Arizona Winter School 2007 • Baker's Group Project Descriptions

1. Consider the following "classical" result in p-adic analysis: *Let* $B = \cup_{j=1}^{m} B(a_j, r_j)$ *be a union of* m *pairwise disjoint closed discs in* $\mathbb{C}_p$*, with centers* $a_j \in \mathbb{C}_p$ *and radii* $r_j \in |\mathbb{C}_p^*| = p^{\mathbb{Q}}$*. Then there exists a polynomial* $f(T) \in \mathbb{C}_p[T]$ *such that* $B = \{z \in \mathbb{C}_p \mid |f(z)| \leq 1\}$. The purpose of this project is to give a self-contained, constructive proof of this result using potential theory on $\mathbb{P}^1_{\mathrm{Berk}}$.

> **(a)** Let $\mathcal{B}(a_j, r_j)$ denote the closed Berkovich disc in $\mathbb{P}^1_{\mathrm{Berk}}$ associated to $B(a_j, r_j)$, let $\mathcal{B} = \cup_{j=1}^{m} \mathcal{B}(a_j, r_j)$, and consider the simple domain $V = \mathbb{P}^1_{\mathrm{Berk}} \setminus \mathcal{B}$. Show using linear algebra, and without using any major results from the lecture notes, that the Poisson-Jensen measure $\mu = \mu_{\infty,V}$ on V relative to the point ∞ exists and is unique, and find an explicit formula for it.
> **(b)** Let $p = p_{\mu,\infty}$ be the potential function for μ relative to the point ∞, as defined in Example 3.22 of the lecture notes. Prove, without just quoting the facts stated in Example 3.22, that p is subharmonic on V, zero on ∂V, and constant on $\mathbb{P}^1_{\mathrm{Berk}} \setminus V$.
> **(c)** Use parts (a) and (b) to construct the desired polynomial $f(T) \in \mathbb{C}_p[T]$.
> **(d)** Compute an explicit example in which B is a union of 3 disjoint closed discs.

2. Consider the normed ring $(\mathcal{O}_K, \|\cdot\|)$, where $\mathcal{O}_K$ is the ring of integers in a number field K, and $\|\cdot\|$ is the norm defined by $\|\alpha\| = \max_\sigma\{|\sigma(\alpha)|_{\mathbb{C}}\}$, the max taken over all embeddings $\sigma : K \hookrightarrow \mathbb{C}$. Here $|\cdot|_{\mathbb{C}}$ denotes the ordinary absolute value on $\mathbb{C}$.

> **(a)** Classify, with a rigorous proof, all bounded multiplicative seminorms on $\mathcal{O}_K$.
> **(b)** Give a detailed description of the Berkovich spectrum $\mathcal{M}(\mathcal{O}_K)$ of $(\mathcal{O}_K, \|\cdot\|)$, including its topology and its structure as a profinite metrized graph.
> **(c)** For each nonzero element $\alpha \in \mathcal{O}_K$, define a function $f_\alpha : \mathcal{M}(\mathcal{O}_K) \to \mathbb{R}$ by $f_\alpha(x) = \log |\alpha|_x$. Describe this function and its analytic properties, and calculate its Laplacian $\Delta_{\mathcal{M}(\mathcal{O}_K)}(f_\alpha)$. Interpret the product formula and the Weil height on K in terms of the analytic behavior of f_α.
> **(d)** Carry out these investigations for the ring $\bar{\mathbb{Z}}$ of all algebraic integers.

3. Let K be an algebraically closed, complete, non-archimedean field, with a countable residue field $\tilde{K}$, and let X/K be a (smooth, proper, geometrically integral) curve. Let ρ denote the canonical metric on $\mathbf{H}(X) = X_{\mathrm{Berk}} \setminus X(K)$ as described in §5. Show that if ρ' is an *arbitrary* metric on $\mathbf{H}(X)$, with associated Laplacian $\Delta'_{X_{\mathrm{Berk}}}$, and if the Poincaré-Lelong formula $\Delta'_{X_{\mathrm{Berk}}}(-\log_v |\varphi|) = \delta_{\mathrm{div}(\varphi)}$ holds for all meromorphic functions φ on X_{Berk}, then $\rho' = \rho$.

> **(a)** First consider $X = \mathbb{P}^1$, and note that given any divisor $D \in \mathrm{Div}^0(\mathbb{P}^1)$, there exists $\varphi \in K(\mathbb{P}^1)$ with $\mathrm{div}(\varphi) = D$. Use this fact to recover ρ' from the Poincaré-Lelong formula.
> **(b)** For general X, of course, one can no longer expect to find meromorphic functions with prescribed divisors. However, a result of R. Rumely (cf. his book "Capacity Theory on Algebraic Curves," pp. 48-49) states that, given distinct points $x, \zeta \in X(K)$ and a neighborhood U of x in $X(K)$, there exists a rational function $\varphi \in K(X)$ whose only pole is at ζ, and whose only zeros lie in U. Use this result to again recover ρ' from the Poincaré-Lelong formula.