Notes on fermionic Fock space for number theorists

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This is the Mar. 8, 2000 version of my "notebook". It is a compilation of exercises, worked examples and key references (along with provocative remarks) that I have compiled in order to help myself and (fingers crossed) others learn their way around fermionic Fock space. Eventually (again, fingers crossed) the notebook will become a monograph suitable for use by, say, second year graduate students with an interest in number theory.

Contents

CHAPTER 1

Essential tools

Throughout this chapter we fix a commutative artinian local ring k with maximal ideal m. We call elements of k *scalars* and elements of k-modules *vectors*. We denote the set of rational integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. We denote the cardinality of a set S by $|S|$. The principal results of this chapter are Theorems 15.10 and 17.1.

1. Linear algebra over an artinian local ring

EXAMPLE 1.1. Since k is artinian and local, the following also hold:

- k is noetherian.
- m is the set of zero divisors of k.
- m is nilpotent.

We refer the reader to [**Matsumura CRT**] for background in commutative algebra.

EXAMPLE 1.2. For any power q of a prime number the quotient $\mathbb{Z}/q\mathbb{Z}$ is an artinian local ring.

EXAMPLE 1.3. Let $t_1, \ldots, t_n$ be independent variables. Let k_0 be a field. Let I be an ideal of the power series ring $k_0[[t_1, \ldots, t_n]]$ contained in the maximal ideal $(t_1, \ldots, t_n)$ and containing the ideal $(t_1^N, \ldots, t_n^N)$ for some positive integer N. The quotient $k_0[[t_1, \ldots, t_n]]/I$ is an artinian local ring.

EXAMPLE 1.4. Let X be an n by n matrix with entries in the maximal ideal m. There exist unique matrices Y and Z with entries in m such that Y is upper triangular ($Y_{ij} \neq 0 \Rightarrow i \leq j$), Z is strictly lower triangular ($Z_{ij} \neq 0 \Rightarrow i > j$), and $(1+X) = (1+Y)(1+Z)$. The idea developed in this example is often exploited in the sequel.

EXAMPLE 1.5. Let n be a positive integer. Put $\mathcal{G} := \mathrm{GL}_n(k)$. Let $\mathcal{B} \subseteq \mathcal{G}$ be the subgroup consisting of upper triangular matrices. Let $\mathcal{U} \subseteq \mathcal{G}$ be the subgroup consisting of matrices differing from the identity matrix by a strictly lower triangular matrix with entries in m. Let $\mathcal{W} \subseteq \mathcal{G}$ be the subgroup consisting of permutation matrices. One has a disjoint union decomposition $\mathcal{G} = \coprod_{W \in \mathcal{W}} \mathcal{U}\mathcal{B}W\mathcal{B}$.

Example 1.6. Let n be a positive integer. If there exists $t \in k^\times$ such that $1 - t \in k^\times$, then $\mathrm{SL}_n(k)$ is the commutator subgroup of $\mathrm{GL}_n(k)$. Hint:

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{x}{t-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & \frac{x}{t-1} \\ 0 & 1 \end{bmatrix}^{-1} \quad (x \in k)$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (y \in k^\times)$$

Definition 1.7. Let n be a positive integer. An $2n$ by $2n$ matrix A with scalar entries is said to be *split orthogonal* if $A \in \mathrm{GL}_{2n}(k)$ and A preserves the quadratic form $h \mapsto \sum_{i=1}^n h_i h_{2n+1-i}$ on the space of column vectors of length $2n$ with scalar entries.

Definition 1.8. Let n be a positive integer. Given an n by n matrix A with scalar entries, let $A^\dagger$ be the matrix given by the rule

$$(A^\dagger)_{ij} := A_{n+1-j,n+1-i}$$

for $i, j = 1, \ldots, n$. In other words, $A^\dagger$ is obtained from A by reflecting in the anti-diagonal $\{i + j = n + 1\}$. We say that an n by n matrix A with scalar entries is *dagger-alternating* if $A^\dagger + A = 0$ and A vanishes on the anti-diagonal.

Example 1.9. Let n be a positive integer. Fix a $2n$ by $2n$ matrix A with scalar entries. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} d^\dagger & b^\dagger \\ c^\dagger & a^\dagger \end{bmatrix}$$

be the decomposition of A and corresponding decomposition of $A^\dagger$ into n by n blocks. The matrix A is split orthogonal if and only if

$$ad^\dagger + bc^\dagger = 1, \quad d^\dagger a + b^\dagger c = 1$$

and the matrices

$$c^\dagger a, \quad d^\dagger b, \quad ab^\dagger, \quad cd^\dagger$$

are dagger-alternating. An $2n$ by $2n$ permutation matrix W is split orthogonal if and only if the $2n$ by $2n$ permutation matrix

$$\begin{bmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{bmatrix}$$

commutes with W.

EXAMPLE 1.10. Let n be a positive integer and let $\mathcal{G} \subseteq \mathrm{GL}_{2n}(k)$ be the group of split orthogonal matrices. Let $\mathcal{B} \subseteq \mathcal{G}$ be the subgroup consisting of upper triangular matrices. Let $\mathcal{U} \subseteq \mathcal{G}$ be the subgroup consisting of matrices differing from the identity matrix by a strictly lower triangular matrix with entries in m. Let $\mathcal{W} \subseteq \mathcal{G}$ be the subgroup consisting of permutation matrices. One has a disjoint union decomposition $\mathcal{G} = \coprod_{W\in\mathcal{W}} \mathcal{U}\mathcal{B}W\mathcal{B}$. Let $\mathcal{P} \subseteq \mathcal{G}$ be the subgroup consisting of matrices vanishing in the lower left n by n block. Let $R \in \mathcal{W}$ be the permutation matrix representing the transposition exchanging n and $n+1$. The group $\mathcal{G}$ is generated by the subgroup $\mathcal{P}$ and the matrix R.

