(t)}$.

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which defines the $\mathbb{F}_q(t)$-subgroup scheme $\Gamma_\Psi \subseteq \mathrm{GL}_{r+1/\mathbb{F}_q(t)}$.
- In our case, this implies first that
$$\Gamma_\Psi \subseteq \left\{ \begin{bmatrix} * & 0 \\ * & \mathrm{id}_r \end{bmatrix} \right\} \subseteq \mathrm{GL}_{r+1/\mathbb{F}_q(t)}\,.$$
Thus we can consider the coordinate ring of Γ_Ψ to be a quotient of $\mathbb{F}_q(t)[X_0, \dots, X_r, 1/X_0]$.

The vector group V

- The homomorphism of $\mathbb{F}_q(t)$-group schemes

$$\begin{bmatrix} \alpha & 0 \\ \delta & \mathrm{id}_r \end{bmatrix} \mapsto \alpha : \Gamma_\Psi \xrightarrow{\mathrm{pr}} \mathbb{G}_m$$

coincides with the surjection,

$$\Gamma_\Psi \twoheadrightarrow \Gamma_C. \quad (\Gamma_C \cong \mathbb{G}_m).$$

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- Thus we have exact sequence of group schemes over $\mathbb{F}_q(t)$:

$$0 \to V \to \Gamma_\Psi \xrightarrow{\mathrm{pr}} \mathbb{G}_m \to 0,$$

and we can consider $V \subseteq (\mathbb{G}_a)^r$ over $\mathbb{F}_q(t)$.

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- For any $u = \left[\begin{smallmatrix} 1 & 0 \\ v & I_r \end{smallmatrix}\right] \in V(\overline{\mathbb{F}_q(t)})$, we find that

$$\gamma^{-1} u \gamma = \begin{bmatrix} 1 & 0 \\ \alpha v & I_r \end{bmatrix} \in V(\overline{\mathbb{F}_q(t)}).$$

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- Now V is smooth over $\mathbb{F}_q(t)$ because $\mathrm{pr} : \Gamma_\Psi \to \mathbb{G}_m$ is surjective on Lie algebras.
- It follows that defining equations for V are linear forms in $X_1, \ldots, X_r$ over $\mathbb{F}_q(t)$.

Definining equations for Γ_ψ

- Pick $b_0 \in \mathbb{F}_q(t)^\times \setminus \mathbb{F}_q^\times$.
- Lift (use Hilbert Thm. 90) to

$$\gamma = \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_r & 0 & \cdots & 1 \end{bmatrix} \in \Gamma_\psi(\mathbb{F}_q(t)).$$

- We can use γ to create defining equations for Γ_ψ using defining forms for V.

Theorem

- *Suppose* $F = c_1X_1 + \cdots + c_rX_r$, $c_1, \ldots, c_r \in \mathbb{F}_q(t)$, *is a defining linear form for* V. *Then*

$$G = (b_0 - 1)F - F(b_1, \ldots, b_r)(X_0 - 1)$$

is a defining polynomial for Γ_ψ.

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$$(b_0(\theta) - 1)\sum_{i=1}^{r} c_i(\theta)\log_C(\alpha_i) - \sum_{i=1}^{r} c_i(\theta)b_i(\theta)\pi_q = 0.$$

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- *Let* N *be the* k*-linear span of* $\pi_q, \log_C(\alpha_1), \ldots, \log_C(\alpha_r)$. *Then*

$$\dim \Gamma_\psi = \dim_k N.$$

Algebraic independence of Carlitz logarithms

- Starting with $\alpha_1, \ldots, \alpha_r \in \overline{k}$ (suitably small), we found $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ and $\Psi \in \mathrm{Mat}_r(\mathbb{E})$ so that

$$\Psi(\theta) = \begin{bmatrix} -1/\pi_q & 0 & \cdots & 0 \\ -\log_C(\alpha_1)/\pi_q & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\log_C(\alpha_r)/\pi_q & 0 & \cdots & 1 \end{bmatrix}.$$

- Since $\mathrm{tr.\,deg}_{\overline{k}}\, \overline{k}(\pi_q, \log_C(\alpha_1), \ldots \log_C(\alpha_r)) = \dim \Gamma_\Psi = \dim_k N$, we can prove the following the following theorem.

