Arizona Winter
School
Lectures

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LECTURE 4
4.1 The family

\[ X : \ t(X_0^5 + \ldots + X_4^5) = X_0 \cdots X_4 \]

\[ \mathcal{S} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \frac{\sqrt[5]{0}}{5}\} \]

This is a family of CY threefolds with

\[ H^3(X_t, \mathbb{Z}) \cong \mathbb{Z} \]

Consider the group

\[ G = \left\{ (z_0, \ldots, z_4) \in \mathbb{R}^5 \mid \prod z_i = 1 \right\} \]

and take

\[ V = \left( \mathbb{R}^3 \times (\mathbb{Z}) \right)^G \]
Proposition 4.1.1

(a) The rank of $V$ is $4$, and the Hodge numbers are $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$.

(b) The variation is not constant.

4.1.2 Another geometric realization

Note that

$$V_{t,G} = H^3(X_t, \mathbb{Q})^G = H^3(X_t/G, \mathbb{Q}).$$

To compute $X_t/G$ note

$$C[x_0, \ldots, x_4]^G = C[y_0, \ldots, y_4]/(y^5 - y_0 - y_4)$$

where

$$Y_i = X_i$$

and

$$Y_5 = X_0 - X_4$$

Thus

$$X_t/G = \text{Proj } C[y_0, \ldots, y_4]/(y^5 - y_0 - y_4)^t$$

$$= \text{Proj } (C[x_0, \ldots, x_4]/(x^5 - y_0 - y_4)^5 - y_0 - y_4)^t.$$
For a parameter $s \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, \frac{1}{3}, 0 \}$ set

$$Y_s : s(y_0 + \cdots + y_4)^5 = y_6 \cdots y_4$$

Then $1 \to Y_s$

Then

$S \xrightarrow{\tau_b} S_{new}$

$1 \mapsto s = s^5$

We see

$$V_\mathbb{Q} = \pi_\ast R^3f_{new \ast} \mathcal{O}$$

and the betti #s of $Y_s$ are

$$1, 0, 1, 4, 1, 0, 1$$

"The only interesting part of $H^3(Y_s)$ is the $H^3$ which is $V_{s^5 \mathbb{Q}}", \mathcal{O}
4.2 A thesis problem

4.2.1 For how many $t \in S$ is the Hodge structure $V_t$ reducible?

Reducible means:

$$V_t = V \otimes V^\perp \text{ over } \mathbb{Q}$$

$$V_1 = V_{1,0} \otimes V_{1,1} \cap V_{1,2}$$

Remarks 4.2.2 (Unexplained)

(a) This is not the same as asking the H.S. to be “CM” (means $MT = TM_c$).

Prof. Oort & me expect that CM happens only for a finite # of $t$’s.

(b) For very general $t \in S$ the HS is neither CM nor reducible.

(c) The generalized Hodge conjecture
implies

\[ \forall_{t, \mathbb{Z}} \text{ reducible } \iff \exists \text{ algebraic family of } \]

\[ \text{cycle-cycles } C_\lambda \subset X_t \]

\[ \lambda \in \mathbb{A} \text{ such that } \]

\[ \mathcal{A}: \mathbb{A} \to J^G(X_t) \to J^G(X) \]

is not constant

In particular we see that such \( t \in \overline{\mathbb{Q}} \).

(d) The implication

\[ \forall_{t, \mathbb{Z}} \text{ reducible } \Rightarrow t \in \overline{\mathbb{Q}} \]

also follows from "\( H \Rightarrow A\mathbb{H} \)."

(e) Assuming "\( H \Rightarrow A\mathbb{H} \)" one can also show

\[ \forall_{t, \mathbb{Z}} \Rightarrow t \in \mathbb{Z}^{[\mathbb{Q}]} \text{ (not so easy).} \]
4.3 PVHS of wt 3 and type (1,1,1)

\((V^2, V_3^0, \psi) / \mathbb{D}\)

Lemma 4.3.1 There exist many bases

\[ e_1, e_2, e_3, e_4 \text{ of } V^2 \quad \text{o.t.} \]

\[ \psi \sim \frac{1}{(2\pi)^3} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]

For a suitable choice of such bases and possibly shrinking \( \mathbb{D} \) we will have

\[ F_5^2 = V_5^3 \cdot 0 = C \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ F_5^3 = V_5^3 \cdot 0 + V_5^{24} = F_5^3 + C \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]

\[ F_5^4 = F_5^2 + C \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

where \( a, a' \ldots \) are functions on \( \mathbb{D} \).
(1.1.4) \( a, a', \ldots \) holom. on \( S \)

(1.1.6.1) \[
\begin{cases}
\dot{a}'' = -a \\
\dot{b}' = b - aa'
\end{cases}
\]

(1.1.6.2) defines a nonempty open \( U \subset \{ (a, \ldots) \text{ s.t. (1.1) } \} \).

Note that \( \dim C U = 4 \)

(1.1.5) Write \( \dot{a} = \frac{da}{ds} \), etc then we get
\[
\begin{pmatrix}
0 \\
\dot{a} \\
\dot{b}' \\
\dot{c}
\end{pmatrix} \in F^2_s, \quad
\begin{pmatrix}
0 \\
\dot{a}' \\
\dot{b}'
\end{pmatrix} \in F^2_s
\]

This is equivalent to
\[
\begin{cases}
\dot{b}' = \dot{a}a' \\
\dot{c} = \dot{a}(b - aa') \quad \left[= \dot{a}b' \right]
\end{cases}
\]

Many solutions (even starting at the same point and having high arc contact)
Prop 4.3.2 Suppose that 
\[ p : D^n \rightarrow U \]
corresponds to a PVHS over \( D^n \).
Then \( \dim p(D^n) \leq 1 \).

Rem No "universal" PVHS's!

Proof. (1.1.5) says
\[
\frac{\partial b}{\partial z_i} = a' \frac{\partial a}{\partial z_i}, \quad \frac{\partial c}{\partial z_i} = b' \frac{\partial a}{\partial z_i}.
\]

Hence the vectors
\[
(\frac{\partial a}{\partial z_i}, \frac{\partial b}{\partial z_i}, \frac{\partial c}{\partial z_i}) \in C.(1, a', b').
\]
Hence
\[ D^n \rightarrow C^3 \]
has image \( \dim \leq 1 \). Finally
\[ a' = \frac{\partial b}{\partial z_i} / \frac{\partial a}{\partial z_i} \]
depends on one parameter also.
Prop 4.3.3 The HS with \( a, b, c, a', b', c' \) is reducible iff \( \exists \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{Q} \) such that

\[
b = \lambda_1 + \lambda_2 a
\]

\[
c = \mu_1 + \mu_2 a
\]

More precisely: \( \dim_{\mathbb{Q}} (\mathbb{Q} \cdot \text{span}(1, a, b, c)) = 2 \).

Reference for CM question: Ciprian Borcea, CY threefolds and CM.

Reference for AH- cycles: P. Deligne (notes by J. Milne), Hodge cycles on Abelian Varieties in LNM 900.