EXAMPLE 1.11. Let E be a k-module and let $V \subseteq E$ be a k-submodule.

- If V is finitely generated over k and E is free over k, then V is contained in a finitely generated free k-submodule of E.
- If E is free and $xV = 0$ for some $0 \neq x \in k$, then $V \subseteq mE$.
- If V is free over k, then $V \cap mE = mV$.
- If $V + mE = E$, then $E = V$.

The last assertion is a version of Nakayama's Lemma; note that it is not necessary to assume that E/V is finitely generated.

DEFINITION 1.12. We say that a k-module E is *flat* if for every positive integer n, column vector e of length n with entries in E, and row vector x of length n with scalar entries such that $xe = 0$, there exists some positive integer N, matrix A of n rows and N columns with scalar entries, and column vector f of length N with entries in E such that $xA = 0$ and $e = Af$.

PROPOSITION 1.13. *Let E be a flat (e. g. free) k-module. For any family $\{e_i\}_{i\in S}$ of vectors of E, the following assertions are equivalent:*

1. *The family $\{e_i + mE\}_{i\in S}$ is linearly independent over k/m.*
2. *The family $\{e_i\}_{i\in S}$ is linearly independent over k.*
3. *The family $\{e_i\}_{i\in S}$ can be extended to a k-basis of E.*

(In particular, the k-module E is free.)

PROOF. ($1 \Rightarrow 2$) Without loss of generality we may assume that $S = \{1, \ldots, n\}$ for some positive integer n. Arrange the family $\{e_i\}_{i=1}^n$ into a column vector e of length n. Let x be any row vector of length n with scalar entries such that $xe = 0$. By hypothesis there exists a positive integer N and a matrix A of n rows and N columns, and a column vector f of length N with entries in E such that $xA = 0$ and $e = Af$. The matrix A reduced modulo m must have rank n, for otherwise we arrive at a contradiction to our assumption that $e_1 + mE, \ldots, e_n + mE$ are (k/m)-linearly independent. It follows that some n by n minor of A is an invertible scalar, and hence that $x = 0$ by Cramer's rule.

($1\Rightarrow 3$) Taking this implication for granted if $m = 0$, i. e., if k is a field, we may assume without loss of generality that the family $\{e_i + mE\}_{i\in S}$ is a (k/m)-basis for E/mE. Then the family $\{e_i\}_{i\in S}$ spans E over k by Example 1.11, is k-linearly independent by what we have already proved, and hence is already a k-basis for E.

($3\Rightarrow 1,2$) Trivial. □

EXAMPLE 1.14. The following hold:

- Each idempotent k-linear endomorphism α of a free k-module E gives rise to a k-linear direct sum decomposition
$$E = \alpha E \oplus (1 - \alpha)E$$
both summands of which are free k-modules.
- A k-linear map $A \stackrel{\phi}{\rightarrow} B$ of free k-modules is injective (resp. surjective) if and only if the induced map $A/mA \stackrel{\phi \bmod m}{\rightarrow} B/mB$ is injective (resp. surjective).
- Given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of k-modules, if two of the k-modules A, B, C are free, then so is the third and the sequence splits.

EXAMPLE 1.15. Let S be a set and let 2^S be the family of all subsets of S. A subfamily $\Phi \subseteq 2^S$ is called a *boolean ideal* under the following conditions:

- $\emptyset \in \Phi$.
- For all $I, J \in \Phi$ one has $I \cup J \in \Phi$.
- For all $I \in \Phi$ and $J \subseteq S$ one has $I \cap J \in \Phi$.

Given a k-module E and a boolean ideal $\Phi \subseteq 2^S$, put
$$E(S, \Phi) := \{e : S \rightarrow E \mid \{s \in S \mid e(s) \neq 0\} \in \Phi\}.$$
The k-module-valued functor $E \mapsto E(S, \Phi)$ of k-modules is exact. If E is a free k-module, the k-module $E(S, \Phi)$ is again free.

2. Algebras and ideals

DEFINITION 2.1. A *k-algebra* is a k-module equipped with a k-bilinear associative (but possibly noncommutative) product with respect to which there exists a (necessarily unique) two-sided identity. The identity element of a k-algebra $\mathcal{A}$ is assumed to act as the identity on each left (resp. right) $\mathcal{A}$-module. We say that a k-algebra $\mathcal{A}$ is *flat* if flat (and hence free) as a k-module. The group consisting of the elements of a k-algebra $\mathcal{A}$ possessing two-sided inverses is denoted by $\mathcal{A}^\times$. A *homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ of k-algebras is a k-linear map such that $\phi(1) = 1$ and $\phi(aa') = \phi(a)\phi(a')$ for all $a, a' \in \mathcal{A}$.

