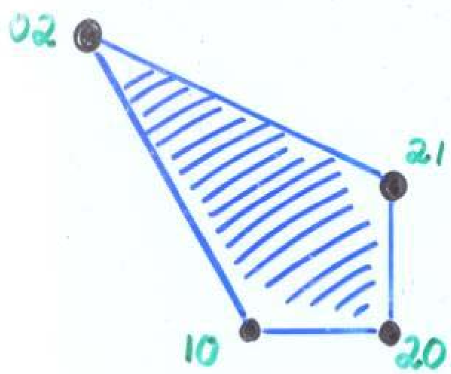
Bernd Sturmfels'
Arizona Lecture #1
Newton Polytopes & Tropical Varieties

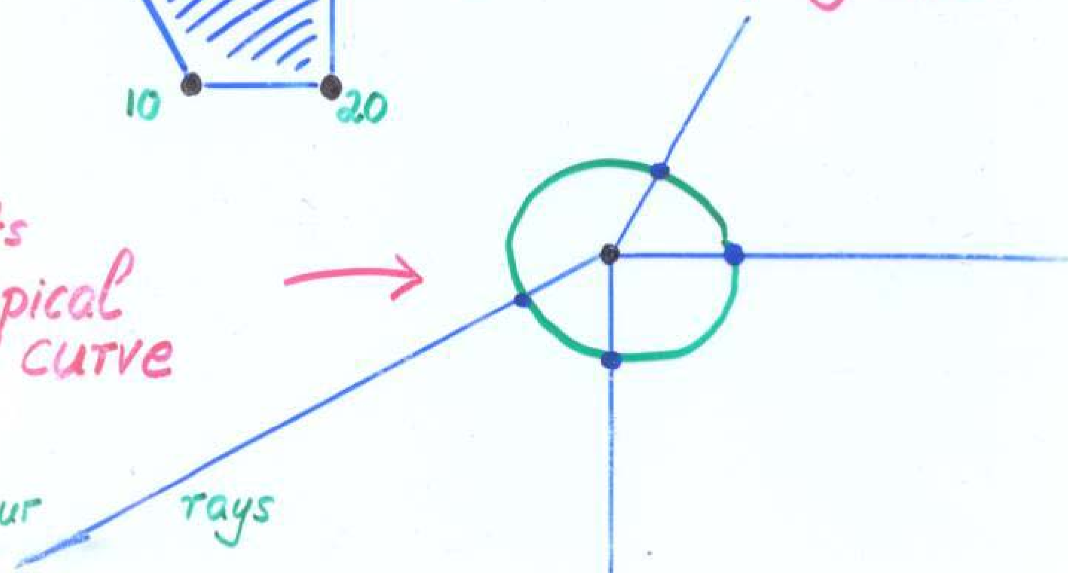
$$f = 3x^2y - y^2 + 8x^2 + x$$

a polynomial



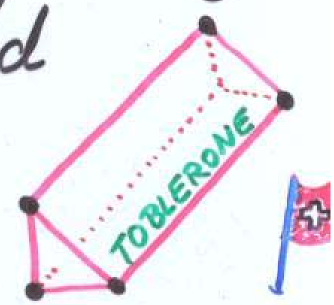
its Newton polytope

its tropical curve



Polytopes

A polytope P in $\mathbb{R}^n$ is the convex hull of finitely many points, or the bounded intersection of finitely many closed halfspaces.



Each $\omega \in \mathbb{R}^n$ defines a face of P:

$$\mathrm{face}_\omega(P) := \{u \in P \mid \forall v \in P : u \cdot \omega \leq v \cdot \omega\}$$

If F is a face then its normal cone is $\mathcal{N}_F(P) := \{\omega \in \mathbb{R}^n \mid F = \mathrm{face}_\omega(P)\}$

The normal fan of P is the set of all normal cones $\mathcal{N}_F(P)$ for all faces F of P.

