1.1 Polarized Variations of Hodge Structures

Let $S$ be a complex (analytic) manifold

$$S = D = \{ z \in \mathbb{C}, \ |z| < 1 \}$$

$$D^* = D \setminus \{0\}$$

$D^n$ (polydisc)

$\mathbb{P}^n(\mathbb{C}) \setminus \{ \text{divisor} \}$

Set

$$\mathbb{Z}(k) = (2\pi i)^k \mathbb{Z} \quad \text{Hodge structure of type } (-k,-k)$$

will also denote the constant sheaf with value $\mathbb{Z}(k)$.

Suppose we are given data

(1.1.1) $V^*_k$ a local system of free abelian groups over $S$.

(i.e. locally constant sheaf with fibre $\mathbb{Z}^*$ some $r$)
(1.1.2) \( \Psi : \mathbb{V}_Z \otimes_{\mathbb{Z}} \mathbb{V}_Z \to \mathbb{Z}(-w) \)

a bilinear pairing, symmetric
if \( w \) even, alternately if \( w \) odd

(1.1.3) For each point \( s \in S \) a Hodge structure

of weight \( w \) on \( V_s, \mathbb{Z} = \) fibre of \( \mathbb{V} \)
at \( s \):

\[ V_{s, \mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=w} V_{s}^{p,q} \]

Definition: Data as above define a

PVHS's iff the following conditions are satisfied

(1.1.4) (Holomorphicity) Setting

\[ F^p_s = \bigoplus_{p \geq p} V_{s}^{p,q} \]

the filtration

\[ 0 = F^0_s \subset F^1_s \subset \ldots \subset F^w_s = V_{s, \mathbb{C}} \]

depends analytically on \( s \in S \) i.e. \( F \)
locally direct summands

\[ F^p \subset V_0 \overset{def}{=} \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_S \]

such that \( F^p = F_\ast^p \).

(1.1.5) (Griffiths Transversality) the canonical connection

\[ \nabla : V_0 \rightarrow \Omega^1_S \otimes_{\mathcal{O}_0} V_0 \]

 sede and \( F^\ast \) satisfy

\[ \nabla(F^p) \subset \Omega^1_S \otimes_{\mathcal{O}_0} F^{p-1} \]

(1.1.6) (Polarization) for every \( s \in S \) the map \( \psi_s \) defines a polarization of the Hodge structure on \( V_{s, \mathbb{Z}} \):

(1.1.6.1) \( \psi_s : V_{s, \mathbb{Z}} \otimes_{\mathbb{Z}} V_{s, \mathbb{Z}} \rightarrow \mathbb{Z}(w) \)

is a morphism of Hodge structures

\[ \psi_s (V_{s, \mathbb{Z}} \otimes V_{s, \mathbb{Z}}^\prime) = (0) \text{ unless } p + p' = q + q' = w \]

(1.1.6.2) \( (2\pi i)^w \psi( i^{p-q} x, x) > 0 \) for \( 0 \neq x \in V_s^{p,q} \).
Example \((1.1.7)\)

Assume \(w = 1\), \(V_Z = \mathbb{Z}^2\) is constant and

\[
\psi : V_Z \otimes V_Z \to \frac{1}{2\pi i} \mathbb{Z},
\]

\[
\psi ((b), (c)) = \frac{1}{2\pi i} (ad - bc)
\]

Then in any point \(s \in S\) we have a weight 1 Hodge structure of rank 2.

Hence \(\dim V_s^{1,0} = \dim V_s^{0,1} = 1\), and

\[
P^1_s = V_s^{1,0} = \left\langle \left(\frac{1}{\tau(s)}\right) \right\rangle, \quad V_s^{0,1} = \left\langle \left(\frac{1}{\bar{\tau}(s)}\right) \right\rangle
\]

for some \(\tau(s) \in \mathbb{C}\).

(1.1.4): \(\tau(s)\) is a holomorphic function.

(1.1.5): \(\tau(s)\) is a constant since \(\psi F^0 = V_0\)

(1.1.6.1): \(\psi ((1), (1)) = 0\)

(1.1.6.2): \(2\pi i \psi ((i), (\bar{1} - i)) = \frac{1}{2\pi i} (i \bar{\tau} - i \tau)\)

\[
= 2 \text{Im}(\tau) > 0
\]

So \(\text{Im}(\tau(s)) > 0\).
Conclusion: In the case \( w = 1 \), \( V_2 = \mathbb{Z}^2 \) and \( H \) as above, the "universal" PHS \( H \) lies over

\[ H = \{ \xi \in C \mid \text{Im}(\xi) > 0 \} \]

with \( H \) itself obtained by pulling back via a "period" map

\[ S \longrightarrow H \]

1.2 PVHS and algebraic geometry

Let \( f : X \longrightarrow S \) be a smooth projective morphism of algebraic varieties over \( C \), with connected fibres of dimension \( d \)

and assume \( S \) non-singular as well.