DEFINITION 2.2. For any positive integer n and k-algebra $\mathcal{A}$, we denote the k-algebra of n by n matrices with entries in $\mathcal{A}$ by $\mathrm{Mat}_n(\mathcal{A})$, and we put $\mathrm{GL}_n(\mathcal{A}) := \mathrm{Mat}_n(\mathcal{A})^\times$. Given positive integers p and q, we denote by $\mathrm{Mat}_{p\times q}(\mathcal{A})$ the set of p by q matrices with entries in $\mathcal{A}$.

DEFINITION 2.3. Fix a k-algebra $\mathcal{A}$. A k-submodule $\mathcal{A}_0 \subseteq \mathcal{A}$ is said to be a *k-subalgebra* if $1 \in \mathcal{A}_0$ and $ab \in \mathcal{A}_0$ for all $a, b \in \mathcal{A}_0$. A k-submodule $\mathcal{I} \subseteq \mathcal{A}$ is said to be a *left ideal* of $\mathcal{A}$ if $ab \in \mathcal{I}$ for all $a \in \mathcal{A}$ and $b \in \mathcal{I}$. A k-submodule $\mathcal{I}^\star \subseteq \mathcal{A}$ is said to be a *right ideal* of $\mathcal{A}$ if $ba \in \mathcal{I}^\star$ for all $a \in \mathcal{A}$ and $b \in \mathcal{I}^\star$. Symbols bearing the superscript $\star$ are reserved for use in denoting right ideals. A k-submodule of $\mathcal{A}$ is said to be a *two-sided* ideal if both a left and a right ideal. We say that a left ideal of $\mathcal{A}$ is *flat* if free as a k-module; similarly we speak of flat right ideals and flat k-subalgebras of $\mathcal{A}$.

EXAMPLE 2.4. Let a k-algebra $\mathcal{A}$ and left ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{A}$ be given. Put

$$(\mathcal{I} : \mathcal{J}) := \{a \in \mathcal{A} \mid \mathcal{J}a \subseteq \mathcal{I}\}.$$

The sequence

$$0 \to \mathcal{I} \subset (\mathcal{I} : \mathcal{J}) \overset{a \mapsto (x+\mathcal{J} \mapsto xa+\mathcal{I})}{\to} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathcal{J}, \mathcal{A}/\mathcal{I}) \to 0$$

is exact.

EXAMPLE 2.5. Put $\mathcal{A} := \mathrm{Mat}_n(k)$. Fix $x \in \mathcal{A}$. Let $\mathcal{J} \subset \mathcal{A}$ be the flat left ideal consisting of matrices with vanishing first column. Let $\mathcal{I}^\star \subset \mathcal{A}$ be the flat right ideal consisting matrices with vanishing first row. The following assertions hold:

- $x - c \in \mathcal{I}^\star + \mathcal{J}$ for a unique scalar c.
- If $\mathcal{A}x\mathcal{A} \subseteq \mathcal{I}^\star + \mathcal{J}$, then $x = 0$.
- If $x\mathcal{A} \subseteq \mathcal{J}$, then $x = 0$.
- If $\mathcal{A}x \subseteq \mathcal{I}^\star$, then $x = 0$.
- If $x\mathcal{A} \subseteq \mathcal{I}^\star + \mathcal{J}$, then $x \in \mathcal{I}^\star$.
- If $\mathcal{A}x \subseteq \mathcal{I}^\star + \mathcal{J}$, then $x \in \mathcal{J}$.
- If $\mathcal{J}x \subseteq \mathcal{J}$, then $x - c \in \mathcal{J}$ for a unique scalar c.
- If $x\mathcal{I}^\star \subseteq \mathcal{I}^\star$, then $x - c \in \mathcal{I}^\star$ for a unique scalar c.
- If x is central in $\mathcal{A}$, then x is a scalar.
- $(x + \mathcal{J}) \cap \mathcal{A}^\times \neq \emptyset$ if and only if $x \notin \mathcal{J} + m\mathcal{A}$.
- $(x + \mathcal{I}^\star) \cap \mathcal{A}^\times \neq \emptyset$ if and only if $x \notin \mathcal{I}^\star + m\mathcal{A}$.

There exist triples $(\mathcal{A}, \mathcal{I}^\star, \mathcal{J})$ consisting of an infinite rank flat k-algebra $\mathcal{A}$, a flat right ideal $\mathcal{I}^\star$ of $\mathcal{A}$ and a flat left ideal $\mathcal{J}$ of $\mathcal{A}$ such that for every $x \in \mathcal{A}$ all the statements above make sense and remain true. The theory of Clifford algebras provides natural examples of such triples.

EXAMPLE 2.6. Let E be a free k-module. Put

$$\mathcal{T}(E) := \bigoplus_{n=0}^{\infty} \underbrace{E \otimes_k \cdots \otimes_k E}_{n}.$$

The k-module $\mathcal{T}(E)$ contains E as a k-submodule, forms a k-algebra with unit under the tensor product operation, and has the following universal property in the category of k-algebras:

- For any k-algebra $\mathcal{A}$ and k-linear map $\phi : E \to \mathcal{A}$ there exists a unique extension of ϕ to a k-algebra homomorphism $\mathcal{T}(E) \to \mathcal{A}$.

