Lecture 4

Frits Beukers

Arithmetic of values of E- and G-function

Two theorems

Theorem, G.Chudnovsky 1984

The minimal differential equation of a G-function is Fuchsian.

Theorem, Y.André 2000

Let $f(z)$ be an E-function. Then $f(z)$ satisfies a differential equation of the form

$$z^m y^{(m)} + \sum_{k=0}^{m-1} q_k(z) y^{(k)} = 0$$

where $q_k(z) \in \overline{\mathbb{Q}}[z]$ for all k.

Connecting E-functions and G-functions

Proposition

Let $a_0, a_1, a_2, \ldots \in \overline{\mathbb{Q}}$. Then the following are equivalent

1. $f(z) = \sum_{k \geq 0} a_k \frac{z^k}{k!}$ is an E-function.
2. $g(z) = \sum_{k \geq 0} a_k z^k$ is a G-function.

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The function g is a formal Laplace transform of f, more precisely

$$\int_0^\infty e^{-xz} f(z) dz = \frac{1}{x} g(\frac{1}{x}).$$

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Recall

$$\int_0^\infty e^{-xz} \frac{z^k}{k!} dz = \frac{1}{x^{k+1}}.$$

Connection between DE's

For any non-negative integers k, m and repeated partial integration we can derive the equality

$$\int_0^\infty e^{-xz}\left(\frac{d}{dz}\right)^k z^m f(z)dz = x^k\left(-\frac{d}{dx}\right)^m \frac{1}{x}g(\frac{1}{x}).$$

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Assume that $g(x)$ is a G-function. Then $g(x)$ satisfies a linear differential equation and so does $\frac{1}{x}g(\frac{1}{x})$. Assume

$$\sum_{m=0}^{M} G_m(x) \left(-\frac{d}{dx}\right)^m \frac{1}{x} g(\frac{1}{x}) = 0.$$

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Then, by our Laplace transform property,

$$0 = \int_0^\infty e^{-xz} \sum_{m=0}^{M} G_m \left(\frac{d^k}{dz^k}\right) z^m f(z) dz.$$

Proof of André's Theorem

Hence

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- But there is more, the equation for g is Fuchsian. So $\deg_x(G_i(x)) < \deg_x(G_m(x))$ for all $i < m$.
- The order of the equation for f becomes $\deg G_m$. The coefficient of this highest derivative is z^m.
- This proves André's theorem stating that an E-function satisfies an equation with only singularity at $z = 0$.

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$$z(z^2 + 11z - 1)y'' + (3z^2 + 22z - 1)y' + (z + \lambda)y = 0$$

and let $y_\lambda(z) = \sum_{k \geq 0} u_k z^k$ be a solution.

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$$(k+1)^2u_{k+1}=(11k(k+1)+\lambda)u_k+k^2u_{k-1}\quad k\geq 1.$$

Taking $u_0=1$ we get $u_1=\lambda$.

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Question

For which $\lambda\in\overline{\mathbb{Q}}$ is $\sum_{k\geq 0}u_k z^k$ a G-function?

Some values of λ

Consider $u_0 = 1,\ u_1 = \lambda$ and

$$(k+1)^2 u_{k+1} = (11k(k+1) + \lambda)u_k + k^2 u_{k-1} \quad k \geq 1$$

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Take $\lambda = 0$, then we get the sequence

$$1, 0, \frac{1}{4}, \frac{11}{6}, \frac{977}{64}, \frac{162613}{1200},$$

$$\frac{14432069}{11520}, \frac{5603179109}{470400}, \frac{2983229567887}{25804800}, \ldots$$

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Take $\lambda = 3$,then we get

$$1, 3, 19, 147, 1251, 11253, 104959, 1004307,$$

$$9793891, 96918753, 970336269, \ldots$$

A conjecture

Conjecture

Let $u(z) = \sum_{k\geq 0} u_k z^k$ and $u_0 = 1,\ u_1 = \lambda \in \overline{\mathbb{Q}}$ and

$$(k+1)^2 u_{k+1} = (11k(k+1) + \lambda)u_k + k^2 u_{k-1} \quad k \geq 1.$$

Then $u(z)$ is a G-function if and only if $\lambda = 3$.

Proposition, FB 1999

The following are equivalent

- $\lambda \in \mathbb{Q}$ and u_k 3-adically integral for all k.
- $\lambda = 3$.

