Situation

$$X = V(I) \subset \mathbb{P}^n$$

$$I \subset S = k[x_0, \dots, x_n], \qquad R = S/I$$

M graded R-module
(finitely generated)

$\mathcal{F} = \widetilde{M}$ coherent sheaf on X

want to find:

- $h^i(\mathcal{F}) := \dim_k H^i(X, \mathcal{F})$
- $H^i(X, \mathcal{F})$ $\qquad H^i_*(\mathcal{F})_{\geq e}$
- module structure:

$$H^i_*(\mathcal{F}) := \bigoplus_{d \in \mathbb{Z}} H^i(X, \mathcal{F}(d))$$

this is an R-module

- $$H^i_*(\mathcal{F})_{\geq e} := \bigoplus_{d \geq e} H^i(X, \mathcal{F}(d))$$

Čech complex + Čech cohomology

$\mathcal{U} = \{U_0, \dots, U_m\}$

open affine cover of X

$$U_\lambda := \bigcap_{i \in \lambda} U_i \qquad \lambda \subset \{0, \dots, m\}$$

Define $$\mathcal{C}^p(\mathcal{F}) := \bigoplus_{|\lambda| = p+1} \mathcal{F}(U_\lambda)$$

$$\sigma_p : \mathcal{C}^p(\mathcal{F}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{F})$$

$$(f_{i_0 \dots i_p}) \longmapsto (g_{j_0 \dots j_{p+1}})$$

$$g_{j_0 \dots j_{p+1}} = \sum_{i=0}^{p+1} (-1)^i f_{j_0 \dots \hat{j}_i \dots j_{p+1}}$$

$\mathcal{C}(\mathcal{F})$ is the complex

$$0 \to \mathcal{C}^0(\mathcal{F}) \xrightarrow{\sigma_0} \mathcal{C}^1(\mathcal{F}) \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{m-1}} \mathcal{C}^m(\mathcal{F}) \to 0$$

Def $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$

$$:= H^i(\mathcal{C}(\mathcal{F}))$$

theorem This is independent of the open cover (as long as it is affine)

$H^i(\widetilde{M})$:

Let M be a f.g. graded S-module

$\widetilde{M}$ coherent sheaf on $\mathbb{P}^n$

Def Let $\mathcal{C}^p(M) := \bigoplus_{|\lambda| = p+1} M \otimes_S S[x_\lambda^{-1}]$

where $\lambda = \{\lambda_0, \ldots, \lambda_p\} \subseteq \{0, \ldots, n\}$

and $x_\lambda = x_{\lambda_0} x_{\lambda_1} \ldots x_{\lambda_p} \in S$

Define $\sigma_p : \mathcal{C}^p(M) \longrightarrow \mathcal{C}^{p+1}(M)$

as above

$$\mathcal{C}(M) : \quad 0 \to \mathcal{C}^0(M) \to \cdots \to \mathcal{C}^n(M)$$

<u>proposition</u>

$$H^i_*(\widetilde{M}) = H^i(\mathcal{C}(M))$$

$$H^i(\widetilde{M}) = H^i(\mathcal{C}(M)_{\deg 0})$$

"<u>proof</u>"

Let $U_i = \mathbb{P}^n \setminus V(x_i)$ $\quad i = 0, \dots, n$

then $\mathcal{C}(M)_d = \mathcal{C}(\widetilde{M}(d))$ $\boxtimes$

note: this complex + cohomology can be used to <u>define</u> the module structure on $H^i_*(\widetilde{M})$

cohomology of $\mathcal{O}_{\mathbb{P}^n} = \tilde{S}$ (Serre)

$$H^i_*(\mathcal{O}_{\mathbb{P}^n}) = \begin{cases} S & i = 0 \\ 0 & 1 \leq i \leq n \\ \frac{1}{x_0 \cdots x_n} k[x_0^{-1}, \ldots, x_n^{-1}] & i = n \end{cases}$$

proof idea: good exercise!

$C(S)$ is $\mathbb{Z}^{n+1}$-graded

$C(S)_m$ is recognizable

$(m \in \mathbb{Z}^{n+1})$

Def if $M = \bigoplus_{d \in \mathbb{Z}} M_d$

is a graded S-module

the graded k-dual $M^\vee$ is

$$M^\vee = \bigoplus_{d \in \mathbb{Z}} M'_{-d}$$

k-vector space dual

So:

$$H^n_*(S) \simeq [\underbrace{S(-n-1)}_{\omega_{\mathbb{P}^n}}]^\vee$$

dualizing sheaf

Local duality (Serre) ($S = k[x_0, \dots, x_n]$)

(a) for $i \geq 1$

$$H^i_*(\widetilde{M}) \simeq \mathrm{Ext}_S^{n-i}(M, S(-n-1))^\vee$$

and so

$$H^i(\widetilde{M}) \simeq \mathrm{Ext}_S^{n-i}(M, S)'_{-n-1}$$

(b)

$$0 \longrightarrow \mathrm{Ext}_S^{n+1}(M, S(-n-1))^\vee \longrightarrow M \longrightarrow H^0_*(\widetilde{M}) \longrightarrow \mathrm{Ext}_S^{n}(M, S(-n-1))^\vee \longrightarrow$$

is exact.

simple, yet useful corollary of local duality:

Let M be a f.g. graded S-module

Then

$$\mathrm{pdim}_S(M) \leq n-1 \iff M = H^0_*(\widetilde{M})$$

[in particular, in this case

$$M_0 = H^0(\widetilde{M})$$]

corollary of local duality

$H^0_*(\widetilde{M})$ is f.g.

$\iff$ every associated component of M has dimension ≥ 1 in $\mathbb{P}^n$

proof

$H^0_*(\widetilde{M})$ f.g.

