ARIZONA WINTER SCHOOL PROJECT ON ANALYTIFICATIONS

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1. Introduction

There is a natural analytification functor from the category of locally separated algebraic spaces locally of finite type over $\mathbf{C}$ to the category of complex-analytic spaces [Kn, Ch. I, 5.17*ff*]. (Recall that a map of algebraic spaces $X \to S$ is *locally separated* if the diagonal $\Delta_{X/S} : X \to X \times_S X$ is an immersion. We require algebraic spaces to have quasi-compact diagonal over $\mathrm{Spec}\,\mathbf{Z}$.) It is natural to ask if a similar theory works over a non-archimedean base field k. This is a non-trivial question because the construction of analytifications in the complex-analytic case is so local that any attempt to carry out the same method in the rigid setting seems to get stuck on unpleasant admissibility issues. The contrast is perhaps better appreciated in view of the surprise that there are counterexamples showing that local separatedness is *not* sufficient for analytifiability of an algebraic space locally of finite type over k; see Example 3.1. Put in more concrete terms, there are locally separated algebraic spaces of finite type over $\mathbf{Q}$ with dimension 2 such that the k-fiber does not admit an analytification for any non-archimedean field k of characteristic 0. Roughly speaking, this dichotomy between the archimedean and non-archimedean worlds is explained by the lack of a Gelfand–Mazur theorem over non-archimedean fields. (That is, any non-archimedean field k admits non-trivial non-archimedean extension fields with a compatible absolute value, even if k is algebraically closed.)

Over $\mathbf{C}$, analytification is defined in terms of a quotient process, and it follows from [SGA1, XII, 3.2(*iv*)] that an algebraic space locally of finite type over $\mathbf{C}$ admits an analytification if and only if it is locally separated over $\mathrm{Spec}(\mathbf{C})$. It is unclear if local separatedness is a necessary condition for the existence of analytifications in the non-archimedean case (for reasons that we explain above Lemma 2.16). Since local separatedness fails to be a sufficient criterion for the existence of non-archimedean analytification, it is natural to seek a reasonable salvage of the situation. It turns out that separatedness suffices. These notes are extracted from a paper [CT] in preparation by the two authors, and omitted details are left as exercises for the Winter School.

These notes explain how the general problem is reduced to the special case of finite free actions by a finite group in the context of Berkovich spaces, but this special case is not addressed because it requires an entirely different viewpoint (Temkin's theory of reduction of germs [T]) for which there will likely not be time to be discussed at the Winter School. We systematically develop the basic definitions and prove some of the basic lemmas that we need, and by stating other results without proof (left as exercises) we hope this will provide a structured framework for the project. An existence theorem for analytifications in the separated case is discussed in §4. Throughout these notes, it is tacitly understood that "algebraic space" means "algebraic space locally of finite type over k" unless we explicitly say otherwise.

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2. Étale equivalence relations and algebraic spaces

In this section we develop basic concepts related to analytification for algebraic spaces $\mathscr{X}$ that are locally of finite type over k. We give the category of rigid spaces (over k) the topology generated by the Tate topology and the class of faithfully flat maps that admit local *fpqc* quasi-sections in the sense of [C2, Def. 4.2.1]; that is, a *covering* of a rigid space X is a collection of flat maps $X_i \to X$ with $\pi : \coprod X_i \to X$ surjective such that locally on the base for the Tate topology (i.e., over the contituents of an admissible covering of X) there exist sections after a faithfully flat and quasi-compact base change.

Exercise 2.1. Consider the following example that is not a covering for the above topology. Let X be the closed unit ball $\{|t| \leq 1\}$ over k, and let $X_1 = \{|t| < 1\}$ and $X_2 = \{|t| = 1\}$ be the corresponding (non-quasi-compact) open unit ball and (affinoid open) "boundary". Using that $\{X_1, X_2\}$ is not an admissible covering of X, prove that it is not a covering in the weaker sense defined above. (Hint: use connectedness considerations.) Generalize this example.

By [C2, Cor. 4.2.5], all representable functors are sheaves for this topology. Let $X' \to X$ be a flat surjection of rigid spaces and assume that it admits local *fpqc* quasi-sections. The maps $R = X' \times_X X' \rightrightarrows X'$ define a monomorphism $R \to X' \times X'$, and we have an isomorphism $X'/R \simeq X$ as sheaves of sets on the category of rigid spaces since the maps $R \rightrightarrows X'$ are faithfully flat and admit local *fpqc* quasi-sections in such cases.

Conversely, given a pair of flat maps $R \rightrightarrows X'$ admitting local *fpqc* quasi-sections such that $R \to X' \times X'$ is functorially an equivalence relation (in which case we call $R \to X' \times X'$ a *flat equivalence relation*), consider the sheafification of the presheaf $Z \mapsto X'(Z)/R(Z)$. If this sheaf is represented by some rigid space X then we call X (equipped with the map $X' \to X$) the *flat quotient* of X' modulo R and we denote it X'/R. By the very definition of the topology used to define the quotient sheaf X'/R, if a flat quotient X exists then the projection map $\pi : X' \to X$ admits local *fpqc* quasi-sections. Moreover, π is automatically faithfully flat. Indeed, arguing as in the case of schemes, choose a faithfully flat map $z : Z \to X$ such that there is a quasi-section $z' : Z \to X'$ over X. The map π is faithfully flat if and only if the projection $q_2 : X' \times_X Z \to Z$ is faithfully flat, and via the isomorphism $X' \times_X Z \simeq X' \times_X X' \times_{X',z'} Z = R \times_{p_2,X',z'} Z$ the map q_2 is identified with a base change of the projection $p_2 : R \to X'$ that is faithfully flat.

When the flat quotient $X = X'/R$ exists, the map $R \to X' \times_X X'$ is an isomorphism and so for every property $\mathbf{P}$ in [C2, Thm. 4.2.7] the map $X' \to X$ satisfies $\mathbf{P}$ if and only if the maps $R \rightrightarrows X'$ satisfy $\mathbf{P}$. Likewise, X is quasi-separated (resp. separated) if and only if the map $R \to X' \times X'$ is quasi-compact (resp. a closed immersion). By descent theory for morphisms, the diagram of sets

$$X(Z) \to X'(Z) \rightrightarrows R(Z) \tag{2.1}$$

is left-exact for any rigid space Z when $X = X'/R$ is a flat quotient.

Definition 2.2. An *étale equivalence relation* on a rigid space X' is a functorial equivalence relation $R \to X' \times X'$ such that the maps $R \rightrightarrows X'$ are étale and admit local étale quasi-sections in the sense of [C2, Def. 4.2.1]. If the flat quotient X'/R exists, it is called an *étale quotient* in such cases.

Example 2.3. If $X' \to X$ is an étale surjection that admits local étale quasi-sections then the étale quotient of X' modulo the étale equivalence relation $R = X' \times_X X'$ exists: it is X.

Lemma 2.4. *Let $R \to X' \times X'$ be a flat equivalence relation on a rigid space X', and assume that the flat quotient X'/R exists. The equivalence relation $R \to X' \times X'$ is étale if and only if the map $X' \to X'/R$ is étale and admits local étale quasi-sections.*

Proof. Let $X = X'/R$. Since $R = X' \times_X X'$ and the map $X' \to X$ is faithfully flat with local *fpqc* quasi-sections, we may use [C2, Thm. 4.2.7] for the property $\mathbf{P}$ of being étale with local étale quasi-sections. ■

Let $\mathscr{X}$ be an algebraic space and let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$. By [C2, Thm. 4.2.2], the maps $\mathscr{R}^{\text{an}} \rightrightarrows \mathscr{U}^{\text{an}}$ admit local étale quasi-sections. Since a map in any category with fiber products is a monomorphism if and only if its relative diagonal is an isomorphism, analytification of algebraic k-schemes carries monomorphisms to monomorphisms. Thus, the morphism $\mathscr{R}^{\text{an}} \to \mathscr{U}^{\text{an}} \times \mathscr{U}^{\text{an}}$ is a monomorphism and so $\mathscr{R}^{\rm an}$ is an étale equivalence relation on $\mathscr{U}^{\rm an}$. It therefore makes sense to ask if the étale quotient $\mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$ exists.

We want such existence and the actual quotient to be independent of the chart in a canonical manner, in which case we define it to be the *analytification* of $\mathscr{X}$. The étale equivalence relations that arise in the problem of analytifying algebraic spaces are rather special, and so one might hope that in such cases it is possible to always construct the required quotient, at least when the algebraic space is locally separated over k (as is necessary and sufficient for the existence of analytifications in the complex-analytic theory). However, we will give counterexamples in Example 3.1: locally separated algebraic spaces that are not analytifiable in the above sense defined via quotients. In the positive direction, the desired quotient will be shown to always exist in the separated case (though in these notes we will just reduce the problem to a concrete special case whose treatment requires other ideas).

We first address the "independence of choice" and canonicity issues for $\mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$ in terms of $\mathscr{X}$. These will go essentially as in the complex-analytic case except that we have to occasionally use properties related to local étale quasi-sections for the Tate topology. In the complex-analytic case it does not seem that the relevant arguments are available in the literature, so for this reason and to ensure that the Tate topology presents no difficulties we have decided to give the arguments in detail (especially so we can see that it carries over to Berkovich spaces, as we shall need later).

Let $\mathscr{R}_1 \rightrightarrows \mathscr{U}_1$ and $\mathscr{R}_2 \rightrightarrows \mathscr{U}_2$ be two étale charts for $\mathscr{X}$. Let $\mathscr{U}_{12} = \mathscr{U}_1 \times_{\mathscr{X}} \mathscr{U}_2$ and let $\mathscr{R}_{12} = \mathscr{R}_1 \times_{\mathscr{X}} \mathscr{R}_2$, so $\mathscr{R}_{12} \rightrightarrows \mathscr{U}_{12}$ is an étale chart dominating each chart $\mathscr{R}_i \rightrightarrows \mathscr{U}_i$.

Lemma 2.5. *If $\mathscr{U}_1^{\rm an}/\mathscr{R}_1^{\rm an}$ exists then so do $\mathscr{U}_2^{\rm an}/\mathscr{R}_2^{\rm an}$ and $\mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$, and the natural maps*

$$\pi_i : \mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an} \to \mathscr{U}_i^{\rm an}/\mathscr{R}_i^{\rm an}$$

are isomorphisms. The induced isomorphism $\phi = \pi_2 \circ \pi_1^{-1} : \mathscr{U}_1^{\rm an}/\mathscr{R}_1^{\rm an} \simeq \mathscr{U}_2^{\rm an}/\mathscr{R}_2^{\rm an}$ is transitive with respect to a third choice of étale chart for $\mathscr{X}$.

Proof. The natural composite map $\mathscr{U}_{12}^{\rm an} \to \mathscr{U}_1^{\rm an} \to \mathscr{U}_1^{\rm an}/\mathscr{R}_1^{\rm an}$ is étale with local étale quasi-sections (as each step in the composite has this property, due to [C2, Thm. 4.2.2] for $\mathscr{U}_{12} \to \mathscr{U}_1$ and the defining properties for $\mathscr{U}_1^{\rm an}/\mathscr{R}_1^{\rm an}$ as an étale quotient).

