Computing with sheaves and sheaf cohomology in algebraic geometry

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Suppose :

We are given $X \subseteq \mathbb{P}^n$

by equations: $X = V(I)$

$I = (f_1, \ldots, f_r) \subseteq S = k[x_0, \ldots, x_n]$

I homogeneous

— — —

Gröbner bases give

dim X

degree X

dim sing X

Suppose X is nonsingular

Might want to know:

- Is X connected?
- If X is a curve:
  - genus of X
  - Mittag-Leffler problem:

given $P_1, \ldots, P_r \in X$

and multiplicities

$m_1, \ldots, m_r \in \mathbb{Z}$

find a rational function

$f \in K(X)^*$

with these zeros + poles

(if possible)

If X is a surface:

- DeRham cohomology
  $H^i(X; \mathbb{C})$
- numerical invariants
  e.g.: geometric genus
  irregularity
  ...
- Is X rational?
  [Castelnuovo's theorem]
- Intersection theory on X

Hodge diamond:

$$h^{p,q} = \dim H^q(X, \Omega^p_X)$$

Another important situation:

families of projective varieties

varieties near X

(deformations)

- Hilbert scheme

- Deformation of subvarieties

  e.g: lines on a quintic 3-fold

Cohomology of sheaves
to the rescue!

Computing cohomology
either solves these problems
or gives useful information
about them

Example: Hilbert scheme of the twisted cubic curve $X \subseteq \mathbb{P}^3$

twisted cubic:

$$I = I_X = (b^2 - ac,\ bc - ad,\ c^2 - bd) \subseteq k[a,b,c,d]$$

Hilbert scheme:

$\mathrm{Hilb}^{3d+1}(\mathbb{P}^3) = H$

parametrizes subschemes of $\mathbb{P}^3$ whose Hilbert polynomial is $P(d) = 3d+1$

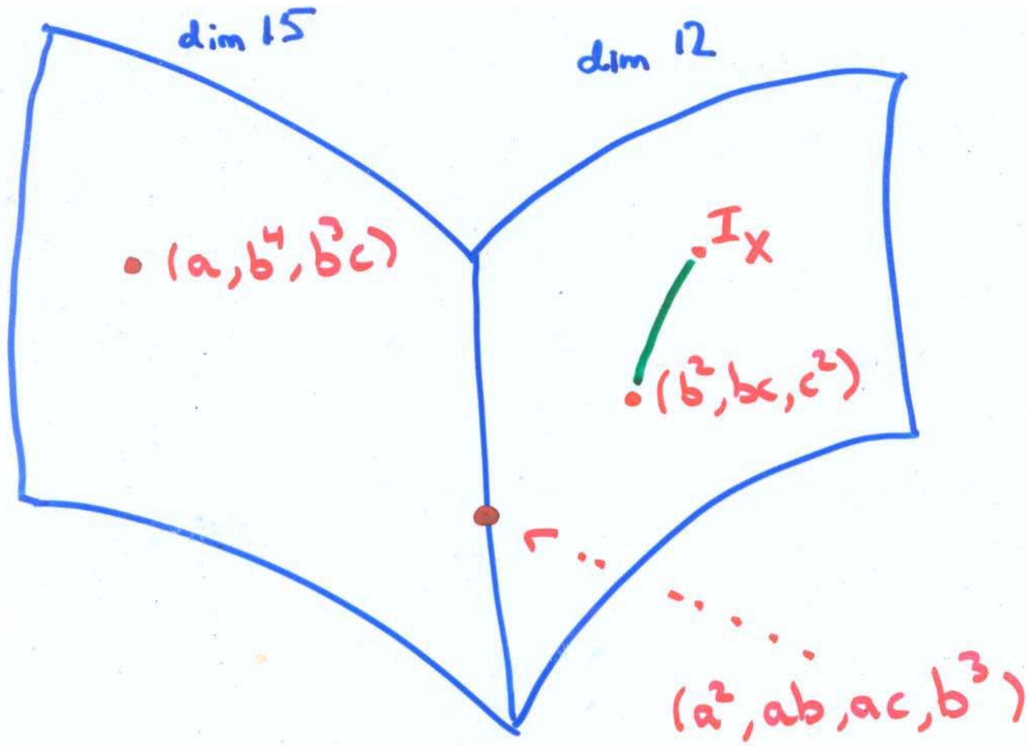
points on H:

$[I_X] \in H$

$[(b^2, bc, c^2)] \in H$ too

$(a, b^4, b^3c) \in H$

H:



A Gröbner basis gives a path on H

$$I_t = (b^2 - tac,\ bc - t^2ad,\ c^2 - tbd)$$

$$I_1 = I_X$$

$I_t = I_X$ up to scaling $t \neq 0$

$$I_0 = (b^2, bc, c^2)$$

theorem

Let $X \subseteq \mathbb{P}^n$ be a projective variety (or scheme) with Hilbert polynomial $p(d)$.

