Transcendence in Positive Characteristic

Introduction to Function Field Transcendence

W. Dale Brownawell
Matthew Papanikolas

Penn State University
Texas A&M University

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Outline

1. Things Familiar
2. Things Less Familiar
3. Things Less Less Familiar
Things Familiar
Arithmetic objects from characteristic 0

- The multiplicative group and exp(z)
- Elliptic curves and elliptic functions
- Abelian varieties
The multiplicative group

We have the usual exact sequence of abelian groups

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \rightarrow 0,$$

where

$$\exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} \in \mathbb{Q}[[z]].$$
The multiplicative group

We have the usual exact sequence of abelian groups

\[ 0 \to 2\pi i \mathbb{Z} \to \mathbb{C}^{\exp} \to \mathbb{C}^\times \to 0, \]

where

\[ \exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} \in \mathbb{Q}[[z]]. \]

For any \( n \in \mathbb{Z}, \)

\[ \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \xrightarrow{\cdot n} \mathbb{C}^\times \xrightarrow{\exp} \mathbb{C}^\times \]

which is simply a restatement of the functional equation

\[ \exp(nz) = \exp(z)^n. \]
The $n$-th roots of unity are defined by
\[
\mu_n := \{ \zeta \in \mathbb{C}^\times \mid \zeta^n = 1 \} = \{ \exp(2\pi i a/n) \mid a \in \mathbb{Z} \}
\]

- $\text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.  
- **Kronecker-Weber Theorem:** The cyclotomic fields $\mathbb{Q}(\mu_n)$ provide explicit class field theory for $\mathbb{Q}$.  
- For $\zeta \in \mu_n$,  
  \[
  \log(\zeta) = \frac{2\pi i a}{n}, \quad 0 \leq a < n.
  \]
Elliptic curves over $\mathbb{C}$

Smooth projective algebraic curve of genus 1.

$$E : y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C}$$

$E(\mathbb{C})$ has the structure of an abelian group through the usual chord-tangent construction.
Weierstrass uniformization

There exist \( \omega_1, \omega_2 \in \mathbb{C} \), linearly independent over \( \mathbb{R} \), so that if we consider the lattice
\[
\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,
\]
then the Weierstrass \( \wp \)-function is defined by
\[
\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda, \omega \neq 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).
\]
The function \( \wp(z) \) has double poles at each point in \( \Lambda \) and no other poles.
We obtain an exact sequence of abelian groups,

\[ 0 \rightarrow \Lambda \rightarrow \mathbb{C} \xrightarrow{\exp_E} E(\mathbb{C}) \rightarrow 0, \]

where

\[ \exp_E(z) = (\varphi(z), \varphi'(z)). \]
We obtain an exact sequence of abelian groups,

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where

$$\exp_E(z) = (\wp(z), \wp'(z)).$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\exp_E} & E(\mathbb{C}) \\
\downarrow{z\mapsto nz} & & \downarrow{P\mapsto [n]P} \\
\mathbb{C} & \xrightarrow{\exp_E} & E(\mathbb{C})
\end{array}$$

where $[n]P$ is the $n$-th multiple of a point $P$ on the elliptic curve $E$. 
Periods of $E$
How do we find $\omega_1$ and $\omega_2$?

An elliptic curve $E$,

$$E : y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C},$$

has the geometric structure of a torus in $\mathbb{P}^2(\mathbb{C})$. Let

$$\gamma_1, \gamma_2 \in H_1(E, \mathbb{Z})$$

be generators of the homology of $E$. 
Periods of $E$

How do we find $\omega_1$ and $\omega_2$?

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be generators of the homology of $E$.

Then we can choose

$$\omega_1 = \int_{\gamma_1} \frac{dx}{\sqrt{4x^3 + ax + b}}, \quad \omega_2 = \int_{\gamma_2} \frac{dx}{\sqrt{4x^3 + ax + b}}.$$
Quasi-periods of \( E \)

- The differential \( dx/y \) on \( E \) generates the space of holomorphic 1-forms on \( E \) (differentials of the first kind).
- The differential \( x \, dx/y \) generates the space of differentials of the second kind (differentials with poles but residues of 0).
Quasi-periods of $E$

- The differential $dx/y$ on $E$ generates the space of holomorphic 1-forms on $E$ (differentials of the first kind).
- The differential $x \, dx/y$ generates the space of differentials of the second kind (differentials with poles but residues of 0).
- We set
  \[ \eta_1 = \int_{\gamma_1} \frac{x \, dx}{\sqrt{4x^3 + ax + b}}, \quad \eta_2 = \int_{\gamma_2} \frac{x \, dx}{\sqrt{4x^3 + ax + b}}, \]

  and $\eta_1, \eta_2$ are called the quasi-periods of $E$.
- $\eta_1, \eta_2$ arise simultaneously as special values of the Weierstrass $\zeta$-function and as periods of extensions of $E$ by $\mathbb{G}_a$. 
Period matrix of $E$