Proof of the Proposition

Let $\lambda \in \mathbb{Q}_3^{\mathrm{unram}}$, the maximal unramified extension of $\mathbb{Q}_3$. Consider the corresponding recurrence and generating function $u_\lambda(z)$.

Theorem FB, 1998

Let $f_1 = 1 + z^2,\ f_2 = 1 + (1+i)z - z^2,\ f_3 = 1 + (1-i)z - z^2$. Suppose $u_\lambda \in \mathbb{Z}_3^{\mathrm{unram}}[[z]]$. Then there exists an infinite sequence $i_1, i_2, \ldots$ with $i_j \in \{1, 2, 3\}$ such that

$$u_\lambda(z) \equiv f_{i_1}(z) f_{i_2}(z)^3 f_{i_3}(z)^{3^2} \cdots \pmod 3.$$

Moreover this gives a 1-1 correspondence.

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Moreover this gives a 1-1 correspondence.

If we want $u_\lambda \in \mathbb{Z}[[z]]$ then we must have $i_j = 1$ for all j. Hence, by the 1-1 correspondence, there is precisely one solution with $u_\lambda \in \mathbb{Z}_3[[z]]$. Since we know that $u_3 \in \mathbb{Z}[[z]]$ this must be our solution.

Bombieri-Dwork conjecture

- Consider a family of algebraic varieties parametrised by z and consider a relative differential r-form Ω_z. We assume everything defined over $\overline{\mathbb{Q}}$. Take a continuous family of suitable cycles γ_z and consider the integral

$$w(z) = \int_{\gamma_z} \Omega_z.$$

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- It is known that $w(z)$ satifies a linear differential equation, the so-called Picard-Fuchs equation of the family.
- By a theorem of N.Katz $w(z)$ a Piard-Fuchs equation has G-function solutions.
- We say that a differential equation 'comes from geometry' if it a (factor of a) Picard-Fuchs equation.

The main conjecture

Conjecture, Bombieri-Dwork

The minimal differential equation of a G-function comes from geometry.

An experiment by Zagier

- Consider the recurrence

$$(k+1)^2 u_{k+1} = (An(n+1)+\lambda)u_k + Bk^2 u_{k-1}$$

for many choices of $A, B, \lambda, u_0, u_1 \in \mathbb{Z}$.
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Verify integrality of u_k for $k \leq 20$, say.

- There are essentially 7 different finds with $B(B^2-4A) \neq 0$.

A	B	λ
0	16	0
7	8	2
9	-27	3
10	-9	3
11	1	3
12	-32	4
17	-72	6

Beauville's list

Theorem, A.Beauville, 1982

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All of the above differential differential equations occur as Picard-Fuchs equation for one of the families in Beauville's list.

Explicit equations and solutions

$(z^3 - z)y'' + (3z^2 - 1)y' + zy = 0$

Solution:

$$b(z)^{1/4} {}_2F_1\left(\begin{matrix} 1/12 \; 5/12 \\ 1 \end{matrix} \,\middle|\, \frac{27z^4(1-z^2)^2}{4b(z)^3}\right)$$

where $b(z) = 1 - z^2 + z^4/16$.

$z(z-1)(8z+1)y'' + (24z^2 - 14z - 1)y' + (8z - 2)y = 0$

Solution:

$$b(z)^{1/4} {}_2F_1\left(\begin{matrix} 1/12 \; 5/12 \\ 1 \end{matrix} \,\middle|\, \frac{1728z^6(z-1)^2(1+8z)}{b(z)^3}\right)$$

where $b(z) = 1 + 8z - 16z^3 + 16z^4$.

Explicit equations and solutions, continued

$z(z^2 + 11z - 1)y'' + (3z^2 + 22z - 1)y' + (z + 3)y = 0$

Solution:

$$b(z)^{1/4} {}_2F_1\left(\begin{matrix} 1/12 \ 5/12 \\ 1 \end{matrix} \middle| \frac{1728z^5(1 - 11z - z^2)}{b(z)^3}\right)$$

where $b(z) = 1 - 12z + 14z^2 + 12z^3 + z^4$.

$z(3z^2 - 3z + 1)y'' + (3z - 1)^2 y' + (3z - 1)y = 0$

Solution:

$$b(z)^{1/4} {}_2F_1\left(\begin{matrix} 1/12 \ 5/12 \\ 1 \end{matrix} \middle| \frac{-64z^3(1 - 3z + 3z^2)^3}{b(z)^3}\right)$$

where $b(z) = (1 - z)(1 - 3z + 3z^2 - 9z^3)$

Projective equivalence and pull-back

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- We say that L_2 is a rational pullback of L_1 if there exists a rational function $S(z)$ such that the solutions of $L_2 y = 0$ are given by $y(S(z))$ where y runs through the solutions of $L_1 y = 0$.