Let $H = \mathrm{Hilb}^{p(d)}(\mathbb{P}^n)$

Then

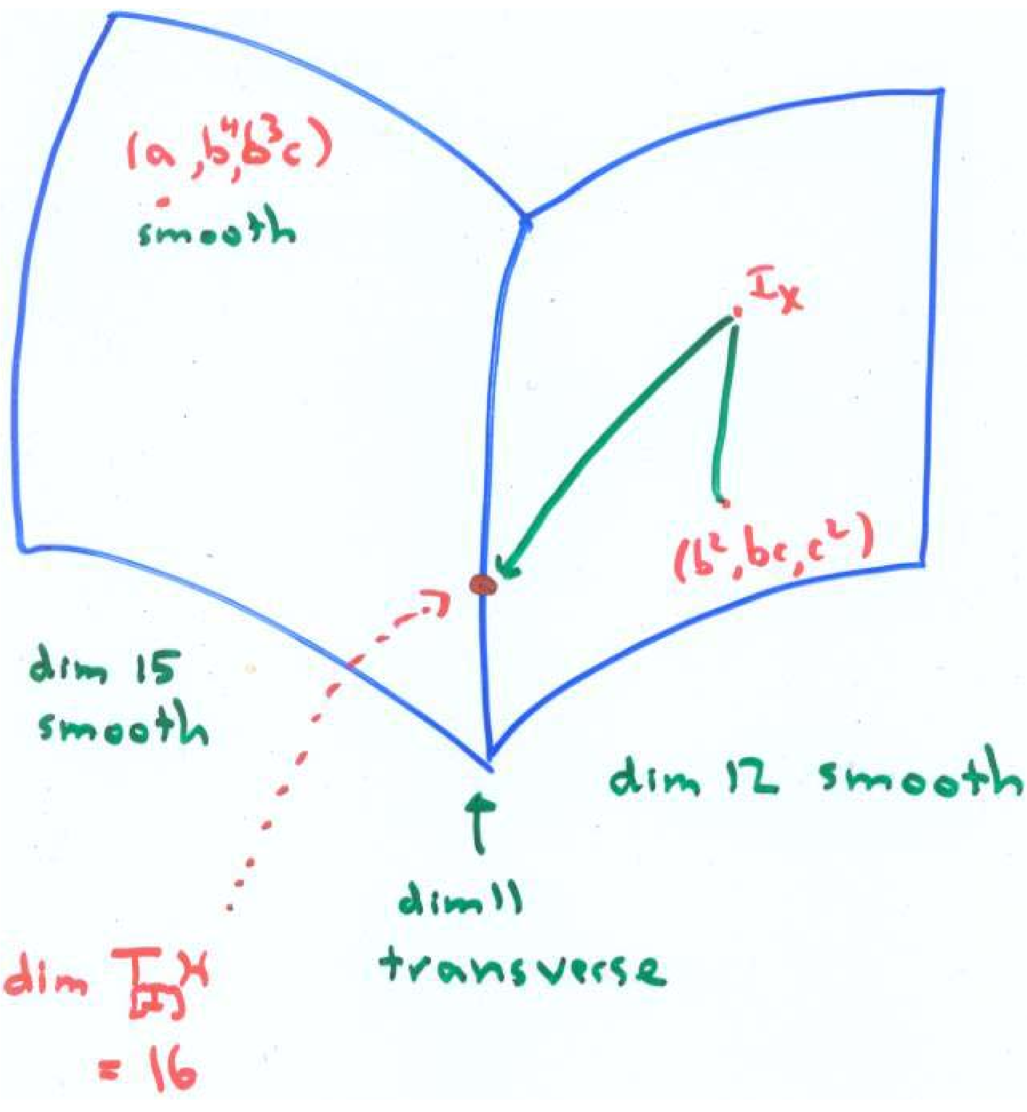
ⓐ The Zariski tangent space

$$T_{[X]} H = H^0(X, \mathcal{N}_{X/\mathbb{P}^n})$$

ⓑ If $X \subseteq \mathbb{P}^n$ is a local complete intersection (e.g: smooth) then every component of H thru $[X]$ has