$\iff$ $\mathrm{Ext}^n_S(M,S)$ has finite dim over k

$\iff$ $\mathrm{codim}\, \mathrm{Ext}^n_S(M,S) = n+1$

But (Eisenbud - Huneke - Vasconcelos)

$\mathrm{codim}\, \mathrm{Ext}^i_S(M,S) \geq i$

and equality holds iff M has an associated prime of codim i

Important example:

Sheaf Ω^1_X of differential forms on $X \subseteq \mathbb{P}^n$.

Two useful exact sequences:

$$\boxed{0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow[(x_0 \dots x_n)]{} \mathcal{O}_{\mathbb{P}^n} \to 0}$$

[think: $dx_0, \dots, dx_n$, on U_i: generated by $dx_0, \dots, \widehat{dx_i}, \dots, dx_n$]

$$\boxed{\tilde{I}_X \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \to 0}$$

$$g \longmapsto dg$$

unwind these:

<u>proposition</u> Let $X = V(I) \subset \mathbb{P}^n$

$R = S/I$. The cotangent sheaf

Ω^1_X is the sheaf associated to

the homology module of

$$F \otimes R \xrightarrow{dj} R(-1)^{n+1} \xrightarrow[(x_0 \ \dots \ x_n)]{} R$$

where if $j : F \longrightarrow \cancel{R}\, S$ is

the generator matrix of I

then dj is the Jacobian of j.

<u>Example</u> Fermat quartic

$$X = V(a^4+b^4+c^4+d^4) \subseteq \mathbb{P}^3$$

K3 surface

Let's find $\Omega^1_X = \widetilde{M}$, $h^i(\Omega^1_X)$.

① $$R(-4) \xrightarrow{\begin{pmatrix} a^3 \\ b^3 \\ c^3 \\ d^3 \end{pmatrix}} R(-1)^4 \xrightarrow{(a\ b\ c\ d)} R$$

② gives M.

$$0 \to \begin{matrix} S(-4) \\ \oplus \\ S(-8) \end{matrix} \longrightarrow \begin{matrix} S(-3)^4 \\ \oplus \\ S(-5)^4 \end{matrix} \longrightarrow S(-2)^6 \longrightarrow M \to$$

So $H^0_*(\Omega^1_X) = M$

$$H^0(\Omega^1_X) = 0$$

get $h^1(\Omega^1_X) = 20$

$$h^2(\Omega^1_X) = 0$$

Question $X = V(I) \subset \mathbb{P}^n$ smooth, say

from above, get M

$$\widetilde{M} = \Omega^1_X$$

when is $\operatorname{pdim}_S(M) \leq n-1$

ie: $H^0_*(\Omega^1_X) = M$?

Ω^p_X :

Given $$G \xrightarrow{\varphi} F \to M \to 0$$

than

$$\begin{tikzcd}
G \otimes \Lambda^{p-1} F \arrow[rr] \arrow[dr, dotted, "\varphi \otimes id"'] & & \Lambda^p F \arrow[r] & \Lambda^p M \arrow[r] & 0 \\
& F \otimes \Lambda^{p-1} F \arrow[ur, dotted, "\wedge"'] & & &
\end{tikzcd}$$

is a presentation of $\Lambda^p M$

and : $$\Omega^p_X = \widetilde{\Lambda^p M}$$

if $$\Omega^1_X = \widetilde{M}$$

Example Hodge diamond

$X \subseteq \mathbb{P}^n$ smooth

dimension d (say $=3$)

$$h^{p,q} := \dim H^q(\Omega^p_X)$$

have: $h^{p,q} = h^{d-p,d-q}$

$h^{p,q} = h^{q,p}$

$$H^i(X;\mathbb{C}) \simeq \bigoplus_{p+q=i} H^q(\Omega^p_X)$$

$$\begin{array}{cccc}
h^0(\mathcal{O}_X) & h^1(\mathcal{O}_X) & h^2(\mathcal{O}_X) & h^3(\mathcal{O}_X) \\
h^0(\underline{\Omega^1}) & h^1(\Omega^1) & h^2(\Omega^1) & h^3(\underline{\Omega^1}) \\
h^0(\underline{\Omega^2}) & h^1(\underline{\Omega^2}) & h^2(\underline{\Omega^2}) & h^3(\underline{\Omega^2}) \\
h^0(\underline{\Omega^3}) & h^1(\underline{\Omega^3}) & h^2(\underline{\Omega^3}) & h^3(\underline{\Omega^3})
\end{array}$$

Want: to compute as *few* of these as possible.

Prop $\quad \chi(\widetilde{M}) := \sum_{i=0}^{n} (-1)^i h^i(\widetilde{M})$

is $P_M(0)$, where

$P_M(d) :=$ Hilbert poly of M

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first row : easiest

second row : $\quad \widetilde{M} = \Omega^1_X$

need only: $\chi(\Omega^1_X)$

$h^1(\Omega^1_X)$

(for $\dim X = 2$ or 3)

Hodge diamond

$X \subseteq \mathbb{P}^4$ quintic 3-fold

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 100 & 0 \\ 0 & 100 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

note:

$$h^1(\widetilde{M}) = \dim \operatorname{Ext}_S^{n-1}(M,S)_{-n-1}$$

$$= \dim \operatorname{Ext}_S^3(M,S)_{-5}$$

__Def__ $X \subseteq \mathbb{P}^n$, smooth, is called rationally connected if $\forall\, p \neq q \in X$, there is a rational curve on X containing p, q

__Conjecture__ X is RC

$$\iff \quad H^0((\Omega^1_X)^{\otimes m}) = 0$$

for all $m \geq 1$

[Mumford, Mori ?]