Outline of the lectures “ Motivic and p-adic integration” by François Loeser

My goal is to explain the basics of p-adic integration and motivic integration and to discuss some connections with Model Theory.

Lecture 1: p-adic integration. We will explain the basics of p-adic integration on smooth varieties, its relation with number of points of reductions modulo p^n (Oesterlé's Theorem) and applications to rationality of Poincaré series (work of Igusa and Denef). We shall conclude by presenting Denef's results on the measure of definable sets.

Lecture 2 : Motivic integration. Arc spaces. Grothendieck rings of varieties. Construction of motivic measures and basic properties. Change of variable formula. Applications to rationality results.

Lecture 3 : Assigning virtual motives to definable sets. We shall explain first Chow motives, Galois stratifications and quantifier elimination for pseudo finite fields. Then we will be able to assign a virtual motive to definable sets. We shall explain how it relates to counting points.

Lecture 4: Arithmetic motivic integration. Using results from the previous lecture, we shall round the loop by explaining how one can construct motivic integrals that specialize to p-adic ones. If time allows we shall show how this fits in a much more general framework.

Prerequisites : Familiarity with the language of Algebraic Geometry (as developed in Hartshorne's book) and with the most elementary part of Model Theory.

References:

Lecture 1
[1] J.Igusa, An introduction to the theory of local zeta functions. AMS/IP Studies in Advanced Mathematics, 14. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000.
[2] J.Denef, Arithmetic and geometric applications of quantifier elimination for valued fields. Model theory, algebra, and geometry, 173–198, Math. Sci. Res. Inst. Publ., 39, Cambridge Univ. Press, Cambridge, 2000
Lecture 2
[3] J.Denef, F. Loeser, Geometry on arc spaces of algebraic varieties. European Congress of Mathematics, Vol. I (Barcelona, 2000), 327–348, Progr. Math., 201, Birkhüser, Basel, 2001
[4] J.Denef, F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration. Invent. Math. 135 (1999), no. 1, 201–232.
Lecture 3, 4
[5] M.Fried, M.Jarden, Field arithmetic. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 11. Springer-Verlag, Berlin, 1986.

[6] J.Denef, F.Loeser, Definable sets, motives and p-adic integrals. J. Amer. Math. Soc. 14 (2001), no. 2, 429–469.
[7] T.Hales, Can p-adic integrals be computed? math.RT/0209001.
[8] J.Denef, F.Loeser, Motivic integration and the Grothendieck group of pseudo-finite fields AG/0207163.
[9] J.Denef, F.Loeser, On some rational generating series occuring in arithmetic geometry, available at http://www.dma.ens.fr/ loeser/.

Project 1) The quantifier elimination Theorem of Fried and Sacerdote plays a basic role in Lecture 3. The proof presented in the book [5] (Proposition 25.9 of [5]) is given in a very algebraic language and could be translated in more geometric terms using basic knowledge of Algebraic Geometry such as Galois Theory for coverings of schemes, The project has 2 steps:

- the first is to present a neat self-contained geometric proof of Proposition 25.9 of [5].
- the second is to find (and to prove) a generalization of that result over a more general base than the spectrum of a field.

Suggested readings for the project: Familiarity with the relevant chapters of [5] and learning about geometric aspects Galois covers in [13].

Project 2) The complete proof of Theorem 6.4.1 in [9] (the main result in lecture 3) is scattered between 3 places ([6], [8] and [9]). The project would be to rearrange the arguments given or sketched in these papers to be able to write down a self contained direct proof of the Theorem.

Suggested readings for the project:

Learn basics about Chow motives in:

[10] A. Scholl, Classical motives. Motives (Seattle, WA, 1991), 163–187, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

Get a look to the paper (without going through proofs)

[11] S.del Bao Rollin, V.Navarro Aznar, On the motive of a quotient variety. Collect. Math. 49 (1998), no. 2-3, 203–226.

The standard modern reference for coverings of varieties and schemes is SGA1, available on the arxiv

[12] A.Grothendieck, M.Raynaud, SGA 1, Revêtements étales et groupe fondamental math.AG/0206203

but this may seem somewhat arid for most. A more leisurely introduction can be found in

[13] J.-P.Serre, Topics in Galois theory, Research Notes in Mathematics, 1, Jones and Bartlett Publishers, Boston, MA, 1992.