One calls $\mathcal{T}(E)$ the *tensor algebra* of E over k. Now let $\{e_i\}_{i \in S}$ be a k-basis for E. We declare a *word* W in S to be an element of the disjoint union $\coprod_{n=0}^{\infty} S^n$. Given a word $W = (i_1, \ldots, i_n) \in S^n$, put

$$e_W := e_{i_1} \otimes \cdots \otimes e_{i_n} \in \mathcal{T}(E).$$

The family $\{e_W\}$ indexed by words in S is a k-basis for $\mathcal{T}(E)$. One has

$$e_W \otimes e_{W'} = e_{WW'}$$

for all words W and W' in S, where WW' denotes the concatenation of W and W'. Every k-algebra is of the form $\mathcal{T}(E)/\mathcal{I}$ for some free k-module E and two-sided ideal $\mathcal{I} \subseteq \mathcal{T}(E)$.

EXAMPLE 2.7. Let E be a free k-module. Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{T}(E)$ generated by the set $\{e \otimes e \mid e \in E\}$. The quotient $\bigwedge(E) := \mathcal{T}(E)/\mathcal{I}$ is called the *exterior algebra* of E over k. The product in $\bigwedge(E)$ is traditionally denoted by the wedge symbol $\wedge$. It is well known that the natural map $(e \mapsto e+\mathcal{I}) : E \to \bigwedge(E)$ is injective; E is traditionally identified via this map with a k-submodule of $\bigwedge(E)$. The exterior algebra has the following universal property:

- For all k-algebras $\mathcal{A}$ and k-linear maps $\phi : E \to \mathcal{A}$ such that
$$\phi(e)^2 = 0$$
for all $e \in E$, there exists a unique k-algebra homomorphism $\bigwedge(E) \to \mathcal{A}$ extending ϕ.

Let $\{e_i\}_{i\in S}$ be a k-basis for E indexed by a linearly ordered set S. Put
$$e_I := e_{i_1} \wedge \cdots \wedge e_{i_r} \in \bigwedge(E)$$
for each finite subset $I = \{i_1 < \cdots < i_r\} \subseteq S$. The family $\{e_I\}$ indexed by finite subsets of S is a k-basis for $\bigwedge(E)$. One has
$$e_I \wedge e_J = \left\{ \begin{array}{ll} (-1)^{|\{(i,j)\in I\times J | i>j\}|} e_{I\cup J} & \text{if } I \cap J = \emptyset \\ 0 & \text{if } I \cap J \neq \emptyset \end{array} \right.$$
for all finite subsets $I, J \subseteq S$.

EXAMPLE 2.8. Suppose E has a k-basis $e_1, \ldots, e_n$. Let $A \in \mathrm{Mat}_n(k)$ be given. Put
$$f_j := \sum_{i=1}^{n} A_{ij} e_i$$
for $j = 1, \ldots, n$. Then
$$f_1 \wedge \cdots \wedge f_n = (\det A) e_1 \wedge \cdots \wedge e_n.$$
Thus the theory of determinants is linked to the theory of exterior algebras.

3. Laurent series

DEFINITION 3.1. Let t be a variable. A series $f = \sum_{i\in\mathbb{Z}} a_i t^i$ with scalar coefficients a_i vanishing for $i \ll 0$ is called a *Laurent series* in t; if the coefficients a_i vanish for $i < 0$ we call f a *power series* in t. We denote the k-module of Laurent series in t with scalar coefficients by $k((t))$ and the k-submodule of power series in t by $k[[t]]$. The k-modules $k((t))$ and $k[[t]]$ are free. Under the standard rule for series multiplication the k-module $k((t))$ becomes a commutative ring with unit and $k[[t]]$ a subring. The ring $k((t))$ is artinian and local. The unique maximal ideal of $k((t))$ is generated by m and consists of all

Laurent series in t with coefficients in m. The ring $k[[t]]$ is noetherian and local but not artinian.

Definition 3.2. Given a Laurent series $f = \sum_i a_i t^i \in k((t))$, put

$$w(f) := \min(\{i \mid a_i \not\equiv 0 \bmod m\} \cup \{+\infty\}),$$

thereby defining the *winding number* of f. One has

$$w(f) < +\infty \;\Leftrightarrow\; f \in k((t))^\times$$

for all $f \in k((t))$. One has

$$w(fg) = w(f) + w(g), \quad w(f+g) \geq \min(w(f), w(g))$$

for all $f, g \in k((t))$.

Example 3.3. Fix a positive integer n and $f \in k[[t]]$ of winding number n. The *Weierstrass Division Theorem* says that for each power series $g \in k[[t]]$ there exist unique $q, r \in k[[t]]$, the latter a polynomial in t of degree $< n$, such that $f = gq + r$. The *Weierstrass Preparation Theorem* says that there exist unique $u, r \in k[[t]]$, the former a power series in t with invertible constant term and the latter a polynomial in t of degree $< n$ all coefficients of which belong to m, such that $f = (t^n + r)u$.

Definition 3.4. For each $f \in k((t))^\times$ there exists a unique family $\{a_i\}_{i\in\mathbb{Z}}$ of scalars with the following properties:

- $a_i = 0$ for $i \ll 0$.
- $a_i \equiv 0 \bmod m$ for $i < 0$.
- $a_0 \not\equiv 0 \bmod m$.
- $f^{-1}t^{w(f)} \displaystyle\prod_{i=-\infty}^{N} \begin{cases} (1 - a_i t^i) & \text{if } i \neq 0 \\ a_0 & \text{if } i = 0 \end{cases} \in 1 + t^{N+1}k[[t]]$ for $N \geq 0$.