- The period matrix of $E$ is the matrix

$$P = \begin{bmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{bmatrix}.$$ 

It provides a natural isomorphism

$$H^1_{\text{sing}}(E, \mathbb{C}) \cong H^1_{\text{DR}}(E, \mathbb{C}).$$
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**Legendre Relation:** From properties of elliptic functions, the determinant of $P$ is

$$\omega_1 \eta_2 - \omega_2 \eta_1 = \pm 2\pi i.$$
An abelian variety $A$ over $\mathbb{C}$ is a smooth projective variety that is also a group variety.

Elliptic curves are abelian varieties of dimension 1.
Abelian varieties
Higher dimensional analogues of elliptic curves

- An abelian variety $A$ over $\mathbb{C}$ is a smooth projective variety that is also a group variety.
- Elliptic curves are abelian varieties of dimension 1.
- Much like for $\mathbb{G}_m$ and elliptic curves, an abelian variety of dimension $d$ has a uniformization,

$$\mathbb{C}^d / \Lambda \cong A(\mathbb{C}),$$

where $\Lambda$ is a discrete lattice of rank $2d$. 
The period matrix of an abelian variety

Let $A$ be an abelian variety over $\mathbb{C}$ of dimension $d$.

- As in the case of elliptic curves, there is a natural isomorphism,

$$H^1_{\text{sing}}(A, \mathbb{C}) \cong H^1_{\text{DR}}(A, \mathbb{C}),$$

defined by period integrals, whose defining matrix $P$ is called the *period matrix of $A$.*
The period matrix of an abelian variety

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- As in the case of elliptic curves, there is a natural isomorphism,
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*period matrix of $A$.*

- We have
  $$P = \begin{bmatrix} \omega_{ij} & \eta_{ij} \end{bmatrix} \in \text{Mat}_{2d}(\mathbb{C}),$$
  where $1 \leq i \leq 2d$, $1 \leq j \leq d$.

- The $\omega_{ij}$’s provide coordinates for the period lattice $\Lambda$.
- The $\eta_{ij}$’s provide periods of extensions of $A$ by $\mathbb{G}_a$. 
Things Less Familiar
Transcendence in characteristic 0

- Theorems of Hermite-Lindemann and Gelfond-Schneider
- Schneider’s theorems on elliptic functions
- Linear independence results
- Grothendieck’s conjecture
Transcendence from $\mathbb{G}_m$

**Theorem (Hermite-Lindemann 1870’s, 1880’s)**

Let $\alpha \in \overline{\mathbb{Q}}$, $\alpha \neq 0$. Then $\exp(\alpha)$ is transcendental over $\mathbb{Q}$.

**Examples**

Each of the following is transcendental:

- $e$ \hspace{1cm} ($\alpha = 1$)
- $\pi$ \hspace{1cm} ($\alpha = 2\pi i$)
- $\log 2$ \hspace{1cm} ($\alpha = \log 2$)
Hilbert’s Seventh Problem

Theorem (Gelfond-Schneider 1930’s)

Let $\alpha, \beta \in \overline{\mathbb{Q}}$, with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$. Then $\alpha^\beta$ is transcendental.

Examples

Each of the following is transcendental:

- $2\sqrt{2}$ ($\alpha = 2, \beta = \sqrt{2}$)
- $e^\pi$ ($e^\pi = (-1)^{-i}$)
- $\log 2 \over \log 3$ ($3^{\log 2 \over \log 3} = 2$)
Periods and quasi-periods of elliptic curves

Theorem (Schneider 1930’s)

Let $E$ be an elliptic curve defined over $\overline{\mathbb{Q}}$,

$$E : y^2 = x^3 + ax + b, \quad a, b \in \overline{\mathbb{Q}}.$$ 

- The periods and quasi-periods of $E$,

$$\omega_1, \omega_2, \eta_1, \eta_2$$

are transcendental.
- Let $\tau = \omega_1/\omega_2$. Then either $\mathbb{Q}(\tau)/\mathbb{Q}$ is an imaginary quadratic extension (CM) or a purely transcendental extension (non-CM).
Linear independence
Linear forms in logarithms

Theorem (Baker 1960’s)

Let $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$. If $\log(\alpha_1), \ldots, \log(\alpha_m)$ are linearly independent over $\mathbb{Q}$, then

$$1, \log(\alpha_1), \ldots, \log(\alpha_m)$$

are linearly independent over $\overline{\mathbb{Q}}$.