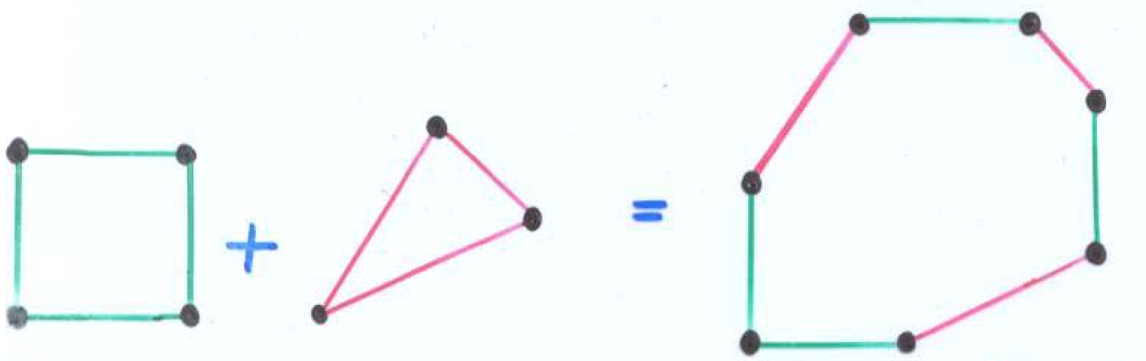
Minkowski Addition

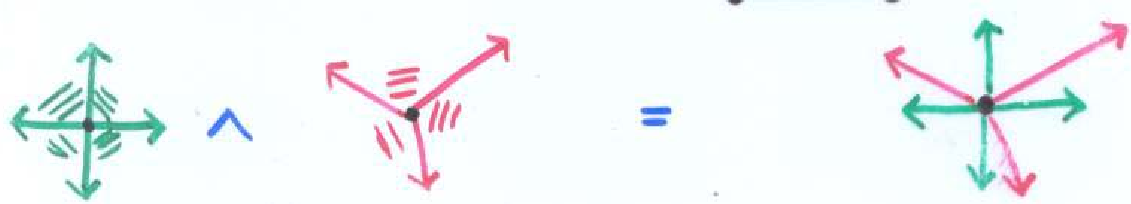
The sum of two polytopes is a polytope

$$P + Q = \{ p + q \mid p \in P,\ q \in Q \}$$

with $\mathrm{face}_\omega(P+Q) = \mathrm{face}_\omega(P) + \mathrm{face}_\omega(Q$

The normal fan of $P+Q$ is the common refinement of the normal fans of P and Q.





From Polynomials to Polytopes

Consider a Laurent polynomial

$$f = \sum_{j=1}^{n} c_i \cdot t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_d^{a_{dj}}$$

where $c_i \in \mathbb{C}^*$ and $a_{ij} \in \mathbb{Z}$.

The Newton polytope $New(f)$ is the convex hull in $\mathbb{R}^d$ of the points $a_j = (a_{1j}, a_{2j}, \dots, a_{dj})$.

Q: If f and g are polynomials what are $New(f+g)$ and $New(f \cdot g)$?

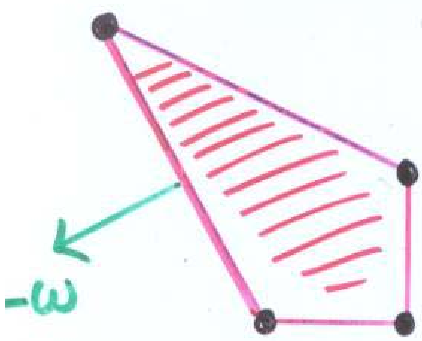
Polytope Algebra

Term orders and initial monomials

Term order for **Gröbner bases** can be represented by **weight vectors** $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$.

If f is a polynomial then the **initial form** $in_\omega(f)$ satisfies

$$New(in_\omega(f)) = face_\omega(New(f))$$



If ω is generic then $in_\omega(f)$ is a **monomial**.

Q: What if ω is not generic?

Tropical Hypersurfaces

Let $f \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_d^{\pm 1}]$.

The tropical hypersurface of f is

$$\mathcal{T}(f) = \left\{ \omega \in \mathbb{R}^d \;\middle|\; \begin{array}{l} in_\omega(f) \text{ is} \\ \text{not a monomial} \end{array} \right\}$$

$$= \left\{ \omega \in \mathbb{R}^d \;\middle|\; \begin{array}{l} face_\omega(New(f)) \\ \text{has dimension} \geq 1 \end{array} \right\}$$

= the union of all cones of codimension ≥ 1 in the normal fan of the Newton polytope of f.

