Example "Normal sheaf"

$$I \subset S = k[x_0, \dots, x_n]$$

$$R = S/I$$

$$N := \mathrm{Hom}_S(I, S/I)$$

$$\left[ \begin{array}{l} = \mathrm{Hom}_S(I/I^2, S/I) \\ = \mathrm{Hom}_R(I/I^2, R) \end{array} \right]$$

Find $N$

① present $I$:

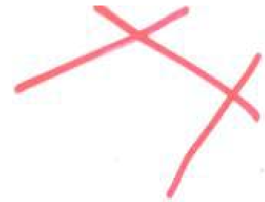
$$G \xrightarrow{\varphi} F \to I \to 0$$

free $S$-modules

② $\mathrm{Hom}_S(-, S/I)$ is left-exact:

$$G^* \otimes R \xleftarrow{\varphi^T} F^* \otimes R \leftarrow N \leftarrow 0$$

③ So $N = \ker(\varphi^T)$.

**Example** Normal sheaf of 

$$I = (a,b) \cap (b,c) \cap (c,d)$$

$$= (bd, bc, ac) \subseteq S = k[a,b,c,d]$$

$$X = V(I) \subseteq \mathbb{P}^3 \qquad R = S/I$$

present $I$:

$$S(-3)^2 \xrightarrow{\begin{pmatrix} c & 0 \\ -d & a \\ 0 & -b \end{pmatrix}} S(-2)^3 \longrightarrow I \longrightarrow 0$$

$$R(3)^2 \xleftarrow{\begin{pmatrix} c & -d & 0 \\ 0 & a & -b \end{pmatrix}} R(2)^3 \longleftarrow \mathrm{Hom}(I, S/I) \longleftarrow$$

$$= \text{image} \begin{pmatrix} 0 & 0 & d & b & 0 & a \\ 0 & 0 & c & 0 & b & 0 \\ d & c & 0 & 0 & a & 0 \end{pmatrix}$$

$$f : I \longrightarrow S/I$$

$$bd, bc \longmapsto 0 \qquad ac \longmapsto d$$

Finding $\mathrm{Ext}^i_S(M, S)$

[can also do $\mathrm{Ext}^i_S(M, N)$]

Steps

① free resolution of $M$

$$0 \to F_r \xrightarrow{d_r} \cdots \to F_1 \xrightarrow{d_1} F_0 \to M.$$

② apply $\mathrm{Hom}_S(-, S)$

ie : transpose.

③

$$\mathrm{Ext}^i_S(M, S) \simeq \frac{\ker(d^*_{i+1})}{\mathrm{im}(d^*_i)}$$

**Example** Rational quartic, again

$\operatorname{Ext}^2(S/I, S(-4))$

$\operatorname{Ext}^3(S/I, S(-4)) \simeq S/(a,b,c,d)\,(1)$

other Ext's are $0$.

$$0 \to S/I \to H^0_*(\mathcal{O}_X) \to k \to 0$$

degree 1

$$h^0(\mathcal{O}_X(1)) = 4+1 = 5$$

$$H^1_*(\mathcal{O}_X) \simeq \operatorname{Ext}^2(S/I, S(-4))^\vee$$

$$H^i_*(\mathcal{O}_X) = 0 \qquad i \geq 2$$

Special case:

$M$ is Cohen-Macaulay

(e.g.: $M = S/I$ complete intersection)

$$0 \to F_c \to \cdots \to F_0 \to M \to 0$$

$c = \operatorname{codim} M$

$H^0_*(\tilde{M}) = M$ $\quad$ $Ext^n$, $Ext^{n+1}$

$H^1_*(\tilde{M}) = 0$ $\quad$ $Ext^{n-1}$

$\vdots$ $\quad$ $\vdots$

$H^{d-1}_*(\tilde{M}) = 0$ $\quad$ $Ext^{c+1}$

$H^d_*(\tilde{M}) \neq 0$ $\quad$ $Ext^c$

**Example** Rational quartic curve

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$(s,t) \longmapsto (s^4, s^3t, st^3, t^4)$$

$$= (a,b,c,d)$$

$X := \text{image}(\varphi)$ quartic curve

$$I_X = (bc - ad,\ c^3 - bd^2,\ ac^2 - b^2d,\ b^3 - a^2c)$$

minimal free resolution of $S/I_X$

$$0 \to S(-5) \xrightarrow{\begin{pmatrix} d \\ -c \\ -b \\ a \end{pmatrix}} S(-4)^4 \xrightarrow{\begin{pmatrix} -b^2 & -ac & -bd & -c^2 \\ c & d & 0 & 0 \\ a & b & -c & -d \\ 0 & 0 & a & b \end{pmatrix}} \begin{matrix} S(-2) \\ \oplus \\ S(-3)^3 \end{matrix} \longrightarrow S \longrightarrow S/I \to$$