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Theorem (P. 2008)

Suppose $\log_C(\alpha_1), \ldots, \log_C(\alpha_r)$ *are linearly independent over* $k = \mathbb{F}_q(\theta)$*. Then they are algebraically independent over* $\overline{k}$

An Example

- Recall

$$\zeta_\theta = \sqrt[q-1]{-\theta}, \quad \exp_C\left(\pi_q/\theta\right) = \zeta_\theta, \quad \log_C(\zeta_\theta) = \frac{\pi_q}{\theta}.$$

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- We take

$$\Phi = \begin{bmatrix} t-\theta & 0 \\ \zeta_\theta^{1/q}(t-\theta) & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Omega & 0 \\ \Omega L_{\zeta_\theta} & 1 \end{bmatrix}.$$

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- We have a relation over $\overline{k}$ on the entries of

$$\Psi(\theta) = \begin{bmatrix} -1/\pi_q & 0 \\ -1/\theta & 1 \end{bmatrix},$$

namely

$$\theta X_{21} + 1 = 0.$$

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- We begin with matrices in $\mathrm{GL}_r(\mathbb{L} \otimes_{\overline{k}(t)} \mathbb{L})$:

$$\Psi_1 = \begin{bmatrix} \Omega \otimes 1 & 0 \\ \Omega L_{\zeta_\theta} \otimes 1 & 1 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 1 \otimes \Omega & 0 \\ 1 \otimes \Omega L_{\zeta_\theta} & 1 \end{bmatrix}.$$

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Then the defining equations over $\mathbb{F}_q(t)$ for Γ_Ψ will be precisely relations among the entries of

$$\Psi_1^{-1}\Psi_2 = \begin{bmatrix} \frac{1}{\Omega} \otimes \Omega & 0 \\ -L_{\zeta_\theta} \otimes \Omega + 1 \otimes \Omega L_{\zeta_\theta} & 1 \end{bmatrix}.$$

- So $\dim \Gamma_\Psi = 1$. (It's at least 1 since $\Gamma_\Psi \twoheadrightarrow \mathbb{G}_m$.)
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- Consider the identity of functions (check!),

$$\zeta_\theta(t-\theta)\Omega(t) - t\Omega(t)L_{\zeta_\theta}(t) - 1 = 0,$$

and substitute into the lower left entry of $\Psi_1^{-1}\Psi_2$.

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- Lower left entry of $\Psi_1^{-1}\Psi_2$ is

$$\begin{aligned} -L_{\zeta_\theta} \otimes \Omega + 1 \otimes \Omega L_{\zeta_\theta} &= -\left(\tfrac{1}{t}\zeta_\theta(t-\theta) - \tfrac{1}{t\Omega}\right) \otimes \Omega \\ &\quad + 1 \otimes \tfrac{1}{t}(\zeta_\theta(t-\theta)\Omega - 1) \\ &= -\tfrac{1}{t}\zeta_\theta(t-\theta) \otimes \Omega + \tfrac{1}{t\Omega} \otimes \Omega \\ &\quad + 1 \otimes \tfrac{1}{t}\zeta_\theta(t-\theta)\Omega - 1 \otimes \tfrac{1}{t} \\ &= \tfrac{1}{t}\left(\tfrac{1}{\Omega} \otimes \Omega\right) - \tfrac{1}{t}(1 \otimes 1). \end{aligned}$$

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- Therefore, Γ_Ψ is defined by

$$\Gamma_\Psi : tX_{12} - X_{11} + 1 = 0.$$

Carlitz zeta values

- Brief review of Carlitz zeta values
- Algebraic independence theorem of Chang-Yu
- Theorem of Chang-P.-Yu for varying q

Applications to Carlitz zeta values

$$\zeta_C(n) = \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ a \text{ monic}}} \frac{1}{a^n} \in k_\infty, \quad n = 1, 2, \ldots.$$

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- As you may recall from the 2nd lecture, using the theory of Anderson and Thakur, one can construct a system of difference equations $\Psi^{(-1)} = \Phi\Psi$ so that $\zeta_C(n)$ appears in $\Psi(\theta)$.