Example: Let $d = 3$ and

$f_1 = t_1 + t_2 + t_3 + 1$

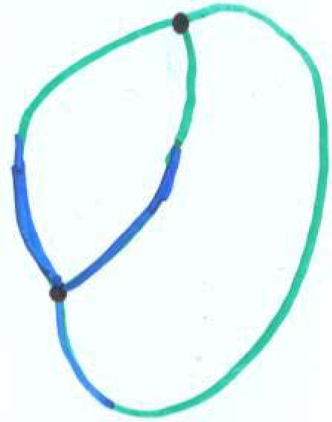
$f_2 = t_1 + t_2 + 2t_3 \ldots$

Can you draw $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$

Tropical Prevarieties

.... are *finite* intersections of tropical hypersurfaces in $\mathbb{R}^n$.

$\mathcal{T}(f_1)$



$\mathcal{T}(f_2)$

This tropical prevariety ...

$\mathcal{T}(f_1) \cap \mathcal{T}(f_2)$

... is **not** a tropical variety

Tropical Varieties

If $I \subset \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ is an ideal then its tropical variety $\mathcal{T}(I)$ is the intersection of the tropical hypersurfaces $\mathcal{T}(f)$ where f runs over **all** polynomials f in I.

Tropical Hilbert Basis Theorem

Every tropical variety is a tropical prevariety.

Transverse Intersection Lemma

We always have $\mathcal{T}(I+J) \subseteq \mathcal{T}(I) \cap \mathcal{T}(J)$.
If the latter intersection is transverse then $\mathcal{T}(I+J) = \mathcal{T}(I) \cap \mathcal{T}(J)$.

» Computing Tropical Varieties «

math.AG/0507563

y T. Bogart, A. Jensen, D. Speyer, BS, R. Thoma

o appear in MEGA '05 issue of JSC

Implementation in GFan

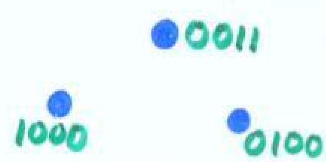
Input: A homog. ideal $I \subset \mathbb{C}[t_0, \dots, t_d]$

Output: The fan $\mathcal{T}(I)$, represented as a spherical polyhedral complex

Example:

INPUT: $I = \langle t_1 + t_2 + t_3 + t_0 \,,\, t_1 + t_2 + 2t_3 \rangle$

OUTPUT: Three points



How does this work ?

Valuations and Connectivity

Let $K = \mathbb{C}\{\{\varepsilon\}\}$ Puiseux series

The valuation $\nu : K^* \to \mathbb{Q}$

induces a map $\nu : (K^*)^d \to \mathbb{Q}^d \hookrightarrow \mathbb{R}^d$.

Theorem

For any ideal $I \subset \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$

the tropical variety $\mathcal{T}(I)$ equals

the closure of the image of the

classical variety $V(I) \subset (K^*)^d$ under ν.

If I is prime of dimension k

then $\mathcal{T}(I)$ is pure of dimension k

and connected in codimension 1.

WHO PROVED THIS ?

Tropicalization of linear spaces

Suppose I is generated by r linearly independent linear forms in $\mathbb{C}[t_0, t_1, \dots, t_d]$

Then $\mathcal{T}(I)$ is easy to compute from the matroid of I (rank r on $\{0, 1, \dots, d\}$)

Example If the linear forms are generic then $\mathcal{T}(I)$ is the $(d-r+1)$-dimensional fan represented by the $(d-r-1)$-skeleton of the simplex on $\{0, 1, \dots, d\}$.

Our Running Example

$$A = \begin{bmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 \\ 53 & 47 & 43 & 41 & 37 & 31 & 29 & 23 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The rows represent $r = 4$ linear for
in eight $(d=7)$ unknowns $t_0, t_1, \ldots, t_7$
generating a linear ideal I.

The tropical linear space $\mathcal{T}(I)$
is a two-dimensional simplicial comple
with 10 vertices and 60 triangles

{•01, •02, •03, •12, •13, •23,
•45, •46, •47, •56, •57, •67,
014, 015, 016, ..., 034, ..., 236, 237,
045, 046 047, 056, ..., 357, 367}

HOMOLOGY?