Exercise 2.6. Show that this composite map serves as the étale quotient for $\mathscr{U}_{12}^{\rm an}$ by $\mathscr{R}_{12}^{\rm an}$ (so the étale quotient $\mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$ exists and π_1 is an isomorphism). The problem is to prove

$$\mathscr{R}_{12}^{\rm an} \stackrel{?}{=} \mathscr{U}_{12}^{\rm an} \times_{\mathscr{U}_1^{\rm an}/\mathscr{R}_1^{\rm an}} \mathscr{U}_{12}^{\rm an}$$

as subfunctors of $\mathscr{U}_{12}^{\rm an} \times \mathscr{U}_{12}^{\rm an}$.

Now we address the existence of $\mathscr{U}_2^{\rm an}/\mathscr{R}_2^{\rm an}$ and the isomorphism property for π_2. The map $\mathscr{U}_{12}^{\rm an} \to \mathscr{U}_2^{\rm an}$ is an étale surjection with local étale quasi-sections, and so by rigid-analytic descent theory the étale quotient map $\mathscr{U}_{12}^{\rm an} \to \mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$ admits at most one factorization through the map $\mathscr{U}_{12}^{\rm an} \to \mathscr{U}_2^{\rm an}$, in which case the resulting map $h : \mathscr{U}_2^{\rm an} \to \mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$ is an étale surjection with local étale quasi-sections.

Exercise 2.7. Prove that h exists by first using (2.1) to reduce to showing that the two maps

$$\mathscr{U}_{12}^{\rm an} \times_{\mathscr{U}_2^{\rm an}} \mathscr{U}_{12}^{\rm an} \rightrightarrows \mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$$

coincide, and then proving this equality of maps.

We now have an étale map $h : \mathscr{U}_2^{\rm an} \to \mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$ with local étale quasi-sections. Since

$$\mathscr{R}_{12} = (\mathscr{U}_{12} \times \mathscr{U}_{12}) \times_{\mathscr{U}_2 \times \mathscr{U}_2} \mathscr{R}_2$$

as subfunctors of $\mathscr{U}_{12} \times \mathscr{U}_{12}$, the natural map $\mathscr{R}_{12} \to \mathscr{R}_2$ is an étale surjection and hence the two composite maps

$$\mathscr{R}_2^{\rm an} \rightrightarrows \mathscr{U}_2^{\rm an} \xrightarrow{h} \mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}$$

are equal if and only if equality holds after composition with the map $\mathscr{R}_{12}^{\rm an} \to \mathscr{R}_2^{\rm an}$. Such equality holds after composition because $\mathscr{R}_{12} \rightrightarrows \mathscr{U}_{12}$ is co-commutative over $\mathscr{R}_2 \rightrightarrows \mathscr{U}_2$. Hence, we have

$$\mathscr{R}_2^{\rm an} \subseteq \mathscr{U}_2^{\rm an} \times_{\mathscr{U}_{12}^{\rm an}/\mathscr{R}_{12}^{\rm an}} \mathscr{U}_2^{\rm an} \tag{2.2}$$

as subfunctors of $\mathscr{U}_2^{\mathrm{an}} \times \mathscr{U}_2^{\mathrm{an}}$. To complete the proof that $\mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ exists and that π_2 is an isomorphism, solve the next exercise.

Exercise 2.8. Show that $\mathscr{U}_2^{\mathrm{an}} \to \mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}$ is an étale quotient by $\mathscr{R}_2^{\mathrm{an}}$. Explain why it suffices to check that the inclusion of subfunctors (2.2) is an equality, which is to say that the natural map

$$g : \mathscr{U}_2^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_2^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}} \times \mathscr{U}_2^{\mathrm{an}}$$

factors through the subfunctor $\mathscr{R}_2^{\mathrm{an}}$. Then make such a factorization. You may find it useful to work with the map

$$f : \mathscr{R}_{12}^{\mathrm{an}} \simeq \mathscr{U}_{12}^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_{12}^{\mathrm{an}} \to \mathscr{U}_2^{\mathrm{an}} \times_{\mathscr{U}_{12}^{\mathrm{an}}/\mathscr{R}_{12}^{\mathrm{an}}} \mathscr{U}_2^{\mathrm{an}}$$

that is an étale surjection with local étale quasi-sections (as $\mathscr{U}_{12} \to \mathscr{U}_2$ is an étale surjection of schemes, so [C2, Thm. 4.2.2] applies) and to use descent theory for rigid-analytic morphisms.

To prove the isomorphism $\pi_2 \circ \pi_1^{-1} : \mathscr{U}_1^{\mathrm{an}}/\mathscr{R}_1^{\mathrm{an}} \simeq \mathscr{U}_2^{\mathrm{an}}/\mathscr{R}_2^{\mathrm{an}}$ is transitive with respect to a third choice of étale chart for $\mathscr{X}$, it suffices to observe that in the preceding considerations we only needed that the étale chart $\mathscr{R}_{12} \rightrightarrows \mathscr{U}_{12}$ dominates the other two charts, and not that it is specifically their "fiber product". ■

Lemma 2.5 permits us to make the following definition.

Definition 2.9. An algebraic space $\mathscr{X}$ is *analytifiable* if the étale quotient $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$ exists for some (and hence any) étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{X}$.

For an analytifiable $\mathscr{X}$, the *analytification* $\mathscr{X}^{\mathrm{an}}$ is defined to be $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$. Up to unique isomorphism, this étale quotient is independent of the specific choice of étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ (by Lemma 2.5). Descent theory for coherent sheaves on rigid spaces (see [C2, Thm. 4.2.8] for the formulation we need) permits us to also define an analytification functor from coherent $\mathscr{O}_{\mathscr{X}}$-modules to coherent $\mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$-modules. We now express the functoriality of analytification when analytifications exist.

Theorem 2.10. *Let $\mathscr{X}$ and $\mathscr{X}'$ be analytifiable algebraic spaces and let $f : \mathscr{X} \to \mathscr{X}'$ be a k-morphism. Let $\mathscr{R} \rightrightarrows \mathscr{U}$ and $\mathscr{R}' \rightrightarrows \mathscr{U}'$ be respective étale charts such that f lifts to a map $F : \mathscr{U} \to \mathscr{U}'$ for which $F \times F$ carries $\mathscr{R}$ into $\mathscr{R}'$ (such a pair of charts always exists). The map of rigid spaces*

$$f^{\mathrm{an}} : \mathscr{X}^{\mathrm{an}} \simeq \mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}} \to (\mathscr{U}')^{\mathrm{an}}/(\mathscr{R}')^{\mathrm{an}} \simeq (\mathscr{X}')^{\mathrm{an}}$$

induced by F^{an} depends only on f and not on the étale charts or the map F lifting f, and this procedure enhances the construction $\mathscr{X} \rightsquigarrow \mathscr{X}^{\mathrm{an}}$ to be a functor from the category of analytifiable algebraic spaces over k to the category of rigid spaces over k. Moreover:

- *The category of analytifiable algebraic spaces is stable under the formation of fiber products and passage to open and closed subspaces.*
- *The functor $\mathscr{X} \rightsquigarrow \mathscr{X}^{\mathrm{an}}$ is compatible with the formation of fiber products and carries open/closed immersions to Zariski-open/closed immersions.*
- *If $f : \mathscr{X}' \to \mathscr{X}$ is a morphism between analytifiable algebraic spaces then f^{an} is separated if and only if f is separated.*

Proof. Any two étale charts are dominated by a third, and any two lifts of f with respect to a fixed choice of charts are $\mathscr{R}'$-equivalent. Thus, the well-definedness of f^{an} is an immediate consequence of Lemma 2.5. The compatibility with composition of morphisms follows from the independence of the choice of charts, so analytification is indeed a functor on analytifiable algebraic spaces over k.

Let $\mathscr{X}$ be an analytifiable algebraic space over k and let $\mathscr{X}' \to \mathscr{X}$ be an open (resp. closed) immersion. We let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$, and let $\mathscr{U}' \to \mathscr{U}$ denote the pullback of $\mathscr{X}'$. The analytification of this inclusion is a Zariski-open (resp. closed) immersion. Rigid-analytic descent theory with respect to coherent ideal sheaves trivially implies that the analytification of $\mathscr{U}'$ descends to a Zariski-open (resp. closed) immersion into $\mathscr{X}^{\mathrm{an}}$, and this descent is easily seen to be an analytification of $\mathscr{X}'$.

Now we consider fiber products. Let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic k-schemes over an algebraic k-scheme $\mathscr{Z}$, and assume that all three are analytifiable.

Exercise 2.11. Check that $\mathscr{P} = \mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$ is analytifiable and that the natural map $\mathscr{P}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}} \times_{\mathscr{Z}^{\mathrm{an}}} \mathscr{Y}^{\mathrm{an}}$ is an isomorphism. To do this, work with choices of étale charts $\mathscr{X} = \mathscr{U}'/\mathscr{R}'$, $\mathscr{Y} = \mathscr{U}''/\mathscr{R}''$, and $\mathscr{Z} = \mathscr{U}/\mathscr{R}$, to build one for $\mathscr{P}$. (The point is to show that $\mathscr{X}^{\mathrm{an}} \times_{\mathscr{Z}^{\mathrm{an}}} \mathscr{Y}^{\mathrm{an}}$ serves as a quotient for an étale equivalence relation in the definition of $\mathscr{P}^{\mathrm{an}}$.)

Also show that if $f : \mathscr{X}' \to \mathscr{X}$ is a morphism between analytifiable algebraic spaces then f^{an} is separated if and only if f is separated. (Hint: first use that the compatibility of analytification and fiber products identifies $\Delta_{f^{\mathrm{an}}}$ and Δ_f^{an} to replace f with Δ_f so as to reduce to checking that if $f : \mathscr{X}' \to \mathscr{X}$ is a quasi-compact immersion between algebraic spaces then f is a closed immersion if and only if f^{an} is a closed immersion. In fact, go a step further with an étale chart to reduce this to the case when f is a quasi-compact immersion between algebraic k-schemes. This case is handled by [C1, 5.2.1(2)].)

■

Corollary 2.12. *Let $\mathscr{X}$ be an algebraic space and let $\{\mathscr{X}_i\}$ be an open covering. Analytifiability of $\mathscr{X}$ is equivalent to that of all of the $\mathscr{X}_i$'s.*

Proof. By Theorem 2.10, if $\mathscr{X}^{\mathrm{an}}$ exists then so does $\mathscr{X}_i^{\mathrm{an}}$ (as a Zariski-open in $\mathscr{X}^{\mathrm{an}}$) for all i. Conversely, assume that $\mathscr{X}_i^{\mathrm{an}}$ exists for all i. The algebraic space $\mathscr{X}_{ij} = \mathscr{X}_i \cap \mathscr{X}_j$ is identified with a Zariski-open subspace of both $\mathscr{X}_i$ and $\mathscr{X}_j$, so by Theorem 2.10 the rigid space $\mathscr{X}_{ij}^{\mathrm{an}}$ exists and is identified with a Zariski-open locus in $\mathscr{X}_i^{\mathrm{an}}$ and $\mathscr{X}_j^{\mathrm{an}}$. Since $\mathscr{X}_{ij} \cap \mathscr{X}_{ij'} = \mathscr{X}_{ij} \times_{\mathscr{X}_i} \mathscr{X}_{ij'}$ for any i, j, j', the fiber-product compatibility in Theorem 2.10 provides the triple-overlap compatibility that is required to glue the $\mathscr{X}_i^{\mathrm{an}}$'s to construct a rigid space X having the $\mathscr{X}_i^{\mathrm{an}}$'s as an admissible covering with $\mathscr{X}_i^{\mathrm{an}} \cap \mathscr{X}_j^{\mathrm{an}} = \mathscr{X}_{ij}^{\mathrm{an}}$ inside of X for all i and j. In particular, $X - \mathscr{X}_i^{\mathrm{an}}$ meets every $\mathscr{X}_j^{\mathrm{an}}$ in an analytic set, and hence $\mathscr{X}_i^{\mathrm{an}}$ is Zariski-open in X.