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- As you may recall from the 2nd lecture, using the theory of Anderson and Thakur, one can construct a system of difference equations $\Psi^{(-1)} = \Phi\Psi$ so that $\zeta_C(n)$ appears in $\Psi(\theta)$.
- Known algebraic relations over $\overline{k}$ among $\zeta_C(n)$:

$$(q-1) \mid n \;\Rightarrow\; \zeta_C(n) = r_n \pi_q^n, \quad r_n \in \mathbb{F}_q(\theta), \qquad \text{(Euler-Carlitz)}$$

$$\zeta_C(np) = \zeta_C(n)^p, \qquad \text{(Frobenius).}$$

The Chang-Yu Theorem

Algebraic independence of $\zeta_C(n)$

Theorem (Chang-Yu 2007)

For any positive integer n, the transcendence degree of the field

$$\overline{k}(\pi_q, \zeta_C(1), \ldots, \zeta_C(n))$$

over $\overline{k}$ *is*

$$n - \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{n}{q-1} \right\rfloor + \left\lfloor \frac{n}{p(q-1)} \right\rfloor + 1.$$

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Question: What can we say about Carlitz zeta values if we allow q to vary?

The Chang-Yu Theorem

Algebraic independence of $\zeta_C(n)$

Theorem (Chang-Yu 2007)

For any positive integer n, the transcendence degree of the field

$$\overline{k}(\pi_q, \zeta_C(1), \ldots, \zeta_C(n))$$

over $\overline{k}$ is

$$n - \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{n}{q-1} \right\rfloor + \left\lfloor \frac{n}{p(q-1)} \right\rfloor + 1.$$

Question: What can we say about Carlitz zeta values if we allow q to vary?

Answer: Even then, the Euler-Carlitz relations and the Frobenius p-th power relations tell the whole stoty....

Zeta values with varying constant fields

For $m \geq 1$, we set

$$\zeta_m(n) = \sum_{\substack{a \in \mathbb{F}_{p^m}[\theta] \\ a \text{ monic}}} \frac{1}{a^n}, \quad n = 1, 2, \ldots.$$

Theorem (Chang-P.-Yu)

For any positive integers s and d, the transcendence degree of the field

$$\overline{k}(\cup_{m=1}^{d}\{\pi_{p^m}, \zeta_m(1), \ldots, \zeta_m(s)\})$$

over $\overline{k}$ is

$$\sum_{m=1}^{d}\left(s - \left\lfloor \frac{s}{p} \right\rfloor - \left\lfloor \frac{s}{p^m - 1} \right\rfloor + \left\lfloor \frac{s}{p(p^m - 1)} \right\rfloor + 1\right).$$

Rank 2 Drinfeld modules

- Periods and quasi-periods
- A Galois group example
- Algebraic independence in the non-CM case

Periods and quasi-periods of rank 2 Drinfeld modules

- Recall that for a rank 2 Drinfeld module $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ with

$$\rho(t) = \theta + \kappa F + F^2,$$

we can take

$$\Phi = \begin{bmatrix} 0 & 1 \\ t - \theta & -\kappa^{1/q} \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 1 \\ 1 & -\kappa \end{bmatrix} \begin{bmatrix} s_1^{(1)} & s_1^{(2)} \\ s_2^{(1)} & s_2^{(2)} \end{bmatrix}^{-1}.$$

- Furthermore,

$$\Psi(\theta)^{-1} = \begin{bmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{bmatrix},$$

where $\omega_1, \omega_2, \eta_1, \eta_2$ are the periods and quasi-periods for ρ.

An example ($\kappa = \sqrt{\theta} + \sqrt{\theta^q}$)

- Assume $p \neq 2$. Consider the Drinfeld module ρ with

$$\rho(t) = \theta + \left(\sqrt{\theta} + \sqrt{\theta^q}\right) F + F^2.$$

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$$\Gamma_\Psi = \left\{ \begin{bmatrix} \alpha & \beta t \\ \beta & \alpha \end{bmatrix} \right\}.$$

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Theorem (Thiery 1992)

The period matrix of a Drinfeld module of rank 2 over $\overline{k}$ with CM has transcendence degree 2 over $\overline{k}$.

Rank 2 Drinfeld modules without CM

In general, we say that a Drinfeld module ρ *does not have complex multiplication* if

$$\mathrm{End}(\rho) = \mathbb{F}_q[t].$$

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Theorem (Chang-P.)

Suppose that $p \neq 2$. Let ρ be a Drinfeld module of rank 2 over $\overline{k}$ without CM. Then

$$\Gamma_\rho \cong \mathrm{GL}_2\,.$$

In particular, the periods and quasi-periods of ρ,

$$\omega_1, \omega_2, \eta_1, \eta_2,$$

are algebraically independent over $\overline{k}$.