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- We say that L_2 is a rational pullback of L_1 if there exists a rational function $S(z)$ such that the solutions of $L_2 y = 0$ are given by $y(S(z))$ where y runs through the solutions of $L_1 y = 0$.
- We say that L_2 is weak (rational) pullback of L_1 if it is projectively equivalent to a rationalpull back of L_1.

A conjecture by Dwork

Example, let $u_k = \sum_{r=0}^{k} \binom{k}{r}^2 \binom{k+r}{r}$ be the sequence of Apéry numbers and $u(z) = \sum_{k \geq 0} u_k z^k$. Then from

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is a weak pullback of the differential equation for ${}_2F_1(1/12, 5/12, 1|z)$.

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Conjecture, Dwork

An irreducible second order linear differential equation which has a G-function solution is a weak pullback of Gaussian hypergeometric function.

A counterexample

Theorem, D.Krammer 1988

The differential equation

$$P(x)f'' + \frac{1}{2}P'(x)f' + \frac{x-9}{18}f = 0,$$

where $P(x) = x(x-1)(x-81)$ has a G-function solution but is *not* a weak pullback of a Gaussian hypergeometric function.

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This equation, and several similar ones, occur in a paper by G.Chudnovsky.

Monodromy group

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- Thus we get a matrix $M_\gamma \in GL(2, \mathbb{C})$ such that

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- M_γ depends only on the class $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, z_0)$ and the map $\gamma \mapsto M_\gamma$ gives a representation $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, z_0) \to GL(2, \mathbb{C})$, the *monodromy representation*. The image is called *monodromy group*.

Monodromy, continued

- Suppose $S = \{s_1, \ldots, s_r, \infty\}$. Then the monodromy group is generated by the simple loops γ_i around the s_i. Moreover, $\gamma_1 \circ \cdots \circ \gamma_r \circ \gamma_\infty = 1$.

Triangle groups

The differential equation for the hypergeometric function ${}_2F_1(\alpha, \beta, \gamma | z)$ reads

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- The monodromy group is subgroup of $SL(2, \mathbb{R})$.
- There exist $p, q, r \in \mathbb{Z}_{\geq 2}$, depending on α, β, γ, such that

$$M_0 M_1 M_\infty = 1, \quad M_0^p = 1, \quad M_1^q = 1, \quad M_\infty^r = 1.$$

This is a Coxeter group and image in $PGL(2, \mathbb{C})$ is a triangle group.

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- Let $\mathcal{O}$ be a maximal order in Q and let $\mathcal{O}^1$ be the unit group of norm 1 elements.
- We can represent Q in $M_2(\mathbb{R})$ and therefore $\mathcal{O}^1$ is represented in $SL(2, \mathbb{R})$.

Proof of Krammer's theorem

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- Discriminant 15 does not occur in his list. Contradiction.

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- This is a one dimensional family parametrised by $z \in \mathbb{P}^1$, say.
- Krammer's equation is the Picard-Fuchs equation for the periods of this family.

The hypergeometric connection

Consider the following period of the general genus 2 curve

$$y^2 = x(1-x)(1-t_1x)(1-t_2x)(1-t_3x),$$

$$\frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-t_1x)(1-t_2x)(1-t_3x)}}.$$

Expand in powers of t_1, t_2, t_3.

$$\sum_{k,l,m\geq 0} \frac{(1/2)_k(1/2)_l(1/2)_m(1/2)_{k+l+m}}{k!l!m!(k+l+m)!}\ t_1^k t_2^l t_3^m$$

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Krammer's equation is obtained by replacing t_i by suitable rational functions $t_i(z) \in \overline{\mathbb{Q}}(z)$

Appell's functions

Some two variable hypergeometric functions.

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}} \, x^m y^n$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n} \, x^m y^n$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m,n \geq 0} \frac{(\alpha)_m(\alpha)_n(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}} \, x^m y^n$$

$$F_4(\alpha, \beta, \gamma, \gamma', x, y) = \sum_{m,n \geq 0} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!n!(\gamma)_m(\gamma')_n} \, x^m y^n$$

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Question

Does every second order equation which is a minimal equation of a G-function arise in this way? (possibly as a factor)