We call $\{a_i\}_{i\in\mathbb{Z}}$ the family of *Witt parameters* of f.

Example 3.5. Let n be a positive integer. Put $\mathcal{G} := \mathrm{GL}_n(k((t)))$. Given $A \in \mathcal{G}$, write $A = \sum_{i\in\mathbb{Z}} A^{(i)}t^i$, where the coefficients $A^{(i)}$ are n by n matrices with scalar coefficients vanishing for $i \ll 0$. Let $\mathcal{B} \subseteq \mathcal{G}$ be the subgroup consisting of matrices A such that $A^{(i)} = 0$ for $i < 0$ and $A^{(0)}$ is upper triangular. Let $\mathcal{U} \subseteq \mathcal{G}$ be the subgroup consisting of matrices A such that $A^{(i)} = 0$ for $i > 0$, $A^{(0)}$ differs from the identity matrix by a strictly lower triangular matrix with entries in m, and $A^{(i)}$ has all entries in m for $i < 0$. Let $\mathcal{W} \subseteq \mathcal{G}$ be the subgroup consisting of matrices factoring as a permutation matrix times a diagonal matrix with power of t on the diagonal. One has a disjoint union decomposition $\mathcal{G} = \coprod_{W\in\mathcal{W}} \mathcal{U}\mathcal{B}W\mathcal{B}$.

EXAMPLE 3.6. For each $f \in k((t))$ of winding number 0 there exists a positive integer N such that $t^N f^i \in k[[t]]$ for all $i \in \mathbb{Z}$.

4. Almost upper triangular matrices

DEFINITION 4.1. Let I and J be subsets of $\mathbb{Z}$ and let A be an I by J matrix with scalar entries. Put

$$\operatorname{supp} A := \{(i,j) \in I \times J \ \mid A_{ij} \neq 0\},$$

thereby defining the *support* of A. We say that A is *finitely supported* if

$$|\operatorname{supp} A| < \infty.$$

We say that A is *upper triangular* if

$$\{(i,j) \in \operatorname{supp} A \mid i > j\} = \emptyset,$$

strictly upper triangular if

$$\{(i,j) \in \operatorname{supp} A \mid i \geq j\} = \emptyset,$$

and *almost upper triangular* if

$$\forall n \in \mathbb{Z} \ \ |\{(i,j) \in \operatorname{supp} A \mid i \geq n \geq j\}| < \infty.$$

The "lower" analogues of the preceding "upper" notions are defined in the obvious way. We denote the set of I by J almost upper triangular matrices with scalar entries by $\mathcal{Q}(I,J)$. If $I = J$ we write $\mathcal{Q}(I)$, and if $I = J = \mathbb{Z}$ we write $\mathcal{Q}$. We permit one or both of I or J to be the empty set, in which case we set $\mathcal{Q}(I,J) = \{0\}$.

DEFINITION 4.2. Let $\mathcal{H}$ be the k-module consisting of column vectors h with scalar entries indexed by $\mathbb{Z}$ such that $h_i = 0$ for all $i \gg 0$. For each subset $I \subseteq \mathbb{Z}$, put

$$\mathcal{H}(I) := \{h \in \mathcal{H} \mid \operatorname{supp} h \subset I\},$$

where

$$\operatorname{supp} h := \{i \in \mathbb{Z} \mid h_i \neq 0\}$$

for each $h \in \mathcal{H}$.

EXAMPLE 4.3. For all subsets $I, J \subseteq \mathbb{Z}$, the corresponding k-module $\mathcal{Q}(I,J)$ is free. For all subsets $I \subseteq \mathbb{Z}$, the corresponding k-module $\mathcal{H}(I)$ is free.

EXAMPLE 4.4. Let $L \subseteq \mathcal{H}$ be a flat k-submodule such that the k-modules $L \cap \mathcal{H}(\mathbb{Z} \setminus \mathbb{N})$ and $\mathcal{H}/(L + \mathcal{H}(\mathbb{Z} \setminus \mathbb{N}))$ are finitely generated. Then there exists a subset $I \subset \mathbb{Z}$ such that $\sup I < +\infty$, $\inf \mathbb{Z} \setminus I > -\infty$ and $\mathcal{H} = \mathcal{H}(I) \oplus L$.

DEFINITION 4.5. A *gauge* is a function $\sigma : \mathbb{Z} \to \mathbb{Z}$ with the following properties:

- $\sigma(n) \leq \sigma(n+1)$ for all $n \in \mathbb{Z}$.
- $\lim\limits_{n \to -\infty} \sigma(n) = -\infty$.
- $\lim\limits_{n \to +\infty} \sigma(n) = +\infty$.

Given subsets $I, J \subseteq \mathbb{Z}$, an I by J matrix A with scalar entries and a gauge σ, we say that σ *dominates* the matrix A if

$$A_{ij} \neq 0 \Rightarrow i \leq \sigma(j)$$

for all $(i, j) \in I \times J$.

EXAMPLE 4.6. For any gauges σ and τ, the functions

$$\left(n \mapsto \left\{ \begin{array}{l} \sigma(\tau(n)) \\ \max(\sigma(n), \tau(n)) \\ \min(\sigma(n), \tau(n)) \\ \min\{j \in \mathbb{Z} \mid n \leq \sigma(j)\} \\ \max\{j \in \mathbb{Z} \mid \sigma(j) \leq n\} \end{array} \right. \right) : \mathbb{Z} \to \mathbb{Z}$$

are again gauges.