- Extension of the Gelfond-Schneider theorem ($m = 2$).
- Work of Bertrand, Masser, Waldschmidt, Wüstholz (1970’s, 1980’s) extended this result to elliptic and abelian integrals.
Linear independence
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- Extension of the Gelfond-Schneider theorem \((m = 2)\).
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Conjecture (Gelfond/Folklore)

Let \( \alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}} \). If \( \log(\alpha_1), \ldots, \log(\alpha_m) \) are linearly independent over \( \mathbb{Q} \), then they are algebraically independent over \( \overline{\mathbb{Q}} \).
Conjecture (Grothendieck)

Suppose $A$ is an abelian variety of dimension $d$ defined over $\overline{\mathbb{Q}}$. Then

\[
\text{tr. deg}(\overline{\mathbb{Q}}(P)/\overline{\mathbb{Q}}) = \dim \text{MT}(A),
\]

where $\text{MT}(A) \subseteq \text{GL}_{2d}/\mathbb{Q}$ is the Mumford-Tate group of $A$. 

Let $A$ be an elliptic curve. One can show

\[
\dim \text{MT}(A) = \begin{cases} \ 4 & \text{if } \text{End}(A) = \mathbb{Z}, \\ \ 2 & \text{if } \text{End}(A) \neq \mathbb{Z}. \end{cases}
\]

(G. Chudnovsky, 1970's) If $\text{End}(A) \neq \mathbb{Z}$, then Grothendieck's conjecture is true.
Grothendieck’s conjecture

Conjecture (Grothendieck)

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Things Less Familiar

- Function fields
- Drinfeld modules
  - The Carlitz module
  - Drinfeld modules of rank 2
- $t$-modules (higher dimensional Drinfeld modules)
- Transcendence results
Let $p$ be a fixed prime; $q$ a fixed power of $p$.

\[ A := \mathbb{F}_q[\theta] \quad \longleftrightarrow \quad \mathbb{Z} \]
\[ k := \mathbb{F}_q(\theta) \quad \longleftrightarrow \quad \mathbb{Q} \]
\[ \overline{k} \quad \longleftrightarrow \quad \overline{\mathbb{Q}} \]
\[ k_{\infty} := \mathbb{F}_q((1/\theta)) \quad \longleftrightarrow \quad \mathbb{R} \]
\[ \mathbb{C}_{\infty} := \overline{k_{\infty}} \quad \longleftrightarrow \quad \mathbb{C} \]
\[ |f|_{\infty} = q^{\deg f} \quad \longleftrightarrow \quad |\cdot| \]
Twisted polynomials

- Let $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the $q$-th power Frobenius map: $F(x) = x^q$.
- For a subfield $F_q \subseteq K \subseteq \mathbb{C}_\infty$, the ring of twisted polynomials over $K$ is

  $$K[F] = \text{polynomials in } F \text{ with coefficients in } K,$$

subject to the conditions

  $$Fc = c^q F, \quad \forall \ c \in K.$$
Twisted polynomials

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- For a subfield $\mathbb{F}_q \subseteq K \subseteq \mathbb{C}_\infty$, the ring of *twisted polynomials* over $K$ is
  \[ K[F] = \text{polynomials in } F \text{ with coefficients in } K, \]
  subject to the conditions
  \[ Fc = c^q F, \quad \forall \ c \in K. \]
- In this way,
  \[ K[F] \cong \{ \mathbb{F}_q\text{-linear endomorphisms of } K^+ \}. \]

For $x \in K$ and $\phi = a_0 + a_1 F + \cdots a_r F^r \in K[F]$, we write
\[ \phi(x) := a_0 x + a_1 x^q + \cdots + a_r x^{q^r}. \]
Drinfeld modules
Function field analogues of $\mathbb{G}_m$ and elliptic curves
Let $\mathbb{F}_q[t]$ be a polynomial ring in $t$ over $\mathbb{F}_q$.