Exercise 2.13. Check that X serves as an analytification of $\mathscr{X}$.

■

Remark 2.14. If $f : \mathscr{X} \to \mathscr{Y}$ is a faithfully flat map between analytifiable algebraic spaces over k, we claim that the induced faithfully flat map $f^{\mathrm{an}} : \mathscr{X}^{\mathrm{an}} \to \mathscr{Y}^{\mathrm{an}}$ has local *fpqc* quasi-sections, and if f is an étale surjection then we claim that f^{an} has local étale quasi-sections (as is proved in the case of maps between algebraic k-schemes in [C2, Thm. 4.2.2]). To prove this, we pick a chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for $\mathscr{Y}$, so $\mathscr{U}^{\mathrm{an}} \to \mathscr{Y}^{\mathrm{an}}$ has local étale quasi-sections (as $\mathscr{Y}^{\mathrm{an}}$ is the étale quotient $\mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$). Hence, we may replace $\mathscr{X} \to \mathscr{Y}$ with its base change by the étale surjection $\mathscr{U} \to \mathscr{Y}$, so we can assume that $\mathscr{Y}$ is an algebraic k-scheme. Running through a similar argument with an étale chart for $\mathscr{X}$ reduces us to the case when $\mathscr{X}$ is also an algebraic k-scheme, and so we are brought to the settled scheme case.

Remark 2.15. If $f : \mathscr{X} \to \mathscr{Y}$ is a map between analytifiable algebraic spaces over k, then f has property **P** if and only if f^{an} property **P**, where **P** is any of the following properties: separated, monomorphism, surjective, isomorphism, open immersion, flat, smooth, and étale. Likewise, if f is finite type then we may take **P** to be: closed immersion, finite, proper, quasi-finite (i.e., finite fibers). To prove this, by [C2, Thm. 4.2.7] and descent theory for schemes we may work étale-locally on $\mathscr{Y}$ and so we can assume that $\mathscr{Y}$ is a scheme of finite type over k. Since flat maps of algebraic spaces are open, the essential properties to consider are isomorphism and properness; the rest then follow exactly as in the case of schemes. By Chow's lemma for algebraic spaces and [C2, §A.1] (for properness), the proper case is reduced to the case of quasi-compact immersions of schemes (more specifically, a quasi-compact immersion into a projective space over $\mathscr{Y}$), and this case follows from [C1, 5.2.1(2)]. If f^{an} is an isomorphism then f is quasi-finite, flat, and (by Theorem 2.10) separated, so $\mathscr{X}$ is necessarily a scheme. Thus, we may use [C1, 5.2.1(1)] to infer that f is an isomorphism.

Now comes a delicate technical point. In algebraic geometry, if a quasi-compact map of schemes is the composite of a closed immersion followed by a quasi-compact open immersion then it can also be expressed as the composition of a quasi-compact open immersion followed by a closed immersion. This is the concept of scheme-theoretic closure, and it leaves no ambiguity about the meaning of a (quasi-compact) immersion between schemes. However, in analytic situations there is no such result and so one has to be attentive to the distinction between the order of composition among open immersions and closed immersions. If X is a complex-analytic space, then since X rests on an ordinary topological space we can find an open subset $V \subseteq X$ such that the diagonal $X \to X \times X$ factors through a closed immersion into $V \times V$. (Take V to be a union of Hausdorff open neighborhoods around the points of X.) This is the key property that underlies the fact that an analytifiable algebraic space over $\mathbf{C}$ is necessarily locally separated. It is unclear if such a result should be true in general in the rigid-analytic case, so necessity of local separatedness as a criterion for analytifiability is unclear over non-archimedean fields. One may still ask if the property of being locally separated has a good relation with the property of being analytifiable for algebraic spaces over k. A relevant notion for this was used in [C2]: a *pseudo-separated* map $f : X \to S$ between rigid spaces is a map whose diagonal $\Delta_f : X \to X \times_S X$ factors as the composite of a Zariski-open immersion followed by a closed immersion. The reason that we choose this order of composition is that in the scheme case it is available in a canonical manner (via scheme-theoretic closure) and hence behaves well with respect to étale localization and descent. We note that a map of rigid spaces is pseudo-separated and quasi-separated (i.e., has quasi-compact diagonal) if and only if it is separated.

Lemma 2.16. *Let $\mathscr{X}$ be an analytifiable algebraic space. The algebraic space $\mathscr{X}$ is locally separated if and only if the rigid space $\mathscr{X}^{\mathrm{an}}$ is pseudo-separated.*

Proof. Since $\mathscr{X}$ is analytifiable, so is $\mathscr{X} \times \mathscr{X}$. Clearly $\Delta_{\mathscr{X}}^{\mathrm{an}} = \Delta_{\mathscr{X}^{\mathrm{an}}}$. If $\mathscr{X}$ is locally separated then (by étale descent) the quasi-compact immersion $\Delta_{\mathscr{X}}$ uniquely factors as a (schematically dense) Zariski-open immersion followed by a closed immersion, and these intervening locally closed subspaces of $\mathscr{X}$ must be analytifiable (by Theorem 2.10). It follows that $\mathscr{X}^{\mathrm{an}}$ must be pseudo-separated in such cases. Conversely, suppose $\mathscr{X}^{\mathrm{an}}$ is pseudo-separated and let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$. The diagram

$$\begin{array}{ccc} \mathscr{R}^{\mathrm{an}} & \longrightarrow & \mathscr{U}^{\mathrm{an}} \times \mathscr{U}^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathscr{X}^{\mathrm{an}} & \xrightarrow[\Delta_{\mathscr{X}^{\mathrm{an}}}]{} & \mathscr{X}^{\mathrm{an}} \times \mathscr{X}^{\mathrm{an}} \end{array}$$

is cartesian because $\mathscr{X}^{\mathrm{an}} = \mathscr{U}^{\mathrm{an}}/\mathscr{R}^{\mathrm{an}}$, so the top side factors as a Zariski-open immersion followed by a closed immersion. This top side is the analytification of the finite type map $\mathscr{R} \to \mathscr{U} \times \mathscr{U}$, so by [C1, 5.2.1(2)] (for the property of being a rigid-analytic immersion in the sense of a composite of an open immersion followed by a closed immersion) we conclude that the scheme map $\mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is an immersion, and hence $\mathscr{X}$ is locally separated. ∎

It is reasonable to ask if analytification for algebraic spaces is compatible with extension of the base field, as is the case for analytification of algebraic k-schemes. To make sense of such a statement it is natural to impose a condition of quasi-separatedness or pseudo-separatedness on the analytifications (each of which suffices to globalize the change of base field functors). Here is the result.

Theorem 2.17. *Let $\mathscr{X}$ be an analytifiable algebraic space over k, and let k'/k be an analytic extension field. If $\mathscr{X}^{\mathrm{an}}$ is either quasi-separated or pseudo-separated (automatic if $\mathscr{X}$ is a locally separated algebraic space, by Lemma 2.16) then the algebraic space $k' \otimes_k \mathscr{X}$ over k' is analytifiable and there is a natural isomorphism $k'\widehat{\otimes}_k \mathscr{X}^{\mathrm{an}} \simeq (k' \otimes_k \mathscr{X})^{\mathrm{an}}$ (so $(k' \otimes_k \mathscr{X})^{\mathrm{an}}$ is quasi-separated or pseudo-separated), and for algebraic k-schemes $\mathscr{X}$ this is the usual isomorphism. These natural isomorphisms are transitive with respect to further extension of the base field and are compatible with the formation of fiber products.*

Proof. Let $\mathscr{R} \rightrightarrows \mathscr{U}$ be an étale chart for $\mathscr{X}$. Since $\mathscr{U}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is étale and admits local étale quasi-sections, the same holds for the map $k'\widehat{\otimes}_k \mathscr{U}^{\mathrm{an}} \to k'\widehat{\otimes}_k \mathscr{X}^{\mathrm{an}}$. Moreover, the natural map

$$k'\widehat{\otimes}_k \mathscr{R}^{\mathrm{an}} \to (k'\widehat{\otimes}_k \mathscr{U}^{\mathrm{an}}) \times_{k'\widehat{\otimes}_k \mathscr{X}^{\mathrm{an}}} (k'\widehat{\otimes}_k \mathscr{U}^{\mathrm{an}})$$

is an isomorphism because it is identified with the extension of scalars of the map $\mathscr{R}^{\mathrm{an}} \to \mathscr{U}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} \mathscr{U}^{\mathrm{an}}$ that is necessarily an isomorphism (due to the defining property of the étale quotient $\mathscr{X}^{\mathrm{an}}$ that we are assuming to exist). Thus, we conclude that $k'\widehat{\otimes}_k \mathscr{X}^{\mathrm{an}}$ serves as an étale quotient for the diagram

$$k'\widehat{\otimes}_k \mathscr{R}^{\mathrm{an}} \rightrightarrows k'\widehat{\otimes}_k \mathscr{U}^{\mathrm{an}} \tag{2.3}$$

that is an étale equivalence relation, due to its identification with the analytification $(k'\otimes_k\mathscr{R})^{\rm an} \rightrightarrows (k'\otimes_k\mathscr{U})^{\rm an}$ of an étale chart for the algebraic space $k'\otimes_k \mathscr{X}$ over k'. This shows that if $\mathscr{X}^{\rm an}$ exists then $k'\widehat{\otimes}_k\mathscr{X}^{\rm an}$ naturally serves as an analytification for the algebraic space $k'\otimes_k\mathscr{X}$ over k'. Moreover, it is clear that this identification $k'\widehat{\otimes}_k\mathscr{X}^{\rm an}\simeq(k'\otimes_k\mathscr{X})^{\rm an}$ is independent of the choice of étale chart $\mathscr{U}\rightrightarrows\mathscr{R}$ for $\mathscr{X}$ and that it is therefore functorial in the analytifiable $\mathscr{X}$. The compatibility with fiber products (when the relevant analytifications exist over k) is now obvious. ∎

3. Analytification counterexamples and constructions

To show that the theory in §2 is not vacuous, we need to prove the analytifiability of an interesting class of algebraic spaces that are not necessarily schemes. Since an algebraic space locally of finite type over $\mathbf{C}$ is analytifiable if and only if it is locally separated, it is reasonable to focus attention on the class of algebraic spaces (locally of finite type over k) that are locally separated. It turns out that all such algebraic spaces are analytifiable in the separated case, but here we give locally separated examples where analytifiability fails.