LEMMA 4.7. *Let subsets $I, J \subseteq \mathbb{Z}$ and an I by J matrix A with scalar entries be given. The matrix A is almost upper triangular if and only if there exists a gauge σ dominating A.*

PROOF. ($\Rightarrow$) For each $n \in \mathbb{Z}$, put

$$S_n := \{\ell \in \mathbb{Z} \mid \ell \leq n\} \cup \bigcup_{\substack{j \in J \\ n \geq j}} \{i \in I \,|A_{ij} \neq 0\} \subseteq \mathbb{Z}.$$

Clearly, one has $S_n \neq \emptyset$, $S_n \subseteq S_{n+1}$ and $\bigcup S_n = \mathbb{Z}$. By hypothesis, S_n is bounded above and one has $\bigcap S_n = \emptyset$. The function

$$(n \mapsto \max S_n) : \mathbb{Z} \to \mathbb{Z}$$

is therefore a gauge dominating A.

($\Leftarrow$) Let σ be a gauge dominating A. For all $n \in \mathbb{Z}$ one has

$$\left\{ (i, j) \in I \times J \,\middle|\, \begin{array}{l} i \geq n \geq j \\ A_{ij} \neq 0 \end{array} \right\} \subseteq \left\{ (i, j) \in I \times J \,\middle|\, \begin{array}{l} i \geq n \geq j \\ i \leq \sigma(n) \\ n \leq \sigma(j) \end{array} \right\},$$

and the set on the right is finite. $\square$

DEFINITION 4.8. Let subsets $I_0, I_1, I_2 \subseteq \mathbb{Z}$ be given. For $\nu = 1, 2$, let a matrix $A^{(\nu)} \in \mathcal{Q}(I_{\nu-1}, I_\nu)$ and a gauge σ_ν dominating $A^{(\nu)}$ be given. One has

$$A^{(1)}_{i_0 i_1} A^{(2)}_{i_1 i_2} \neq 0 \Rightarrow \begin{cases} i_1 \geq \min\{j \in \mathbb{Z} \mid i_0 \leq \sigma_1(j)\} > -\infty \\ i_1 \leq \sigma_2(i_2) \\ i_0 \leq \sigma_1(\sigma_2(i_2)) \end{cases}$$

for all $(i_0, i_1, i_2) \in I_0 \times I_1 \times I_2$. We refer to this estimate as the *gauge trick*. It follows that the product $A^{(1)} A^{(2)}$ is a well defined I_0 by I_2 matrix with scalar entries dominated by the gauge $\sigma_1 \circ \sigma_2$. Thus a natural product

$$\mathcal{Q}(I_0, I_1) \times \mathcal{Q}(I_1, I_2) \to \mathcal{Q}(I_0, I_2)$$

is defined.

EXAMPLE 4.9. Consider the $\mathbb{N}$ by $\mathbb{N}$ matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \\ \vdots & \vdots & & \ddots & \ddots \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

One has

$$(B^T A)B = \begin{bmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \end{bmatrix}, \quad B^T(AB) = \begin{bmatrix} 0 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \end{bmatrix}.$$

Moral: Associativity of products of infinite matrices cannot be taken for granted.

DEFINITION 4.10. Let subsets $I, J \subseteq \mathbb{Z}$ be given. Let an I by J matrix A with scalar entries and a sequence $\{A^{(n)}\}_{n=1}^\infty$ of I by J matrices with scalar entries be given. The sequence $\{A^{(n)}\}$ is said to be *uniformly dominated* if there exists a gauge dominating all of the matrices $A^{(n)}$. We say that the sequence $\{A^{(n)}\}$ *converges entrywise* to A or that the *entrywise limit* of the sequence $\{A^{(n)}\}$ is A if for every pair $(i, j) \in I \times J$ there exist only finitely many n such that $A^{(n)}_{ij} \neq A_{ij}$. We say that the sequence $\{A^{(n)}\}$ is *entrywise convergent* if there exists some (necessarily unique) I by J matrix with scalar entries to which the sequence $\{A^{(n)}\}$ converges entrywise.

EXAMPLE 4.11. A sequence $\{A^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{Q}$ is uniformly dominated if and only if for all $(i_0, j_0) \in \mathbb{Z}^2$ the set

$$\left(\bigcup_{n=1}^{\infty} \operatorname{supp} A^{(n)}\right) \bigcap \left\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, \begin{array}{l} i \geq i_0 \\ j \leq j_0 \end{array}\right\}$$

is finite.

LEMMA 4.12. *Let matrices $A, B \in \mathcal{Q}$ and uniformly dominated sequences $\{A^{(n)}\}_{n=1}^{\infty}$ and $\{B^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{Q}$ be given. Assume that $\{A^{(n)}\}$ converges entrywise to A and that $\{B^{(n)}\}$ converges entrywise to B. Then the sequence $\{A^{(n)}B^{(n)}\}_{n=1}^{\infty}$ is uniformly dominated and converges entrywise to AB.*