**Definition**

A *Drinfeld module* over $\mathbb{F}_q$ is an $\mathbb{F}_q$-algebra homomorphism,

$$\rho : \mathbb{F}_q[t] \to \mathbb{C}_\infty[F],$$

such that

$$\rho(t) = \theta + a_1 F + \cdots + a_r F^r.$$
Drinfeld modules

Function field analogues of $\mathbb{G}_m$ and elliptic curves

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**Definition**

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such that

$$\rho(t) = \theta + a_1F + \cdots a_rF^r.$$

- $\rho$ makes $\mathbb{C}_\infty$ into a $\mathbb{F}_q[t]$-module in the following way:

  $$f \ast x := \rho(f)(x), \quad \forall f \in \mathbb{F}_q[t], x \in \mathbb{C}_\infty.$$

- If $a_1, \ldots, a_r \in K \subseteq \mathbb{C}_\infty$, we say $\rho$ is defined over $K$.
- $r$ is called the rank of $\rho$. 
The Carlitz module

The analogue of $\mathbb{G}_m$

We define a Drinfeld module $C : F_q[t] \rightarrow C_\infty[F]$ by

$$C(t) := \theta + F.$$ 

Thus, for any $x \in C_\infty$,

$$C(t)(x) = \theta x + x^q.$$
Carlitz exponential

We set

\[
\exp_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta q^i - \theta)(\theta q^i - \theta q) \cdots (\theta q^i - \theta q^{i-1})}.
\]

- \(\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty\) is entire, surjective, and \(\mathbb{F}_q\)-linear.

- Functional equation:

\[
\exp_C(\theta z) = \theta \exp_C(z) + \exp_C(z)^q,
\]
\[
\exp_C(f(\theta)z) = C(f)(\exp_C(z)), \quad \forall f(t) \in \mathbb{F}_q[t].
\]
Carlitz uniformization and the Carlitz period

We have a commutative diagram of $\mathbb{F}_q[t]$-modules,

\[
\begin{array}{ccc}
\mathbb{C}_\infty & \xrightarrow{\exp_C} & \mathbb{C}_\infty \\
\downarrow & & \downarrow \\
\mathbb{C}_\infty & \xrightarrow{\exp_C} & \mathbb{C}_\infty
\end{array}
\]

$z \mapsto \theta z$ 

$\ker(\exp_C(z)) = \mathbb{F}_q[\theta]$,

$\pi_q = \theta^{q-1} - \theta^{-\infty} \prod_{i=1}^{\infty} (1 - \theta^{-q^i})^{-1}$. 

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\mathbb{C}_\infty & \xrightarrow{\exp_C} & \mathbb{C}_\infty
\end{array}
\]

$z \mapsto \theta z \quad x \mapsto \theta x + x^q$

The kernel of $\exp_C(z)$ is

\[
\ker(\exp_C(z)) = \mathbb{F}_q[\theta] \pi_q,
\]

where

\[
\pi_q = \theta^{q-1} \sqrt{-\theta} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}.
\]
Wade’s result

Thus we have an exact sequence of $F_q[t]$-modules,

$$0 \rightarrow F_q[\theta] \pi_q \rightarrow C_\infty \xrightarrow{\exp_c} C_\infty \rightarrow 0.$$ 

Theorem (Wade 1941)

The Carlitz period $\pi_q$ is transcendental over $\overline{k}$. 
Drinfeld modules of rank 2

- Suppose \( \rho : \mathbb{F}_q[t] \to \overline{k}[F] \) is a rank 2 Drinfeld module defined over \( \overline{k} \) by
  \[
  \rho(t) = \theta + \kappa F + \lambda F^2.
  \]
- Then there is an unique, entire, \( \mathbb{F}_q \)-linear function
  \[
  \exp_\rho : \mathbb{C}_\infty \to \mathbb{C}_\infty,
  \]
so that
  \[
  \exp_\rho(f(\theta)z) = \rho(f)(\exp_\rho(z)), \quad \forall f \in \mathbb{F}_q[t].
  \]
Periods of Drinfeld modules of rank 2

- Furthermore, there are $\omega_1, \omega_2 \in \mathbb{C}_\infty$ so that

$$\ker(\exp_\rho(z)) = \mathbb{F}_q[\theta]\omega_1 + \mathbb{F}_q[\theta]\omega_2 =: \Lambda,$$

where $\Lambda$ is a discrete $\mathbb{F}_q[\theta]$-submodule of $\mathbb{C}$ of rank 2.
Furthermore, there are \( \omega_1, \omega_2 \in C_\infty \) so that

\[
\ker(\exp_\rho(z)) = \mathbb{F}_q[\theta]\omega_1 + \mathbb{F}_q[\theta]\omega_2 =: \Lambda,
\]

where \( \Lambda \) is a discrete \( \mathbb{F}_q[\theta] \)-submodule of \( C \) of rank 2.