Example 3.1. Consider a quasi-separated scheme $\mathscr{X}$ equipped with a closed subscheme $\mathscr{T}$ and a quasi-compact étale surjection $\pi:\mathscr{U}\to\mathscr{X}$. In [Kn, Intro., Ex. 2, pp.10–12] this data is used to construct a locally separated algebraic space $\mathscr{X}'$ equipped with a quasi-compact étale surjection $i:\mathscr{X}'\to\mathscr{X}$ such that i is an isomorphism over $\mathscr{X}-\mathscr{T}$ but has pullback to $\mathscr{T}$ given by the étale covering $\mathscr{T}'=\pi^{-1}(\mathscr{T})\to\mathscr{T}$. To give some specific examples, let k be an abstract field, let $\mathscr{T}\subseteq\mathbf{A}^2_k$ be a dense open subset of the x-axis, and let $\mathscr{T}'\to\mathscr{T}$ be the geometrically connected finite étale covering with degree $d>1$ given by extracting the dth root of a monic separable polynomial $f\in k[x]$ whose zeros are away from $\mathscr{T}$; we assume d is not divisible by the characteristic of k. By shrinking $\mathscr{T}$ near its generic point, we can find an open $\mathscr{X}\subseteq\mathbf{A}^2_k$ in which $\mathscr{T}$ is closed and over which there is a quasi-compact étale cover $\mathscr{U}\to\mathscr{X}$ restricting to $\mathscr{T}'\to\mathscr{T}$ over $\mathscr{T}$. Applying the general construction, we get a locally separated algebraic space $\mathscr{X}'$ that is an étale cover of the open set $\mathscr{X}$ in the plane and restricts to an isomorphism over $\mathscr{X}-\mathscr{T}$ but restricts to a degree-d geometrically connected finite étale covering over the non-empty open set $\mathscr{T}$ in a line. The degree-jumping behavior of this quasi-finite map is opposite what happens for quasi-finite separated étale maps of schemes (via the structure theorem for such maps [EGA, IV_4, 18.5.11]) in the sense that the fiber-degree goes up (rather than down) at special points since $d>1$. Hence, $\mathscr{X}'$ cannot have an open scheme neighborhood (or equivalently, a separated open neighborhood) around any point of $\mathscr{T}'$ since if it did then such a neighborhood would contain the 1-point generic fiber over $\mathscr{T}$, yet no fiber over $\mathscr{T}$ can have a separated open neighborhood in $\mathscr{X}'$ (e.g., an affine open subscheme).

If this construction were considered over $\mathbf{C}$ then an analytification of $\mathscr{X}'$ does exist (since $\mathscr{X}'$ is locally separated) and the local structure over an open neighborhood of a point of $\mathscr{T}(\mathbf{C})$ is very easy to describe: it is simply a gluing of d copies of an open disc to itself along the complement of the origin. In particular, this analytification is non-Hausdorff over such a neighborhood. In the non-archimedean setting, if $\mathscr{T}'\to\mathscr{T}$ has some non-split k-fiber then this local gluing cannot be done near there, and even if k is algebraically closed there are global admissibility problems with the gluing.

Exercise 3.2. Explain how admissibility problems intervene when trying to make the analytification via gluing in the non-archimedean case.

We shall show that the admissibility problems are genuine. The key is that there is an obstruction to analytifiability caused by the failure of the Gelfand–Mazur theorem over non-archimedean fields: k may admit analytic extension fields k'/k such that the étale cover $\mathscr{T}'\to\mathscr{T}$ has a non-split fiber over some $t\in\mathscr{T}(k')$, even if k is algebraically closed.

Let us now prove that if k is a non-archimedean field then $\mathscr{X}'$ is not analytifiable. Assume that an analytification X' of $\mathscr{X}'$ exists. Since $\mathscr{X}'$ is locally separated, by Lemma 2.16 the rigid space X' must be pseudo-separated, and so by Theorem 2.17 if k'/k is any non-archimedean extension field then $k'\otimes_k\mathscr{X}'$ is analytifiable with analytification $k'\widehat{\otimes}_kX'$. Hence, to get a contradiction it suffices to consider the situation after a preliminary extension of the base field $k\to k'$ (which is easily checked to automatically commute with the formation of $\mathscr{X}'$ in terms of $\mathscr{X}$ and $\mathscr{T}'\to\mathscr{T}$). First increase k a finite amount so that f splits, and then consider the extension $K' = k(\mathscr{T}')$ of $K = k(\mathscr{T}) = k(x)$ is defined by adjoining a root to the irreducible polynomial $u^d - f \in K[u]$. We then make a linear change of variable on x so that $f = x\prod_{i>1}(1 - r_i x)$ with $|r_i| < 1$ for each i. In case of mixed characteristic we also require $|r_i|$ to be so small that $1 - r_i x$ has a dth root as a power series (a condition that is automatic in case of equicharacteristic k).

Exercise 3.3. Taking k' to be the completion $\widehat{K}$ of $K = k(x)$ with respect to the Gauss norm (i.e., the completion of the fraction field of the Tate algebra in one variable over k with respect to its multiplicative sup-norm), find $t_0 \in \mathscr{T}(k')$ with no k'-rational point in its non-empty fiber in $\mathscr{T}'$. Use the local structure theorem for quasi-finite maps of rigid spaces [C2, Thm. A.1.3] at a point over t_0 to prove that an analytification of $\mathscr{X} \otimes_k k'$ does not exist over k', and hence $\mathscr{X}$ is not analytifiable.

Returning to the general situation, we are now motivated to focus attention on the problem of analytifying separated algebraic spaces. A key technical issue in the proof that separated algebraic spaces are analytifiable will be to show that locally (in the rigid sense) we can describe the quotient problem in such cases as that of forming the quotient of an affinoid by a *finite* étale equivalence relation. This issue is subtle for two reasons: the theory of products for rigid spaces (and Berkovich spaces) is not as straightforward as for complex-analytic spaces, and saturation with respect to an equivalence relation is a problematic operation with respect to the property of admissibility for subsets of a rigid space.

In the finite étale case with affinoid spaces, the construction of quotients goes as in algebraic geometry except that there is the additional issue of checking that various k-algebras are also k-affinoid:

Lemma 3.4. *Let $f : U \to X$ be a finite étale surjective map of rigid spaces. The rigid space X is affinoid if and only if the rigid space U is affinoid. Moreover, if $R' \rightrightarrows U'$ is a finite étale equivalence relation on an affinoid rigid space U' then the étale quotient $X' = U'/R'$ exists and $U' \to X'$ is a finite étale cover.*

The étale hypothesis for the first part of the lemma is essential; in [Liu] there is an example of a non-affinoid quasi-compact separated surface (over any k) such that the normalization is affinoid. The proof of Lemma 3.4 carries over verbatim to the case of k-affinoid Berkovich spaces that are not necessarily strictly k-analytic, the key point being that if a k-affinoid algebra A is endowed with a continuous action by a finite group G then the closed subalgebra A^G is k-affinoid and A is finite and admissible as an A^G-module [Ber1, 2.1.14(*ii*)]. This is an important ingredient in the problem of étale descent for Berkovich spaces (which in turn is needed to solve the problem of analytifying separated algebraic spaces via rigid spaces).

Proof. Let $R = U \times_X U$, so the two projections $R \rightrightarrows U$ are finite étale covers. If X is affinoid then certainly U is affinoid, so now assume that U is affinoid. Hence, the U-finite R is affinoid and we have to prove that U/R is affinoid when it exists. More generally, we suppose that we are given a finite étale equivalence relation $R' \rightrightarrows U'$ with R' and U' affinoid rigid spaces over k, and we seek to prove that the étale quotient U'/R' exists as an affinoid rigid space with $U' \to U'/R'$ a finite étale covering.

We have $U' = \mathrm{Sp}(A')$ for some k-affinoid A', and likewise $R' = \mathrm{Sp}(A'')$ for some k-affinoid A''. Since the maps $p_1, p_2 : R' \rightrightarrows U'$ are finite, the groupoid conditions may be expressed in opposite terms using k-affinoid algebras with only ordinary tensor products intervening in the description. The resulting pair of maps of affine k-schemes $\mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$ is therefore a finite étale equivalence relation in the category of k-schemes provided that the natural map

$$\delta : \mathrm{Spec}(A'') \to \mathrm{Spec}(A') \times_{\mathrm{Spec}\, k} \mathrm{Spec}(A')$$

is a monomorphism.

Exercise 3.5. Prove that δ is a closed immersion. (Hint: first prove that $R' \to U' \times U'$ is a closed immersion, but then be careful about the distinction between algebraic tensor products and completed tensor products.)

By [SGA3, 4.1, Exp. V], the étale quotient of $\mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$ exists as an affine scheme $\mathrm{Spec}(A)$ over k, with $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ a finite étale covering (and $A' \otimes_A A' \to A''$ an isomorphism). If we can show that A is a k-affinoid algebra, then for $X' = \mathrm{Sp}(A)$ the finite étale covering $U' \to X'$ yields *equal* composites $R' \rightrightarrows X'$ and the induced map $R' \to U' \times_{X'} U'$ is an isomorphism since $A' \otimes_A A' = A' \widehat{\otimes}_A A'$. Thus, X' would serve as the étale quotient U'/R' in the category of rigid spaces.

Exercise 3.6. To show that such an A must be k-affinoid, consider more generally an affine k-scheme $\operatorname{Spec} A$ equipped with a finite étale covering $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ with A' a k-affinoid algebra. Prove that A is k-affinoid. (Hint: reduce to the case when $\operatorname{Spec} A$ and $\operatorname{Spec} A'$ are connected, and then reduce to the case when $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is a Galois finite étale covering with Galois group G. Use [BGR, 6.3.3/3] to deduce from this that A is k-affinoid.)

■

To generalize beyond the case of finite étale equivalence relations on affinoids as in Lemma 3.4, a fundamental issue is the possibility that the rigid-analytic morphism $R \to U \times U$ may not be quasi-compact. For example, if $\mathscr{X}$ is a locally separated algebraic space then its diagonal is a quasi-compact immersion that is not a closed immersion if $\mathscr{X}$ is not separated, and so when working over an étale chart of the algebraic space the pullback of this diagonal morphism has analytification that is not quasi-compact in the sense of rigid geometry when $\mathscr{X}$ is not separated. Lack of such quasi-compactness on the rigid side presents a difficulty because forming saturations under the equivalence relation thereby involves the image of a non-quasi-compact admissible open under a flat morphism of rigid spaces, and the admissibility of such images is difficult to control (even when the flat morphism is quasi-compact). This is what happens in Example 3.1 if we try to use gluing to build the non-existent analytification there. We are therefore led to restrict our attention to the analytic quotient problem when $R \to U \times U$ is quasi-compact (e.g., a closed immersion, as is the case when trying to construct analytification for separated algebraic spaces).

4. Analytification via Berkovich spaces

We are going to now consider analytification in the category of k-analytic Berkovich spaces (to be called *k-analytic spaces* from now on), and then use such spaces to overcome admissibility problems in the rigid case. Beware that the concept of an étale map for k-analytic spaces is much more restrictive than in the rigid-analytic case. For example, the inclusion of an affinoid subdomain into an affinoid space is étale in rigid geometry, but almost never étale in the sense of k-analytic spaces (since a map of affinoid k-analytic spaces has non-empty relative boundary unless it is a finite map).

In order to make sense of this, we briefly digress to discuss how the methods in §2 carry over to the category of k-analytic spaces, endowed with their natural étale topology. An *étale equivalence relation* in the category of k-analytic spaces is a pair of étale morphisms $R \rightrightarrows U$ such that the map $R \to U \times U$ (called the *diagonal*) is a functorial equivalence relation; in particular, it is a monomorphism. As one example, if $\mathscr{R} \rightrightarrows \mathscr{U}$ is an étale chart for an algebraic space $\mathscr{X}$ over k then the analytification functor [Ber2, 2.6.1] to the category of good strictly k-analytic spaces yields an étale equivalence relation $R \rightrightarrows U$ on k-analytic spaces. (By [T, 4.10], the category of strictly k-analytic spaces is a full subcategory of the category of k-analytic spaces, so there is no ambiguity about where the morphisms $R \rightrightarrows U$ take place when R and U are strictly k-analytic.)