PROOF. Let σ be a gauge dominating all the matrices $A^{(n)}$ and hence also the matrix A. Let τ be a gauge dominating all the matrices $B^{(n)}$ and hence also the matrix B. By the gauge trick, the gauge $\sigma \circ \tau$ dominates all the matrices $A^{(n)}B^{(n)}$ and hence the sequence $\{A^{(n)}B^{(n)}\}$ is uniformly dominated. Now fix $(i_0, j_0) \in \mathbb{Z} \times \mathbb{Z}$ and let S be the finite set consisting of all $\ell \in \mathbb{Z}$ such that $\ell \leq \tau(j_0)$ and $i_0 \leq \sigma(\ell)$. By hypothesis there exists $n_0 \in \mathbb{N}$ such that $A_{i_0\ell}^{(n)} = A_{i_0\ell}$ and $B_{\ell j_0}^{(n)} = B_{\ell j_0}$ for all $\ell \in S$ and all $n \geq n_0$. By the gauge trick one has

$$(A^{(n)}B^{(n)})_{i_0j_0} = \sum_{\ell \in S} A_{i_0\ell}^{(n)} B_{\ell j_0}^{(n)} = \sum_{\ell \in S} A_{i_0\ell} B_{\ell j_0} = (AB)_{i_0j_0}$$

for all $n \geq n_0$, and hence $\{A^{(n)}B^{(n)}\}$ converges entrywise to AB as claimed. $\square$

DEFINITION 4.13. One can deduce the associativity of multiplication of almost upper triangular matrices from the associativity of multiplication of finitely supported matrices via Lemma 4.12. Thus, in particular, $\mathcal{Q}$ becomes a ring with unit and $\mathcal{H}$ a left $\mathcal{Q}$-module under the standard rule for matrix multiplication. We sometimes refer to $\mathcal{Q}^{\times}$ as the *Japanese group* since in various versions this group figures prominently in the works of the Kyoto school of soliton theory.

EXAMPLE 4.14. One has

$$A\mathcal{H}\left(\{n \in \mathbb{Z} \mid n \leq n_0\}\right) \subseteq \mathcal{H}\left(\{n \in \mathbb{Z} \mid n \leq \sigma(n_0)\}\right)$$

for all $A \in \mathcal{Q}$, gauges σ dominating A, and $n_0 \in \mathbb{Z}$.

EXAMPLE 4.15. Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a sequence of strictly upper triangular elements of $\mathcal{Q}$ with pairwise disjoint supports. Define a sequence $\left\{A^{(n)}\right\}_{n=1}^{\infty}$ inductively by the rules

$$A^{(1)} := X^{(1)}, \quad A^{(n+1)} := \left(1 + X^{(n)}\right) A^{(n)}.$$

The sequence $\{A^{(n)}\}$ is uniformly dominated and entrywise convergent. The entrywise limit of the sequence $\{A^{(n)}\}$ differs from the identity by a strictly upper triangular matrix.

DEFINITION 4.16. Given subsets $I, J \subseteq \mathbb{Z}$, we denote by $\mathcal{Q}(I,J)^\times$ the subset of $\mathcal{Q}(I,J)$ consisting of matrices A such that for some (necessarily unique) $B \in \mathcal{Q}(J,I)$ one has $AB = 1 \in \mathcal{Q}(I)$ and $BA = 1 \in \mathcal{Q}(J)$. If $I = J$, we write $\mathcal{Q}(I)^\times$, and if $I = J = \mathbb{Z}$, we write $\mathcal{Q}^\times$.

EXAMPLE 4.17. Let gauges σ and τ dominating matrices $A, B \in \mathcal{Q}$, respectively, be given. The composition $\sigma \circ \tau$ dominates the product AB. Moreover, if $AB = 1$, then $n \leq \sigma(\tau(n))$ for all $n \in \mathbb{Z}$.

EXAMPLE 4.18. Let I be any subset of $\mathbb{Z}$ and let $X \in \mathcal{Q}(I)$ be a matrix all entries of which belong to the maximal ideal m. The matrix X is nilpotent and $1 - X \in \mathcal{Q}(I)^\times$.

EXAMPLE 4.19. Let I and J be any subsets of $\mathbb{Z}$. A matrix $A \in \mathcal{Q}(I,J)$ has a left inverse in $\mathcal{Q}(J,I)$ if and only if A has a left inverse modulo m. Similarly, A has a right (resp. two-sided) inverse in $\mathcal{Q}(J,I)$ if and only if A has a right (resp. two-sided) inverse modulo m.

EXAMPLE 4.20. Let I be any subset of $\mathbb{Z}$ and let X be a strictly upper triangular I by I matrix with scalar entries. One has $(1 - X) \in \mathcal{Q}(I)^\times$. The sequence

$$\left\{1 + \sum_{n=1}^{N} X^n\right\}_{N=1}^{\infty}$$

is uniformly dominated and converges entrywise to $(1 - X)^{-1}$.

EXAMPLE 4.21. Let I be any subset of $\mathbb{Z}$ and let $A \subseteq \mathcal{Q}(I)$ be an upper triangular matrix. One has $A \in \mathcal{Q}(I)^\times$ if and only if every diagonal entry of A is an invertible element of k. If $A \in \mathcal{Q}(I)^\times$, then the inverse $A^{-1} \in \mathcal{Q}(I)$ is again upper triangular.

