The p-adic upper half plane
Course and Project Description
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The p-adic upper half plane $\mathcal{X}$ over a p-adic field K is a one-dimensional rigid analytic space whose points in any complete extension L/K are given by

$$\mathcal{X}(L) = \mathbf{P}^1(L) \backslash \mathbf{P}^1(K).$$

This space was first introduced by Mumford, where it plays a key role in the generalization to higher genus of Tate's theory of p-adic uniformization of elliptic curves with semistable reduction. Slightly later, Drinfeld and Cerednik showed that appropriate quotients of this space by discrete arithmetic subgroups of $\mathrm{PGL}_2(K)$ coming from quaternion algebras yield Shimura curves. Since that time, through work of Morita, Schneider, Bertolini, Darmon, Iovita, the authors, and many others, this space and its relationships to arithmetic have been the subject of intensive study. In this course, we will study some aspects of this recent work.

The first part of the course will be a construction of the p-adic upper half plane $\mathcal{X}$ as a rigid space and a study of its relationship to the Bruhat-Tits tree T of PGL_2. This tree T classifies norms on a fixed two dimensional vector space V up to scaling, and there is a map

$$r : \mathcal{X} \to T$$

that commutes with the action of PGL_2 on both spaces. We will also study the compactification of T obtained by adding its set of ends, a set homeomorphic to $\mathbf{P}^1(K)$.

In the second part of the course, we will explore the relationship, first described by Schneider, between rigid analytic one-forms on $\mathcal{X}$, "harmonic" functions on the edges of the tree T, and distributions on the (common) boundary $\mathbf{P}^1(K)$ of $\mathcal{X}$ and T. The key ideas in this part of the course are the residue map, which carries rigid one-forms to harmonic functions on T, and the integral transform, or "Poisson Kernel," which carries measures on $\mathbf{P}^1(K)$ back to rigid one-forms.

In the third part, we will discuss a second, entirely different, construction of measures on $\mathbf{P}^1(K)$ due to Darmon that also uses the tree, but that blends classical harmonic forms and rigid geometry in a remarkable way.

In the last part of the course, we will examine some of the arithmetic applications of the theory developed so far. In particular, we will describe the construction of two different $\mathcal{L}$ invariants. Such invariants, first observed experimentally in [4] , relate the special values of p-adic L-functions and classical L-functions of a modular form f in cases where the two functions have functional equations of different signs; in that case, the p-adic L-function has an extra order of vanishing, and its critical value differs from that of the classical one by a p-adic number $\mathcal{L}(f)$. We will describe the quaternionic construction of an $\mathcal{L}(f)$-invariant due to the second author, and the "double-integral" construction due to Darmon and Orton. As time permits, we will discuss Orton's proof that her $\mathcal{L}$-invariant depends only on the Galois representation of f locally at p, as originally conjectured in [4]. Also time permitting, we will describe Breuil's definition of the $\mathcal{L}$-invariant [1].

Suggested Reading

[1] C. Breuil, Série Spéciale p-adique et Cohomolologie Étale Complétée. IHES preprint, 2003.

[2] H. Darmon, Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications. Ann. of Math. (2) 154 (2001), no. 3, 589–639.

[3] L. Gerritzen and M. van der Put, *Schottky groups and Mumford curves*, Lecture Notes in Math., 817, Springer , Berlin, 1980; (see esp. Chapter IX).

[4] B. Mazur, J. Tate and J. Teitelbaum, On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer , Invent. Math. **84** (1986), no. 1, 1–48.

[5] L. Orton, An elementary proof of a weak exceptional zero conjecture. Canad. J. Math. 56 (2004), no. 2, 373–405.

[6] J. T. Teitelbaum, Values of p-adic L-functions and a p-adic Poisson kernel , Invent. Math. **101** (1990), no. 2, 395–410.

Project Description

Let $\mathbf{H}$ be the Hamilton quaternion algebra over $\mathbf{Q}$, and let A be the Hurwitz maximal order, with 5 inverted:

$$A = \mathbf{Z}[\frac{1}{5}, i, j, k, \epsilon]$$

where $\epsilon = (1 + i + j + k)/2$. Let $\Gamma = A^*/(\mathbf{Z}[1/5])^*$. Since $\mathbf{H}$ is split at 5, we may choose an algebra embedding $\mathbf{H} \to M_2(\mathbf{Q}_5)$, so that Γ becomes a subgroup of $\mathrm{PGL}_2(\mathbf{Q}_5)$.

We can identify (at least) the following set of interesting subgroups of Γ:

1. $\Gamma_+ = \Gamma \cap \mathrm{PSL}_2(\mathbf{Q}_5)$
2. $\Gamma(P) = \{\gamma \in \Gamma : \gamma \equiv 1 \pmod{P}\}$ where P is the unique prime ideal of A above 2.
3. $\Gamma_+(P) = \Gamma_+ \cap \Gamma(P)$.
4. $\Gamma(2) = \{\gamma \in \Gamma : \gamma \equiv 1 \pmod{2}\}$.
5. $\Gamma_+(2) = \Gamma_+ \cap \Gamma(2)$

The goal of the project is to explicitly compute as many as possible of the following objects as $\mathbf{\Gamma}$ runs through the groups given above:

1. The genus of the quotient curve $X_{\mathbf{\Gamma}} = X/\mathbf{\Gamma}$. By Cerednik-Drinfeld, these curves are Shimura curves classifying abelian varieties with quaternionic multiplication by the indefinite algebra of discriminant 10 over $\mathbf{Q}$, with some level structure.
2. The structure of the stable reduction of $X_{\mathbf{\Gamma}}$.
3. The Hecke action on the Jacobian of $X_{\mathbf{\Gamma}}$.
4. A basis for the space of harmonic cocycles of weights 2 and 4.
5. The relationship of these Shimura curves to the elliptic curves of conductor 20 and 40.
6. The p-adic period matrix of the Jacobians of $X_{\mathbf{\Gamma}}$.
7. The quaternionic $\mathcal{L}$ invariant for the unique new form of weight 4 and level 10.
8. The special values of the classical and p-adic L-function for the elliptic curve of conductor 40, compared with the previous computation.