Definition 4.1. Let $R \rightrightarrows U$ be an étale equivalence relation on k-analytic spaces. A *quotient* of $R \rightrightarrows U$ is a k-analytic space X equipped with an étale surjection $U \to X$ such that the composite maps $R \rightrightarrows U \to X$ coincide and the resulting map $R \to U \times_X U$ is an isomorphism.

In order to check that the quotient (when it exists) is unique up to unique isomorphism (and in fact represents a specific sheaf functor), we can use the usual descent theory argument as in the case of schemes *provided* that representable functors on the category of k-analytic Berkovich spaces are étale sheaves. This sheaf property is true within the full subcategories of good k-analytic spaces and strictly k-analytic spaces by [Ber2, 4.1.5], according to which the general case holds once we prove the next result.

Theorem 4.2. *Let $f : X' \to X$ be a finite étale map between k-analytic spaces. If $V' \subseteq X'$ is a quasi-compact k-analytic subdomain then $f(V') \subseteq X$ is a finite union of k-affinoid subdomains. In particular, $f(V')$ is a k-analytic subdomain in X. If X, X', and V' are strictly k-analytic then so is $f(V')$.*

Proof. The image of f is open and closed in X, so we may and do assume that f is surjective. Since X is locally Hausdorff and V' is compact, there is a finite collection of Hausdorff open subsets $U_1, \ldots, U_n$ in

X that cover $f(V')$. The open cover $\{f^{-1}(U_i)\}$ of the quasi-compact V' has a finite refinement consisting of k-affinoid subdomains $V'_j \subseteq V'$, so if we can settle the case of a Hausdorff target then applying this to $f^{-1}(U_i) \to U_i$ and each V'_j mapping into U_i gives the result for $f(V')$. Hence, we now may and do assume that X is Hausdorff, so X' is also Hausdorff.

Let $W_1, \dots, W_m \subseteq X$ be a finite collection of k-affinoid subdomains whose union covers $f(V')$ (with all W_j strictly k-analytic when X', X, and V are so). The pullback subdomains $W'_j = f^{-1}(W_j)$ are k-affinoid in X', and are strictly k-analytic when X', X, and V' are so. Moreover, $V' \cap f^{-1}(W_j)$ is quasi-compact since the graph morphism $\Gamma_f : X' \to X' \times X$ is quasi-compact (as it is a base change of the diagonal morphism $\Delta_X : X \to X \times X$ that is proper since $|X|$ is Hausdorff and $|X \times X| \to |X| \times |X|$ is proper). Hence, we may reduce to the case when $f(V') \subseteq W$ for some k-affinoid subdomain $W \subseteq X$. It is harmless to make the base change by $W \to X$, so we can assume that X and X' are k-affinoid and even connected. Say $X' = \mathscr{M}(A')$ and $X = \mathscr{M}(A)$.

By the theory of the étale fundamental group as in the proof of Lemma 3.4, now applied to $\operatorname{Spec} A' \to \operatorname{Spec} A$, we may find a connected finite étale cover $X'' \to X'$ that is Galois over X. In particular, if X' is strict then so is X''. The preimage of V' in X'' is quasi-compact (and strict when X' and V' are strict), so we may assume that X' is Galois over X, say with Galois group G. The union $W' = \cup_{g \in G} g(V')$ is a quasi-compact k-analytic subdomain whose image in X is the same as that of V', so we can rename it as V' to get to the case when V' is G-stable.

Exercise 4.3. By considering isotropy groups in G at points of V', use quasi-compactness of V' and the locally Hausdorff property of k-analytic spaces to reduce to the case when $V' = \mathscr{M}(B')$ is k-affinoid.

By [Ber1, 2.1.14(*ii*)], the closed k-subalgebra $B = {B'}^G$ is k-affinoid. It is moreover a strict k-affinoid algebra if V' is strict [BGR, 6.3.3]. The map $V' \subseteq X' \to X$ factors through the surjection $V' = \mathscr{M}(B') \to \mathscr{M}(B)$, so it suffices to check that the natural map $V = \mathscr{M}(B) \to \mathscr{M}(A) = X$ is a k-analytic subdomain.

Exercise 4.4. By using maps from k-affinoids $Z = \mathscr{M}(C)$, prove that $V \to X$ is indeed a k-analytic subdomain. (Hint: Rephrase the required mapping property in terms of an auxiliary map of k-affinoids being an isomorphism, and use extension of the base field to reduce to the strictly k-analytic case. In this case the image $f(V')$ is a k-analytic subdomain by Raynaud's theory, and $V' \to f(V')$ is a finite mapping because V' is the full preimage of $f(V')$ in X', due to the G-stability of V' in X'. Explain why $f(V')$ is k-affinoid.)

■

Example 4.5. In the setup of Theorem 4.2, if $V' \subseteq X'$ is a quasi-compact k-analytic subdomain whose two pullbacks to X'' coincide then it descends uniquely to a k-analytic subdomain $V \subseteq X$. Indeed, if we let V be the quasi-compact k-analytic subdomain $f(V') \subseteq X$ then to check that the preimage of V in X' is no larger than (and hence is equal to) V' it suffices to check this after base change on X by geometric points of V. This case is trivial.

By Theorem 4.2, if $R \rightrightarrows U$ is an étale equivalence relation on k-analytic spaces and X is a quotient in the sense that we have defined for k-analytic spaces, then X represents the quotient sheaf of sets U/R on the étale site for the category of k-analytic spaces. Thus, such an X is unique up to unique isomorphism. We can also use descent arguments as in the classical rigid case to run this in reverse: if the quotient sheaf U/R on the étale site for the category of k-analytic spaces is represented by a k-analytic space X then the natural map $U \to X$ is automatically an étale surjection that equalizes the maps $R \rightrightarrows U$ and yields an isomorphism $R \simeq U \times_X U$. In particular, the formation of the quotient is compatible with arbitrary analytic extension of the base field (when the quotient exists over the initial base field).

The arguments in §2 may now be carried over essentially *verbatim* to show that when $R \rightrightarrows U$ arises from an étale chart $\mathscr{R} \rightrightarrows \mathscr{U}$ for an algebraic space $\mathscr{X}$ then whether or not an analytic quotient $X = U/R$ exists is independent of the choice of étale chart for $\mathscr{X}$, and its formation (when it does exist) is Zariski-local on $\mathscr{X}$. In particular, when X exists it is canonically independent of the chart and is *functorial* in $\mathscr{X}$ in a manner that respects the formation of fiber products and Zariski-open and Zariski-closed immersions. We call such an X (when it exists) the *analytification* of $\mathscr{X}$ in the sense of Berkovich spaces, and we say that $\mathscr{X}$ is analytifiable (in the sense of Berkovich spaces).

Here is the main result in the classical setting.

Theorem 4.6. *If $\mathscr{X}$ is a separated algebraic space over k then $\mathscr{X}$ is analytifiable in the sense of rigid spaces. Moreover, $\mathscr{X}^{\rm an}$ is separated.*

Once $\mathscr{X}^{\rm an}$ is proved to exist, it must be separated since $\Delta_{\mathscr{X}^{\rm an}} = \Delta_{\mathscr{X}}^{\rm an}$ is a closed immersion (as $\mathscr{X}$ is separated). By Corollary 2.12 we can work locally on $\mathscr{X}$ to prove analytifiability, so we may and do assume that $\mathscr{X}$ is quasi-compact. Choose an étale surjection $\mathscr{U} \to \mathscr{X}$ from a scheme, and let $\mathscr{R} \rightrightarrows \mathscr{U}$ be the resulting étale chart for $\mathscr{X}$, so $\mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is a closed immersion. Note that we can choose $\mathscr{U}$ to be affine, so $\mathscr{R}$ is also affine. For a separated algebraic space, we will prove analytifiability in the sense of rigid spaces by deducing it from a stronger existence theorem for étale quotients in the setting of k-analytic spaces. Let U and R be the good strictly k-analytic spaces associated to $\mathscr{U}$ and $\mathscr{R}$. The dictionary relating Berkovich spaces and algebraic schemes [Ber2, 3.3.11] ensures that $R \rightrightarrows U$ is an étale equivalence relation on U and that $R \to U \times U$ is a closed immersion. Theorem 4.6 will be deduced from the following purely analytic result.

Theorem 4.7. *Let $R \rightrightarrows U$ be an étale equivalence relation on k-analytic spaces such that $R \to U \times U$ is quasi-compact. Also assume that U is separated. The quotient U/R exists and is a quasi-separated k-analytic space; it is separated if and only if $R \to U \times U$ is a closed immersion. If U is strictly k-analytic* (*resp. good*) *then so is U/R.*

It seems difficult to detect when U/R is Hausdorff (aside from cases when it is known to be separated or $R \rightrightarrows U$ is finite with U Hausdorff). The separatedness hypothesis on U in Theorem 4.7 is harmless for the purposes of the intended application to equivalence relations coming from algebraic spaces, though in general it is an unpleasant hypothesis. However, some kind of separatedness assumption on U is necessary, because there are examples when U is not separated and U/R does not exist; such examples exist in dimension 2 over any algebraically closed base field. (These examples will be discussed in Arizona if time permits.) But in such examples that are known, it turns out that U fails to be locally separated (in the sense that not every point of U has a separated neighborhood). In view of this, it is natural to pose the following question (to which we do not know the answer).

Exercise 4.8. (Open question). Is local separatedness of U a necessary condition for the existence of U/R? (We expect it to be sufficient, which is to say that in Theorem 4.7 it should suffice to require U to be locally separated.)

The separatedness hypothesis on U (as opposed to a weaker Hausdorff hypothesis) will not be used until near the end of the proof of Theorem 4.7, after we have reduced the problem to the case of a free action by a finite group. Before we proceed to global considerations, let us first show that the existence problem for U/R is local on U (setting aside for now the matter of relating separatedness of U/R and the map $R \to U \times U$ being a closed immersion). To this end, suppose U is covered by open subsets $\{U_i\}$ such that for $R_i = R \times_{U \times U} (U_i \times U_i) = R \cap (U_i \times U_i)$ the quotient $X_i = U_i/R_i$ exists (with X_i strictly k-analytic when U_i is, and likewise for the property of being good); note that $R_i \to U_i \times U_i$ is quasi-compact. We need to define "overlaps" along which we shall glue the X_i's to build a k-analytic quotient U/R. The open overlap $R_{ij} = p_1^{-1}(U_i) \cap p_2^{-1}(U_j)$ in R classifies equivalence among points of U_i and U_j, so its open image U_{ij} in U_i under the étale morphism $p_1 : R \to U$ classifies points of U_i that are equivalent to points of U_j. Let $X_{ij} \subseteq X_i$ be the open image of U_{ij}, so $p_1 : R_{ij} \to X_{ij}$ is an étale surjection. Geometrically, the points of X_{ij} are the R-equivalence classes that meet U_i and U_j (viewed within $X_i = U_i/R_i$).