gens of $I_X$

## Example  Rational quartic curve

recall

$$0 \to S(-5) \xrightarrow{d_3} S(-4)^4 \xrightarrow{d_2} \begin{matrix} S(-2) \\ \oplus \\ S(-3)^3 \end{matrix} \xrightarrow{d_1} S \to S/I \to 0$$

$$\mathrm{Ext}^2(S/I, S) = \frac{\ker d_3^T}{\operatorname{im} d_2^T}$$

- generated by 3 elements
- "canonical" module of $S/I$

$$\mathrm{Ext}^3(S/I, S) = S/(a,b,c,d)\,(5)$$

Which $\mathrm{Ext}^i_S(M,S)$ are nonzero?

let $c = \operatorname{codim} M$ (in $S$)

$d = \operatorname{pd}_S M$

then $c \le d$ and

$$\begin{cases} \mathrm{Ext}^c(M,S) \neq 0 \\ \mathrm{Ext}^{c+1}(M,S) \quad ? \\ \vdots \\ \mathrm{Ext}^{d-1}(M,S) \quad ? \\ \mathrm{Ext}^d(M,S) \neq 0 \end{cases}$$

$M$ is <u>Cohen-Macaulay</u> iff

$c = d$ : only one

non-zero Ext

## Defining sheaves

Let $\{U_f\}$ be a base for the topology of $X$.

example $X \subseteq \mathbb{P}^n$ Zariski topology

$f \in S = k[x_0, \dots, x_n]$ homogeneous

$U_f := X \setminus V(f)$

form a base for Z-topology

To specify a sheaf on $X$, $\mathcal{F}$

give

- $\mathcal{F}(U_f)$ each $f$
- if $U_f \subset U_g$

$$\rho_{gf} : \mathcal{F}(U_g) \longrightarrow \mathcal{F}(U_f)$$

satisfying the usual sheaf axioms

- $\rho_{hf} = \rho_{gf}\rho_{hg}$
- locally $0 \Rightarrow 0$
- can glue sections which agree on overlaps

# Coherent sheaves

$X \subseteq \mathbb{P}^n$

$M$ f.g. graded $S/I$-module

Define $\widetilde{M}$ sheaf on $X$ by

- $\widetilde{M}(U_f) := \left(M \otimes_S S[\frac{1}{f}]\right)_0$

degree 0 part

$U_{fg} \subset U_f$

$\widetilde{M}(U_f) \longrightarrow \widetilde{M}(U_{fg})$

natural map

**Def** A <u>coherent sheaf</u> on $X = V(I) \subseteq \mathbb{P}^n$ is a sheaf of the form

$\widetilde{M}$

for $M$ a graded, f.g. $S/I$-module.

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Your choice:

- (coherent) sheaves
- graded modules

## Examples

- $\mathcal{O}_X = \widetilde{R}$ $\qquad R = S/I$, $\quad X = V(I) \subset \mathbb{P}^n$

- $\mathcal{O}_X(d) = \widetilde{R(d)}$ $\qquad d \in \mathbb{Z}$

- if $D \subset X$ is a codim 1 subvariety with ideal $J \subset S/I$ then $\mathcal{O}_X(-D) := \widetilde{J}$

- normal bundle (sheaf) of $X \subseteq \mathbb{P}^n$

$$N_{X/\mathbb{P}^n} = \widetilde{\mathrm{Hom}_S(I, S/I)}$$

## Two operations

$$M \longmapsto \widetilde{M}$$

$$\mathcal{F} \longmapsto H^0_*(\mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} H^0(\mathcal{F}(d))$$

## Key facts

- $\widetilde{M} = 0 \iff M_d = 0 \quad d \gg$
- $\widetilde{\ }$ is an exact functor
- $\widetilde{M} \simeq \widetilde{N} \iff M_{\geq d} \simeq N_{\geq d}$ for some $d$
- $\widetilde{H^0_*(\mathcal{F})} = \mathcal{F}$

**Question**

$\widetilde{M}$ on $X$ or on $\mathbb{P}^n$ ?

**Answer**

doesn't matter (much)

$H^i(\mathbb{P}^n, \widetilde{M})$

$= H^i(X, \widetilde{M})$

denote by $H^i(\widetilde{M})$