$$\text{dimension} \geq h^0(\mathcal{N}_{X/\mathbb{P}^n}) - h^1(\mathcal{N}_{X/\mathbb{P}^n})$$

where $\mathcal{N}_{X/\mathbb{P}^n}$ is the normal sheaf of $X \subseteq \mathbb{P}^n$


(a, b^4, b^3c)
smooth
I_X
(b^2, bc, c^2)
dim 15
smooth
dim 12 smooth
dim 11
transverse
dim $T_{[I]}X$
$= 16$

theorem (Reeves, –)

Let $L \subset S = k[x_0, \ldots, x_n]$ be *the* lexicographic monomial ideal with Hilbert polynomial $p(d)$. Then

$$[L] \in \mathrm{Hilb}^{p(d)}(\mathbb{P}^n)$$

is a nonsingular point.

Syzygies

$\boxed{\text{Key computation}}$: given a matrix

$$f : R^b \longrightarrow R^a$$

find a generating set for the syzygy module

$$\ker f = \{ v \in R^b : f(v) = 0 \}$$

Schreyer: this is a byproduct of the Buchberger algorithm for finding the Gröbner basis of

$$\operatorname{image}(f) = (f_1, \ldots, f_b) \subseteq R^a$$

__Example__ twisted cubic curve

$$I = (f_1, f_2, f_3) \subseteq k[a,b,c,d] = R$$

$$f_1 = b^2 - ac$$

$$f_2 = bc - ad$$

$$f_3 = c^2 - bd$$

$$f : R^3 \xrightarrow[(f_1\ f_2\ f_3)]{} R$$

$$\ker f = \left\{ \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in R^3 : \sum g_i f_i = 0 \right\}$$

Buchberger algorithm

$$c f_1 - b f_2 + a f_3 = 0$$

$$c f_2 - b f_3 - d f_1 = 0$$

So:

$$\ker f = \left\langle \begin{pmatrix} c \\ -b \\ a \end{pmatrix}, \begin{pmatrix} -d \\ c \\ -b \end{pmatrix} \right\rangle$$

byproduct of GB construction

Aside: graded modules

$S = k[x_0, \dots, x_n]$ $\qquad \deg(x_i) = 1$

M a graded S-module:

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

such that $S_d \cdot M_e \subseteq M_{d+e}$

<u>Def</u> $M(e)$ is the same S-module as M, but with grading

$$M(e)_d := M_{e+d}$$

<u>Example</u> $S(-3)$

has one generator in degree 3

$S(-3)_3 = S_0 = k$

$$0 \longrightarrow R^2 \xrightarrow[\begin{pmatrix} c & -d \\ -b & c \\ a & -b \end{pmatrix}]{d_2} R^3 \xrightarrow[(f_1\ f_2\ f_3)]{d_1} R \longrightarrow R/I \longrightarrow 0$$

is exact : "the" minimal free resolution of R/I

graded free resolution :

$$0 \to R(-3)^2 \xrightarrow{d_2} R(-2)^3 \xrightarrow{d_1} R \longrightarrow R/I \longrightarrow 0$$

<u>Hilbert</u> <u>syzygy</u> <u>theorem</u>

$S = k[x_0, \ldots, x_n]$

M f.g. graded S-module

then

the minimal free resolution of M

has length $\mathrm{pd}_S(M) \leq n+1$

$$0 \to F_r \to \cdots \to F_1 \to F_0 \to M \to 0$$

$r \leq n+1$

A good exercise

(which we will assume, and use)

$R = S = k[x_0, \dots, x_n]$

or $R = S/I$

M, N, P f.g. R-modules

- Given $f : M \longrightarrow N$

  construct $\ker f$, $\operatorname{image} f$, $\operatorname{coker} f$

- Given $M \xrightarrow{f} N \xrightarrow{g} P$ $gf = 0$

  construct $\dfrac{\ker g}{\operatorname{im} f}$

- Rest of homological algebra...!