The canonical involution $R \simeq R$ restricts to an isomorphism $\phi_{ij} : R_{ij} \simeq R_{ji}$ such that $\phi_{ji} = \phi_{ij}^{-1}$, and it is easy to check that the resulting isomorphism $R_{ij} \times R_{ij} \simeq R_{ji} \times R_{ji}$ restricts to an isomorphism of subfunctors $R_{ij} \times_{X_{ij}} R_{ij} \simeq R_{ji} \times_{X_{ji}} R_{ji}$. Hence, since representable functors on the category of k-analytic spaces are étale sheaves (due to Theorem 4.2 and [Ber2, 4.1.5]), the isomorphisms ϕ_{ij} uniquely descend to isomorphisms $X_{ij} \simeq X_{ji}$ between open subsets $X_{ij} \subseteq X_i$ and $X_{ji} \subseteq X_j$. These descended isomorphisms among opens in X satisfy the triple overlap condition, and so we can glue the X_i's along these isomorphisms to build a k-analytic space X. Moreover, if U is strictly k-analytic (resp. good) and the U_i's can be chosen to be so then so are all X_i and hence so is the space X that has an open covering by the X_i's. The étale composites $U_i \to X_i \subseteq X$ glue to define an étale morphism $U \to X$ such that the two composite maps $R \rightrightarrows U \to X$ coincide and $R \to U \times_X U$ is an isomorphism (as it is an étale monomorphism that is surjective on geometric points). It follows that as an étale sheaf of sets on the category of k-analytic spaces, X represents the sheafified quotient U/R.

The diagonal map $X \to X \times X$ is quasi-compact (i.e., X is quasi-separated) since étale surjective base change by $U \times U \to X \times X$ yields the map $R \to U \times U$ that is quasi-compact by hypothesis. As long as finiteness descends through extension of the base field, the equivalence of separatedness of X and $R \to U \times U$ being a closed immersion (prior to the passage to the Hausdorff case) is easily reduced to the strict case, where it follows from descent theory for rigid spaces [C2, Thm. 4.2.7]. Thus, to complete the reduction of Theorem 4.7 to a local problem on U it remains to prove the following general lemma.

Lemma 4.9. *A map $h : Y' \to Y$ between k-analytic spaces is finite if $h_K : Y'_K \to Y_K$ is finite for some analytic extension field K/k.*

Proof. We easily reduce to the case when Y is quasi-compact and Hausdorff (so Y_K is too). Since h_K is finite, the relative interior $\mathrm{Int}(Y'_K/Y_K)$ is empty. Thus, by [Ber2, 1.5.5(*iv*)] and surjectivity of $Y'_K \to Y'$ we see that the relative interior $\mathrm{Int}(Y'/Y)$ is empty. Likewise, each point in Y' is isolated in h-fiber (since this holds over K). Thus, by [Ber2, 3.1.10] the map h is finite at each point $y' \in Y'$ in the sense that there are open neighborhoods $U' \subseteq Y'$ around y' and $U \subseteq Y$ around $h(y')$ such that $U' \subseteq h^{-1}(U)$ and $h : U' \to U$ is finite.

Since finiteness is local on the base, we can restrict our attention near a choice of point $y \in Y$. The fiber $h^{-1}(y)$ is certainly finite (since h_K is finite), say consisting of points $\{y'_1, \dots, y'_n\}$ (it may be empty), so by finiteness of h near each y'_j we may find an open subset $U \subseteq Y$ around y and an open subset $U' \subseteq Y'$ around $h^{-1}(y)$ so that $U' \subseteq h^{-1}(U)$ with $h : U' \to U$ a finite map. (If $h^{-1}(y)$ is empty then we may take U' to be empty.) The map h_K is finite, so the open immersion $U' \to h^{-1}(U)$ over U becomes finite upon extending scalars to K. Thus, $U'_K \subseteq h_K^{-1}(U_K)$ is both open and closed. Since $Y'_K \to Y'$ is a quotient map on topological spaces, it follows that $U' \subseteq h^{-1}(U)$ is open and closed. Hence, if $Z' = h^{-1}(U) - U'$ then $h_K(Z'_K)$ is a closed subset of $h^{-1}(U)_K$, so $h(Z')$ is a closed subset of U not containing y. Replacing U with $U - h(Z')$ therefore brings us to the case where $U' = h^{-1}(U)$. That is, $h : Y' \to Y$ is finite over an open neighborhood U of y. ■

Now we return to the global construction problem for $R \rightrightarrows U$. We have already seen via Lemma 4.9 that the separatedness of U/R (if it exists) is indeed equivalent to $R \to U \times U$ being a closed immersion. Thus, we shall no longer pay attention to this condition, so we can work locally on U to solve the existence problem. Localizing in this way allows us to assume that U is Hausdorff (and we do not lose the property of being strictly k-analytic or good when the original U is so). The map $|U \times U| \to |U| \times |U|$ is separated, so $U \times U$ is also Hausdorff. The map $R \to U \times U$ is a monomorphism, hence separated, so $|R|$ is Hausdorff as well. Having U and R now be Hausdorff will permit us to use arguments with limits of nets to probe the topology. (Note that it does not seem possible to reduce to the case when U and R are separated.) The key to the construction of U/R is general is to reduce the problem to the case when the maps $R \rightrightarrows U$ are finite étale, and to build the quotient in this latter case by reduction to the case of equivalence relations defined by the free action of a finite group. Our first result is to pass to the case of finite étale equivalence relations, via the following lemma (and the local nature of the existence problem for U/R).

Lemma 4.10. *With notation and hypotheses as in Theorem* 4.7, *and U Hausdorff, for every $u \in U$ there exists a base of* (*connected*) *open neighborhoods N around u in U such that $R_N = p_1^{-1}(N) \cap p_2^{-1}(N)$ is a* finite *étale cover of N under both projections, with both covering maps having constant degree equal to the finite degree n of $p_1^{-1}(u) \cap p_2^{-1}(u)$ over u.*

For each $u \in U$, the overlap $p^{-1}(u) \cap p_2^{-1}(u)$ is necessarily finite. To see this finiteness, since the spaces $p_j^{-1}(u)$ are étale over u, it is enough to prove that their overlap is compact. It is the preimage of (u, u) under the composite map of topological spaces $|R| \to |U \times U| \to |U| \times |U|$ in which the preimage of a quasi-compact subset under each map is quasi-compact (see [Ber2, 1.4] for the second step). We view this overlap as an $\mathscr{H}(u)$-analytic space via its open subspace structure in either $p_j^{-1}(u)$; in both cases this is the finite étale space given by the completed residue fields on R at all points of the overlap.

Proof. Fix a compact k-analytic subdomain $K \subseteq U$ that is a neighborhood of u. For any compact k-analytic subdomain $K' \subseteq K$ that is a neighborhood of u we have that $p_1^{-1}(u) \cap (K' \times K')$ is a compact subset of the space $p_1^{-1}(u)$ that is discrete, so this overlap is finite. For K' small enough, this overlap must be the finite set $p_1^{-1}(u) \cap p_2^{-1}(u)$. Indeed, if not then one of the finitely many points $r \in p_1^{-1}(u) \cap (K \times K)$ not in $p_2^{-1}(u)$ must map into $K' \times K'$ for all K', so $p_2(r) \in K'$ for all K' and hence $p_2(r) = u$, a contradiction. We replace K with such a K' that is small enough for both p_1 and p_2. For any open $U' \subseteq K$ containing u we define $R' = p_1^{-1}(U') \cap p_2^{-1}(U')$, so $p_2'({p_1'}^{-1}(u)) = \{u\} = p_1'({p_2'}^{-1}(u))$ with $p_1', p_2' : R' \rightrightarrows U'$ the two projections, since ${p'}_1^{-1}(u) = {p'}_2^{-1}(u) = p_1^{-1}(u) \cap p_2^{-1}(u)$.

Fix such a U'. Since p_1' is étale, by [Ber2, 3.4.1] (an analogue of the structure theorem [C2, Thm. A.1.3] for locally quasi-finite rigid maps) if $U_1' \subseteq U'$ is any sufficiently small open set around u then there exists an open subset $W \subseteq R' \cap {p_1'}^{-1}(U_1')$ such that W is a p_1'-finite étale cover of the open U_1' with constant degree and W contains ${p_1'}^{-1}(u) = {p_2'}^{-1}(u)$, so the degree of W over U_1' (via p_1') is equal to the degree of $p_1^{-1}(u) \cap p_2^{-1}(u)$ over u. Fix such a U_1', so $V' = {p_2'}^{-1}(U_1')$ is an open set in R' containing ${p_1'}^{-1}(u)$. Our first main task is to get to the case $U' = U_1'$.

The *quasi-compact* map $R \to U \times U$ between locally compact Hausdorff spaces is injective and hence is a closed embedding. Observe that (i) R' is open in the overlap $R_K = R \cap (K \times K)$ that is closed in $K \times K$, and (ii) R' is closed in $U' \times U'$.

Exercise 4.11. Prove that for any sufficiently small open set $U'' \subseteq U_1'$ around u, the open set ${p_1'}^{-1}(U'') \subseteq R'$ is contained inside of $V' = {p_2'}^{-1}(U_1')$.

We conclude that for sufficiently small open $U'' \subseteq U_1'$ around u we have ${p_1'}^{-1}(U'') \subseteq V' = {p_2'}^{-1}(U_1')$, so the R'-saturated open set $U_2' = p_2'({p_1'}^{-1}(U''))$ is contained in U_1'. Since

$${p_1'}^{-1}(U_2') = R' \cap {p_1'}^{-1}(U_2') \cap {p_2'}^{-1}(U_2') = {p_2'}^{-1}(U_2'),$$

by renaming the open subset $U_2' \subseteq U_1'$ as U' and $W \cap {p_1'}^{-1}(U_2')$ as W we arrange that R' contains an open subset W that is a p_1'-finite étale cover of U' with constant degree and that contains ${p_1'}^{-1}(u) = {p_2'}^{-1}(u) = p_1^{-1}(u) \cap p_2^{-1}(u)$.

Exercise 4.12. Using that W is p_1'-finite over U', prove that the open inclusion $\iota : W \to R'$ over U' (via p_1') must be a closed mapping of topological spaces, so there is a disjoint-union decomposition $R' = W \coprod W_\eta$ with W_η having empty p_1'-fiber over u. (The easier case is when p_1' is separated, but you have to bypass this restriction; focus on the topology rather than the analytic structure, especially the Hausdorff property.)

Having built the k-analytic decomposition $R' = W \coprod W_\eta$ over U', a subtle issue is that the open set W may not be uniquely determined by this condition because the R'-saturated U' may not be connected. In particular, W may not be invariant under the canonical involution on R'.

Exercise 4.13. Construct a cofinal family of connected open subsets U_i in U' around u such that $R_i = p_1^{-1}(U_i) \cap p_2^{-1}(U_i)$ contains a unique open and closed subset W_i that is stable under the involution on R_i and is finite étale of degree n over U_i with respect to p_1 and p_2, where n is the degree of $p_1^{-1}(u) = p_2^{-1}(u)$.

Exercise 4.14. With notation as in the previous exercise, prove that for sufficiently small U_i, the complement $W_{i,\eta}$ of W_i in R_i is empty. (Hint: First prove $W_{i,\eta} \cap W$ is empty for all i. Then note that if $W_{i,\eta}$ is non-empty for all i, for any net of points $w_i' \in W_{i,\eta}$, the nets $\{p_1(w_i')\}$ and $\{p_2(w_i')\}$ in U each converge to u. But the map of topological spaces $|U \times U| \to |U| \times |U|$ is proper [Ber2, 1.4], so if we choose a compact k-analytic domain $K \subseteq U$ that is a neighborhood of u then $|K| \times |K|$ has compact preimage in $|U \times U|$. By passing to a suitable subnet it may therefore be arranged that the net $\{w_i'\}$ in $U \times U$ has a limit w'. Show $w_i' \in W$ for large i, a contradiction since $w_i' \in W_{i,\eta}$ and $W_{i,\eta}$ is disjoint from W for all i.)