Consider

\[ V_2 = R^d f_*(\mathbb{Z}) / \text{torsion} \]

Since \( f \) is topologically a fibration, this is a local system on \( S \), and

\[ V_{s, 2} = H^w(X_s, \mathbb{Z}) / \text{torsion} \]
The Hodge structures: $V^p,q_s = H^q(X_s, \Omega^p)$ where we use that $H^q(X_s, \mathbb{C}) \otimes \mathbb{C} = H^q(X_s, \mathbb{C}) = H^q(X_s, \Omega^p)$. The cup product gives a map $f$

$$V^p \otimes V^q \to R^{2w} f^* \mathbb{Z} \otimes \mathbb{Z}$$

and we set $w(w-1)$

$$\psi(x, y) := (-1)^{w-1} \frac{1}{(2\pi i)^w} \int_{x+y}$$

All axioms are satisfied except for possibly (1.1.6.2). To get this we take a relatively ample sheaf $\mathcal{L}$ on $X$

eul set $V^p \to V'_p = \operatorname{Ker}(R^w f^* \mathbb{Z} \to R^{w+2} f^* \mathbb{Z})$

with induced Hodge structures.

Theorem: $(V^p'_s, V^q'_s, \psi'_s)$ is a PHHS.
Example 1.2.2. Consider the family of curves 

\[ X : \quad t \left( x_0^3 + x_1^3 + x_2^3 \right) = x_0 x_1 x_2 \]

over \( S = \mathbb{P}^3(\mathbb{C}) \setminus \{ \frac{1}{3}, \frac{e^{2\pi i}}{3}, \frac{e^{4\pi i}}{3}, \frac{e^{6\pi i}}{3} \} \).

For each \( t \in S \), the curve \( X_t \) is a nonsingular projective curve of genus 1. Hence we obtain a PVHS \( (V, V, \psi) \) over \( S \).

For any \( D \subset S \), we obtain by pull back a PVHS over \( D \) as in example 1.1.7 (since we know \( \psi_t \) is unimodular \( \forall t \in S \)) and hence

\[ D \to T \to \tilde{T} \to \hat{T} \to I \to S \]

Claim 1.2.3: \( T \) is not constant.

Before we prove the claim, let us observe the consequence:
Consequence 1.2.4

For countably many values of $t$ the curve $X_t$ has CM (i.e. corresponds to a $\mathcal{E}\in\mathcal{F}_g$ which is imaginary quadratic).

Reason: such points are dense in $\mathcal{F}_g$.

1.2.5

Three proofs of the claim (1.2.3)

$I)$ Constant $\implies$ Isom. class of $\mathfrak{g}(X_t; \mathbb{Z})$ independent of $t$.

Torrelli

$\implies$ " " " $J(X_t)$ " " "

$\implies$ " " " $X_t$ " " "

If $\omega \in \mathcal{D}$ the false as $X_{\infty} \neq X_{t_0}$ (e.g. compute $j$-invariant or autom prop).

$II)$ Constant $\implies$ $F$ does not move over $D$.

Monodromy

$\implies$ $\mathfrak{g}(F) \subset \mathfrak{g} \cap F \subset \mathcal{D}$

$\implies$ $\mathfrak{g}(F) \subset \mathfrak{g} \cap F \subset S$

$\implies$ $\mathfrak{g}$ has a rank 1 local subsystem over $S$, namely $\mathfrak{g}$.

$\implies$ The monodromy representation

$\rho : \pi_1(S, t_0) \rightarrow \text{Aut}(V_{t_0, \mathbb{Z}})$. 
preserves \( V_{t_0}^{1,0} \subset V_{t_0, \mathbb{C}} \)

Easy

\[ \Rightarrow \text{Im}(\rho) \text{ does not have a nontrivial unipotent element} \]

False: Look at \( \rho(\text{loop around } t=0) = (0, 1) \).

(1.253) (III) (Infinitesimal) For Griffiths transversality

\[ \nabla(F^p) \subset \Omega^1 \otimes F^{p-1}, \text{ and this gives linear maps} \]

\[
\begin{align*}
F^p_{/F^p_s} &\rightarrow \Omega^1_{/F^p_s} \otimes F^{p+1}_{/F^p_s} \\
\text{or} \quad T_{S, s} \otimes V_{p, q}^{p-1, q+1} &\rightarrow V_{s}
\end{align*}
\]

If the variation is "constant" then these are zero. In the set up of this line is equal to the composition

\[
T_{S, s} \otimes H^q(X_s, \Omega^p) \rightarrow H^q(X_s, T_{X_s}) \otimes H^q(X_s, \Omega^p)
\]

\[ \downarrow \quad \text{Koszul} \]

\[ \text{H}^{q+1}(X_s, \mathbb{C}^{p+1}) \]
where
\[ \kappa : T_{s_0} \to H^d(X_{s_0}, T_{X_{s_0}}) \]
is the Kodaira–Spencer map. A computation shows this to be nonzero in this case.

1.2.6

Remark In case of a Lefschetz pencil
\[ X_t : tF = G, \quad t, G \in \Gamma(P, \mathcal{L}) \]
inside \( P \) we get
\[ 0 \rightarrow T_{X_{t_0}} \rightarrow T_P|_{X_{t_0}} \rightarrow N_{X_{t_0}} \rightarrow 0 \]
\[ N_{X_{t_0}} \cong \mathcal{L}|_{X_{t_0}} \]
and
\[ \kappa(\frac{\partial}{\partial t}) = \text{Image of } F \text{ via boundary} \]
\[ H^0(N_{X_{t_0}}^* P) \to H^1(T_{X_{t_0}}). \]

Reference P. Deligne, Local behavior of Hodge structures at infinity.