**Chicken vs. Egg:**

\[
\exp_\rho(z) = z \prod_{0 \neq \omega \in \Lambda} \left(1 - \frac{z}{\omega}\right).
\]

Again we have a uniformizing exact sequence of \( \mathbb{F}_q[t] \)-modules

\[
0 \rightarrow \Lambda \rightarrow C_\infty \xrightarrow{\exp_\rho} C_\infty \rightarrow 0.
\]
Transcendence results for Drinfeld modules of rank 2

**Quasi-periods:** It is possible to define quasi-periods $\eta_1, \eta_2 \in \mathbb{C}_\infty$ for $\rho$ with the following properties (see notes):

- $\eta_1, \eta_2$ arise as periods of extensions of $\rho$ by $\mathbb{G}_a$.
- Legendre relation: $\omega_1 \eta_2 - \omega_2 \eta_1 = \zeta \pi_q$ for some $\zeta \in \mathbb{F}_q^\times$. 

Theorem (Yu 1980’s) For a Drinfeld module $\rho$ of rank 2 defined over $k$, the four quantities $\omega_1, \omega_2, \eta_1, \eta_2$ are transcendental over $k$. 

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Transcendence results for Drinfeld modules of rank 2

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- $\eta_1, \eta_2$ arise as periods of extensions of $\rho$ by $\mathbb{G}_a$.
- Legendre relation: $\omega_1 \eta_2 - \omega_2 \eta_1 = \zeta \pi_q$ for some $\zeta \in \mathbb{F}_q^\times$.

**Theorem (Yu 1980’s)**

*For a Drinfeld module $\rho$ of rank 2 defined over $\overline{k}$, the four quantities $\omega_1, \omega_2, \eta_1, \eta_2$ are transcendental over $\overline{k}$.*
A *-module* $A$ of dimension $d$ is an $\mathbb{F}_q$-linear homomorphism,

$$A : \mathbb{F}_q[t] \to \text{End}_{\mathbb{F}_q}(\mathbb{C}^d_\infty) \cong \text{Mat}_d(\mathbb{C}_\infty[F]),$$

such that

$$A(t) = \theta \text{Id} + N + a_0 F + \cdots + a_r F^r,$$

where $N \in \text{Mat}_d(\mathbb{C}_\infty)$ is nilpotent.

Thus $\mathbb{C}^d_\infty$ is given the structure of an $\mathbb{F}_q[t]$-module via

$$f \ast x := A(f)(x), \quad \forall f \in \mathbb{F}_q[t], \; x \in \mathbb{C}^d_\infty.$$
Exponential functions of $t$-modules

- There is a unique entire $\exp_A : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$ so that
  \[
  \exp_A((\theta\text{Id} + N)z) = A(t)(\exp_A(z)).
  \]

- If $\exp_A$ is surjective, we have an exact sequence
  \[
  0 \to \Lambda \to \mathbb{C}_\infty^d \xrightarrow{\exp_A} \mathbb{C}_\infty^d \to 0,
  \]
  where $\Lambda$ is a discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{C}_\infty^d$.

- $\Lambda$ is called the period lattice of $A$.

- Quasi-periods can also be defined (see notes).
Yu’s Theorem of the Sub-\(t\)-module
Analogue of Wüstholz’s Subgroup Theorem

**Theorem (Yu 1997)**

Let \(A\) be a \(t\)-module of dimension \(d\) defined over \(\overline{k}\). Suppose \(u \in \mathbb{C}_\infty^d\) satisfies \(\exp_A(u) \in \overline{k}^d\). Then the smallest vector space \(H \subseteq \mathbb{C}_\infty^d\) defined over \(k\) which is invariant under \(\theta \text{Id} + N\) and which contains \(u\) has the property that

\[
\exp_A(H) \subseteq A(\mathbb{C}_\infty),
\]

is a sub-\(t\)-module of \(A\).
Yu’s Theorem of the Sub-$t$-module

Analogue of Wüstholz’s Subgroup Theorem

Theorem (Yu 1997)

Let $A$ be a $t$-module of dimension $d$ defined over $\overline{k}$. Suppose $u \in \mathbb{C}^d_\infty$ satisfies $\exp_A(u) \in \overline{k}^d$. Then the smallest vector space $H \subseteq \mathbb{C}^d_\infty$ defined over $k$ which is invariant under $\theta \text{Id} + N$ and which contains $u$ has the property that

$$\exp_A(H) \subseteq A(\mathbb{C}_\infty),$$

is a sub-$t$-module of $A$.

Theorem (Yu 1997 (Linear independence of Carlitz logarithms))

Suppose $\alpha_1, \ldots, \alpha_m \in \overline{k}$. If $\log_C(\alpha_1), \ldots, \log_C(\alpha_m) \in \mathbb{C}_\infty$ are linearly independent over $k = \mathbb{F}_q(\theta)$, then

$$1, \log_C(\alpha_1), \ldots, \log_C(\alpha_m)$$

are linearly independent over $\overline{k}$.