Hence, by taking $N = U_i$ for sufficiently large i we may arrange that both maps $R_N \rightrightarrows N$ are finite étale with degree equal to the degree of $p_1^{-1}(u) \cap p_2^{-1}(u)$. ■

By Lemma 4.10, we can cover U by open (even connected) subsets U' such that for $R' = p_1^{-1}(U') \cap p_2^{-1}(U')$ the étale equivalence relation $p_1', p_2' : R' \rightrightarrows U'$ is finite étale with ${p_1'}^{-1}(u) = {p_2'}^{-1}(u) = p_1^{-1}(u) \cap p_2^{-1}(u)$ (equality as $\mathscr{H}(u)$-analytic spaces). Thus, to prove Theorem 4.7 we may and do now assume that the maps $R \rightrightarrows U$ are finite. The next lemma, which is an analogue of Lemma 3.4, will be useful for analyzing properties of the map $U \to U/R$ when the quotient has been constructed.

Lemma 4.15. *Let $f : X' \to X$ be a finite étale surjection between k-analytic spaces. If X' is k-affinoid then so is X, and if in addition X' is strictly k-analytic then so is X.*

Proof. Since $X'' = X' \times_X X'$ is finite over X' under either projection, it is k-affinoid (and strict when X' is so). Also, the map $X'' \to X' \times X'$ between k-affinoid spaces is a closed immersion because a finite monomorphism between k-analytic spaces is a closed immersion (as we may check after first using extension of the base field to reduce to the strict case; the monomorphism property is preserved by such extension since it is equivalent to the relative diagonal map being an isomorphism). Carry over method of proof of Lemma 3.4 (using [Ber1, 2.1.14(i)] to replace [BGR, 6.3.3]) to construct a k-affinoid quotient for the finite étale equivalence relation $X'' \rightrightarrows X'$, and note that this quotient is (by construction) even strict when X' is strict. But X is also such a quotient, so it must be k-affinoid. ■

5. Descent on fibers

To complete the proof of Theorem 4.7, and also of the general existence results in the case of Berkovich spaces (for étale equivalence relations with quasi-compact diagonal) and for rigid analytification of separated algebraic spaces, we have to bring in a closer study of descent on fibers. Here we shall use the well-understood theory of étale descent in the context of fields (such as completed residue fields at points on k-analytic spaces), where there are no subtle existence problems.

Theorem 4.7 is a consequence of the next result, whose proof reduces to a concrete existence problem involving the free action of a finite group (and this latter problem is not proved in these notes, since its proof requires entirely different and more difficult techniques).

Theorem 5.1. *If $R \rightrightarrows U$ is a finite étale equivalence relation on k-analytic spaces with U separated then the quotient U/R exists and $U \to U/R$ is a finite étale covering. If U is strictly k-analytic* (*resp. good, resp. k-affinoid*) *then so is U/R. Moreover, U/R is Hausdorff.*

The separatedness hypothesis on U will be used near the end in a special case; for most of the proof it will suffice to assume that U is Hausdorff. In general the map $R \to U \times U$ is automatically quasi-compact because composing the map on topological spaces with the proper map $|U \times U| \to |U| \times |U|$ and the first projection $|U| \times |U| \to |U|$ between Hausdorff spaces yields the map $p_1 : |R| \to |U|$ that is proper (due to finiteness of the equivalence relation). Hence, the existence of U/R in Theorem 5.1 is a special case of Theorem 4.7.

Proof. First we assume that $\pi : U \to U/R$ exists (for Hausdorff U) and we deduce finer structural claims. The base change of π by the étale covering $\pi : U \to U/R$ is a finite map (it is a projection $R \to U$), so finiteness of π follows from the next exercise.

Exercise 5.2. Prove that if a map $h : Y' \to Y$ becomes finite after an étale surjective base change on Y then it is finite. (Hint: Use Lemma 4.15 and that étale maps are open and are finite locally on the source.)

With π now known to be finite étale (and surjective) when it exists, if U is k-affinoid then Lemma 4.15 ensures that U/R must be k-affinoid. To see that U/R must be Hausdorff, the finite surjection $\pi \times \pi : U \times U \to (U/R) \times (U/R)$ induces a closed map on topological spaces, so properness and surjectivity of $|T_1 \times T_2| \to |T_1| \times |T_2|$ for k-analytic spaces T_1 and T_2 implies that $|U| \times |U| \to |U/R| \times |U/R|$ is closed. The diagonal $|U| \subseteq |U| \times |U|$ is closed since U is Hausdorff, so we conclude that $|U/R|$ has closed diagonal image in $|U/R| \times |U/R|$ as desired. That is, U/R must be Hausdorff when it exists. As for strict k-analyticity (resp. goodness) of U/R when U is strictly k-analytic (resp. good), this will follow from how we construct U/R below.

Now we turn to the task of constructing U/R. As we have already noted below the statement of Lemma 3.4 (and in the proof of Lemma 4.15) if U is k-affinoid (so R is as well) then we know that U/R exists and is k-affinoid. To infer the general case from the k-affinoid case, we will work locally on U. More precisely, we pick a point $u \in U$ and we will find an open neighborhood $N \subseteq U$ around u such that for $R_N = p_1^{-1}(N) \cap p_2^{-1}(N)$ the quotient N/R_N exists, and it is strictly k-analytic (resp. good) when N is so. This will suffice to settle the proof of Theorem 5.1. By Lemma 4.10, we may choose such an N (promptly renamed as U) so that $p_1^{-1}(u) = p_2^{-1}(u)$ as subsets of R (and so also as finite étale $\mathscr{H}(u)$-analytic spaces).

Let $\{r_1, \ldots, r_n\}$ denote the common set $p_1^{-1}(u) = p_2^{-1}(u)$. Viewing $p_j^{-1}(u)$ as a finite étale $\mathscr{H}(u)$-analytic space, it is naturally identified with $\coprod_i \mathscr{M}(\mathscr{H}(r_i))$. The involution on R restricts to an isomorphism $p_1^{-1}(u) \simeq p_2^{-1}(u)$ as $\mathscr{H}(u)$-analytic spaces, and hence an involution of $\prod_i \mathscr{H}(r_i)$ that exchanges the two $\mathscr{H}(u)$-structures. Moreover, there is an evident common section to the structure maps $\mathscr{H}(u) \rightrightarrows \prod_i \mathscr{H}(r_i)$ induced by the identity section $U \to R$. The finite étale groupoid axioms are satisfied by these maps since $\prod_{i,j} \mathscr{H}(r_i) \otimes_{\mathscr{H}(u)} \mathscr{H}(r_j)$ is identified with the product of the residue fields on $R \times_U R$ at the points over $u \in U$ (as we may check by passing to an affinoid subdomain of U containing u and then using the description of analytic local rings on fiber products of finite morphisms in the affinoid case [Ber2, 2.1.6]). To conclude that the map of affine k-schemes associated to the finite étale groupoid $\mathscr{H}(u) \rightrightarrows \prod_i \mathscr{H}(r_i)$ is a finite étale equivalence relation it remains to prove that the map of k-algebras

$$\mathscr{H}(u) \otimes_k \mathscr{H}(u) \to \prod_i \mathscr{H}(r_i)$$

induces a monomorphism of k-schemes. This monomorphism property follows from the next lemma.

Lemma 5.3. *The natural map $\mathscr{H}(u) \otimes_k \mathscr{H}(u) \to \prod_i \mathscr{H}(r_i)$ is surjective.*

Proof. Let $F_i' = \mathscr{H}(r_i)$ and $F = \mathscr{H}(u)$, so if we view $\prod_i F_i'$ as a finite étale F-algebra via the p_1-structure then we have to prove that it is generated by the image of F with respect to the p_2-structure. This is a problem concerning intermediate finite étale F-algebras between F and $\prod_i F_i'$.

The fields F and F_i' are equipped with natural absolute values with respect to which they are complete. Since any linear subspace of a finite-dimensional F-vector space is closed, it is equivalent to show that the natural map of rings $F \widehat{\otimes}_k F \to \prod_i F_i'$ is surjective.

Exercise 5.4. Prove this surjectivity claim. (The source ring is generally not an F-affinoid algebra, so you will need to exercise some care.)

■

We have now proved that $\mathscr{H}(u) \rightrightarrows \prod_i \mathscr{H}(r_i)$ corresponds to a finite étale equivalence relation in the category of k-schemes. By the theory of quotients in such a scheme-theoretic situation [SGA3, Exp. V, 4.1], the equalizer $\mathscr{H}_0 \subseteq \mathscr{H}(u)$ of the maps $\mathscr{H}(u) \rightrightarrows \prod_i \mathscr{H}(r_i)$ is a subfield over which $\mathscr{H}(u)$ is finite separable, and the natural map

$$\mathscr{H}(u) \otimes_{\mathscr{H}_0} \mathscr{H}(u) \to \prod_i \mathscr{H}(r_i)$$

is an isomorphism. We endow $\mathscr{H}_0$ with the restriction of the absolute value on $\mathscr{H}(u)$, so it is complete (due to [Ber2, 2.3.1, 2.4.1]). Finally, we let $\mathscr{H}'/\mathscr{H}(u)$ be a finite Galois extension that is Galois over $\mathscr{H}_0$, say with $G = \mathrm{Gal}(\mathscr{H}'/\mathscr{H}_0)$ and $H = \mathrm{Gal}(\mathscr{H}'/\mathscr{H}(u))$. We assume that the extension $\mathscr{H}'/\mathscr{H}(u)$ is sufficiently large so that it splits all of the fields $\mathscr{H}(r_i)$ when each is viewed as a finite separable extension of $\mathscr{H}(u)$ with respect to either of p_1 or p_2.

The construction of $\mathscr{H}_0$ and $\mathscr{H}'$ is unaffected by further shrinking of U around u (such as by using Lemma 4.10 again around u). By the equivalence of categories between finite étale $\mathscr{H}(u)$-algebras and germs of finite étale covers of open neighborhoods of u in U [Ber2, 3.4.1], another application of Lemma 4.10 permits us to arrange that there is a connected finite étale cover $U' \to U$ having one physical point u' over u and completed residue field $\mathscr{H}(u')$ at this point that is identified with $\mathscr{H}'$ as an extension of $\mathscr{H}(u)$. By further shrinking we may also arrange that there is a (necessarily unique) right H-action on U' over U inducing the canonical action of H on $\mathscr{H}(u') = \mathscr{H}'$ over $\mathscr{H}(u)$. This is a Galois covering, with covering group H. Since G acting on $\mathscr{H}(u')$ does not generally even preserve the subfield $\mathscr{H}(u)$, it may not be immediately evident if the H-action on U' can be extended to a G-action (that fixes u' physically and induces the given G-action on $\mathscr{H}(u')$) by further shrinking around u.

Exercise 5.5. Show that such an action does exist after replacing U with a suitable connected open subset N around u and U' with the preimage N' of N in U', and that such an N' is necessary connected. You may find it useful to consider the geometry of the covering $U' \to U$, especially that u' is the only point over u. (Hint: Let $R' = R \times_{U \times U} (U' \times U') = U' \times_{U,p_1} R \times_{p_2,U} U'$, so we get a finite étale equivalence relation $p'_1, p'_2 : R' \rightrightarrows U'$ since $U' \to U$ is finite étale. The fibers of R' over u' under either projection are totally split as finite étale $\mathscr{H}(u')$-analytic spaces since the u-fiber of R' under either projection is the same analytic space (recall $p_1^{-1}(u) = p_2^{-1}(u)$), namely the one with coordinate ring

$$\prod_i \mathscr{H}(u') \otimes_{\mathscr{H}(u)} \mathscr{H}(r_i) \otimes_{\mathscr{H}(u)} \mathscr{H}(u')$$

that is totally split for both $\mathscr{H}(u')$-algebra structures. You may find it useful to apply the equivalence of categories between finite étale $\mathscr{H}(u')$-schemes and germs of finite étale covers of open neighborhoods of u' in U' [Ber2, 3.4.1], as well as Lemma 4.10.)

By replacing U with a connected open subset around u that is contained in such an N (in accordance with Lemma 4.10), we may thereby to get the situation in which $R' \rightrightarrows U'$ has R' totally split over U' with respect to both projections p'_j. Hence, every connected component of R' maps isomorphically to U' under both projections. We use p'_1 to identify every such component with U', so R' is identified with $\Sigma \times U'$ for a finite set $\Sigma = \pi_0(R')$ and p'_1 is the canonical projection to U'.

Exercise 5.6. By chasing connected components, explain why the data of the equivalence relation $R' \rightrightarrows U'$ arises from a unique structure of finite group on Σ such that p'_2 expresses a free action of this group on U' (as a k-analytic space). Also explain why running through the entire preceding analysis with $U' \to U$ and $R \rightrightarrows U$ replaced with $\mathscr{M}(\mathscr{H}(u')) \to \mathscr{M}(\mathscr{H}(u))$ and $\coprod \mathscr{M}(\mathscr{H})(r_i)) \rightrightarrows \mathscr{M}(\mathscr{H}(u))$ yields the same group structure on Σ. Finally, show that this latter equivalence relation has quotient $\mathscr{M}(\mathscr{H}_0)$ when considered from an algebraic point of view, so the construction in this fibral situation is simply realizing $\mathscr{H}(u')$ as a Galois splitting field for all $\mathscr{H}(r_i)$ and $\mathscr{H}(u)$ over $\mathscr{H}_0$. That is, it is precisely the equivalence relation defined by the G-action on $\mathscr{H}(u')$.

To summarize, we have proved that Σ may be identified with G in a way that extends the G-action at $u' \in U'$. That is, after suitably shrinking U around u we have in fact built a free right G-action on U' that fixes the physical point u' and induces the canonical action of G on $\mathscr{H}(u')$. In particular, the H-action is the given one for U' as a cover of U (since this may be checked on the u-fiber $\mathscr{M}(\mathscr{H}(u'))$). This argument may be redone more locally around u, and so it proves the next lemma.

Lemma 5.7. *There is a base of connected open neighborhoods $N' \subseteq U'$ around u' that are H-stable and on which there is a free right G-action fixing u' and inducing the canonical G-action on $\mathscr{H}(u')$ such that there is a map of k-analytic spaces $q : N' \times G \to R$ inducing a map of finite étale equivalence relations*

$$(N' \times G \rightrightarrows N') \to (R \rightrightarrows U).$$

Choose an H-stable open subset $N' \subseteq U'$ as in Lemma 5.7, and let $N \subseteq U$ be its open image around u. If U is strictly k-analytic (resp. good) then it is clear from the construction that we may take N' to also be strictly k-analytic (resp. good), and the open subspace $N \subseteq U$ certainly is too. Clearly N' is the full preimage of N under the connected finite Galois covering $U' \to U$. The étale equivalence relation $R_N = p_1^{-1}(N) \cap p_2^{-1}(N) \rightrightarrows N$ may not be finite (though it is separated), and if we can prove that N/R_N exists (and is strictly k-analytic, resp. good, when N' is), then since the initial choice of $u \in U$ was arbitrary we will have solved the existence problem for U/R locally on U, and so we will be done. The construction problem for N/R_N (at least for N small enough around u) is reduced to a very classical kind of quotient problem, as we now record.

Lemma 5.8. *If a quotient Q exists for the free G-action on N' then for some open $N_1 \subseteq N$ around u there is an open subspace of Q that serves as a quotient N_1/R_{N_1}.*

Note in particular that if Q is strictly k-analytic (resp. good) when N' is then N_1/R_{N_1} has this property as well in such cases.

Proof. Since N serves as a quotient for the finite étale equivalence relation $N' \times H \rightrightarrows N'$, the G-invariant quotient map $N' \to Q$ (which is necessarily finite étale) factors uniquely through a map $\pi : N \to Q$ that is necessarily an étale surjection. Moreover, π is finite since N' is Q-finite and $N' \to N$ is a finite étale surjection (so we can apply Lemma 4.15 to $N' \to N$ over any k-affinoid subdomain of Q). The map $q : N' \times G \to R_N$ is étale because it covers the étale quotient map $N' \to N$ and respects the étale first projections to N' and N. The two composite maps $R_N \rightrightarrows N \to Q$ become the same when pulled back to $N' \times G$, so they coincide on the open image W of $N' \times G$ in R_N under q. We seek an open $N_0 \subseteq N$ around u such that R_{N_0} is contained in W, so the two maps $R_{N_0} \rightrightarrows Q$ coincide and hence for the open image $Q_0 \subseteq Q$ of N_0 there is a natural map $R_{N_0} \to N_0 \times_{Q_0} N_0$ (which must be shown to be an isomorphism if N_0 is small enough around u, thereby yielding that Q_0 serves as the quotient N_0/R_{N_0}). The key to finding N_0 is that the map q is not only étale but is also a closed map on topological spaces.

Exercise 5.9. Prove that q is a closed map on topological spaces by using the commutative diagram

$$\begin{array}{ccc} N' \times G & \xrightarrow{q} & R_N \\ \downarrow & & \downarrow \\ N' \times N' & \longrightarrow & N \times N \end{array}$$

in which the bottom side is a finite morphism (hence is proper on topological spaces) and all four of these spaces are Hausdorff. Using this, find N_0. (Hint: If $K \subseteq N$ is a compact neighborhood of u then the Hausdorff space $R_K = R \cap (K \times K)$ is compact because $R \to U \times U$ is quasi-compact!)

By shrinking N_0 we may arrange (again by Lemma 4.10) that $R_{N_0} \rightrightarrows N_0$ is a finite étale equivalence relation. Recall that $\pi : N \to Q$ is a finite étale surjection. If we let Q_0 be the open image of N_0 in Q then $N_0 \to Q_0$ is an étale separated map. Consider the commutative diagram

$$\begin{array}{ccc} R_{N_0} & \xrightarrow{\alpha} & N_0 \times_{Q_0} N_0 \\ & {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle 1\times\pi} \\ & & N_0 \end{array}$$

The diagonal map is finite étale due to how we chose N_0, so since the vertical map $1 \times \pi$ is étale and separated it follows that the horizontal map α is finite étale. Lemma 4.15 then implies the preimage of a k-affinoid subdomain under $1 \times \pi$ is k-affinoid, from which it is immediate that $1 \times \pi$ is even finite étale. Thus, α is an N_0-map between k-analytic spaces that are finite étale over N_0. The induced map on fibers over $u \in N_0$ is the canonical morphism

$$\coprod \mathscr{M}(\mathscr{H}(r_i)) \simeq \mathscr{M}(\mathscr{H}(u) \otimes_{\mathscr{H}(\overline{u})} \mathscr{H}(u))$$

where $\overline{u} \in Q$ is the image of $u \in N_0$. The point $\overline{u}$ is also the image of $u' \in N'$ with respect to the quotient map $N' \to Q$ for the finite étale equivalence relation $N' \times G \rightrightarrows N'$.

Exercise 5.10. Deduce that $\mathscr{H}(\overline{u})$ is equal to the subfield of G-invariants in $\mathscr{H}(u')$: this is the subfield $\mathscr{H}_0$. Since the natural map

$$\mathscr{H}(u) \otimes_{\mathscr{H}_0} \mathscr{H}(u) \to \prod_i \mathscr{H}(r_i)$$

is an isomorphism due to how $\mathscr{H}_0$ was originally defined, conclude that α induces an isomorphism on u-fibers. Find an open subspace $Q_1 \subseteq Q$ around $\overline{u}$ such that α restricts to an isomorphism over the open preimage $N_1 = \pi^{-1}(Q_1) \subseteq N$, and prove that Q_1 serves as the quotient N_1/R_{N_1}.

■

By Lemma 5.8, the verification of Theorem 5.1 (and hence Theorem 4.7) is reduced to the special case of finite étale equivalence relations of the form $U \times G \rightrightarrows U$ induced by the free action of a finite group G on a separated k-analytic space U. (It is easy to check that for any such situation, the diagonal map $U \times G \to U \times U$ is necessarily quasi-compact.) The quotient in such cases (if it exists) is denoted U/G. The treatment of this crucial special case requires an entirely different ingredient, Temkin's theory of reduction for germs of k-analytic spaces [T], but it seems far too much of a digression to go into this topic. It is at this step that we use that U is separated rather than just Hausdorff. Accept this special case as proved, though if you want to appreciate some of the difficulties in forming the quotient then try the following exercise:

Exercise 5.11. Assume that $u \in U$ is fixed by the G-action. There exists a G-stable open $U' \subseteq U$ around u such that U'/G exists if and only if there are finitely many G-stable good analytic subdomains $U_1, \dots, U_n$ in U containing u such that $\cup U_j$ is a neighborhood of u.

Now we return to the situation $\mathscr{R}^{\rm an} \rightrightarrows \mathscr{U}^{\rm an}$ considered for proving Theorem 4.6. In particular, we may assume that $\mathscr{X}$ is quasi-compact, so we can and do take $\mathscr{U}$ to be affine. This forces $\mathscr{R}$ to also be affine since $\mathscr{X}$ is separated. Hence, U and R are paracompact and separated. By using analytification with values in the category of good strictly k-analytic spaces, we conclude that $\mathscr{X}$ admits an analytification X in the sense of k-analytic spaces, and X is both good and strictly k-analytic. Since $\mathscr{X}$ is separated, it follows that X is separated (and in particular Hausdorff).

Exercise 5.12. Prove that X is paracompact.

Since the paracompact k-analytic space X is also strictly k-analytic (and good), under the equivalence of categories in [Ber2, 1.6.1] there is a quasi-separated rigid space X_0 uniquely associated to X, and the étale surjective map $U \to X$ (which is in the full subcategory of strictly k-analytic spaces) arises from a unique morphism $\mathscr{U}^{\rm an} \to X_0$.

Exercise 5.13. Prove that $\mathscr{U}^{\rm an} \to X_0$ is étale and surjective, and that the two maps $\mathscr{R}^{\rm an} \rightrightarrows \mathscr{U}^{\rm an}$ are equalized by the map $\mathscr{U}^{\rm an} \to X_0$ (Hint: use that $X = U/R$). Also prove that the resulting map $\mathscr{R}^{\rm an} \to \mathscr{U}^{\rm an} \times_{X_0} \mathscr{U}^{\rm an}$ is an isomorphism.

By Example 2.3, it follows that X_0 represents $\mathscr{U}^{\rm an}/\mathscr{R}^{\rm an}$ as long as the map $\mathscr{U}^{\rm an} \to X_0$ admits local étale quasi-sections. Since X is paracompact and Hausdorff, by [Ber2, 1.6.1] the rigid space X_0 has an admissible covering arising from a locally finite collection of strictly k-analytic affinoid subdomains D that cover X. Using this, construct the required collection quasi-sections (locally for the Tate topology!).

■

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