Definition 4.22. Let subsets $I, J \subseteq \mathbb{Z}$ and an I by J matrix A with scalar entries be given. Let $I = \coprod_{i=1}^{p} I_i$ and $J = \coprod_{j=1}^{q} J_j$ be partitions of I and J, respectively. We define the

$$\begin{bmatrix} I_1 \\ \vdots \\ I_p \end{bmatrix} \times \begin{bmatrix} J_1 \\ \vdots \\ J_q \end{bmatrix}^T$$

block decomposition of A to be the p by q matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1q} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pq} \end{bmatrix}$$

in which a_{ij} is the I_i by J_j block of A. One has $A \in \mathcal{Q}(I, J)$ if and only one has $a_{ij} \in \mathcal{Q}(I_i, J_j)$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$.

Example 4.23. Let $A \in \mathcal{Q}^\times$ be given. Let $\mathbb{Z} = I \coprod J$ be any partition of $\mathbb{Z}$ into two sets. Suppose that the

$$\begin{bmatrix} I \\ J \end{bmatrix} \times \begin{bmatrix} I \\ J \end{bmatrix}^T$$

block decompositions of A and A^{-1} take the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}.$$

The block a is invertible if and only if the block $\bar{d}$ is invertible.

Definition 4.24. The *bilateral shift* $\mathbf{t} \in \mathcal{Q}$ is defined to be the $\mathbb{Z}$ by $\mathbb{Z}$ matrix with 1's along the superdiagonal $\{i = j - 1\}$ and 0's elsewhere. Given a Laurent series $f = \sum_i a_i t^i \in k((t))$ and a positive integer n, put

$$f(\mathbf{t}^n)_{ij} := \begin{cases} a_{\frac{j-i}{n}} & \text{if } i \equiv j \bmod n \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in \mathbb{Z}$, thereby defining a matrix $f(\mathbf{t}^n) \in \mathcal{Q}$.

Example 4.25. The map

$$(f \mapsto f(\mathbf{t}^n)) : k((t)) \to \mathcal{Q}$$

is an injective k-linear ring homomorphism under which the variable t maps to the n^{th} power $\mathbf{t}^n$ of the bilateral shift. The k-module $\mathcal{H}$ viewed as a left $k((t))$-module via the homomorphism $f \mapsto f(\mathbf{t}^n)$ is free of rank n. A matrix $A \in \mathcal{Q}$ commutes with $\mathbf{t}$ if and only if $A = f(\mathbf{t})$ for some (necessarily unique) $f \in k((t))$.

Example 4.26. Let $\{A^{(n)}\}_{n=1}^\infty$ be a sequence in $\mathcal{Q}^\times$. If the sequences $\{A^{(n)}\}$ and $\{(A^{(n)})^{-1}\}$ are both uniformly dominated and entrywise convergent, the entrywise limit of the sequence $\{A^{(n)}\}$ belongs to $\mathcal{Q}^\times$. The sequence $\{\mathbf{t}^n\}_{n=1}^\infty$ is uniformly dominated and converges entrywise to 0. The sequence $\{\mathbf{t}^{-n}\}_{n=1}^\infty$ fails to be uniformly dominated.

Example 4.27. Fix a positive integer n. Let $\phi \in k((t))$ be a Laurent series of winding number n. There exists unique $A \in \mathcal{Q}$ with the following two properties:

- $\phi(\mathbf{t})A = A\mathbf{t}^n$.
- $A_{ij} = \delta_{ij}$ for all $i, j \in \mathbb{Z}$ such that $-n < j \leq 0$.

Modulo m the matrix A is upper triangular and each diagonal entry is nonzero. It follows that $A \in \mathcal{Q}^\times$.

Example 4.28. Fix a positive integer n. Given $A \in \mathcal{Q}$, let $A^{[n]}$ be the n by n matrix with entries in $\mathcal{Q}$ given by the rule

$$(A^{[n]}_{\mu\nu})_{ij} := A_{in-\mu+1,jn-\nu+1}$$

for $\mu, \nu = 1, \ldots, n$ and $i, j \in \mathbb{Z}$. The map

$$(A \mapsto A^{[n]}) : \mathcal{Q} \to \mathrm{Mat}_n(\mathcal{Q})$$

is an isomorphism of k-algebras. Given $h \in \mathcal{H}$, let $h^{[n]}$ be the column vector of length n with entries in $\mathcal{H}$ given by the rule

$$(h^{[n]}_\mu)_i := h_{in-\mu+1}$$

for $\mu = 1, \ldots, n$ and $i \in \mathbb{Z}$. The map

$$\left(h \mapsto h^{[n]}\right) : \mathcal{H} \to \mathrm{Mat}_{n\times 1}(\mathcal{H})$$

is bijective. One has

$$(Ah)^{[n]} = A^{[n]}h^{[n]}$$

for all $A \in \mathcal{Q}$ and $h \in \mathcal{H}$.

Example 4.29. Fix a positive integer n. One has

$$((\mathbf{t}^n)^{[n]})_{\mu\nu} = \mathbf{t}\delta_{\mu\nu}$$

for $\mu, \nu = 1, \ldots, n$ and hence one has a k-algebra isomorphism

$$(A \mapsto A^{[n]}) : \{A \in \mathcal{Q} \mid A\mathbf{t}^n = \mathbf{t}^n A\} \to \mathrm{Mat}_n(\{A \in \mathcal{Q} \mid A\mathbf{t} = \mathbf{t}A\}).$$

Thus the k-algebras

$$\mathrm{Mat}_n(k((t)))$$

and

$$\{A \in \mathcal{Q} \mid A\mathbf{t}^n = \mathbf{t}^n A\} = \text{commutant of } \mathbf{t}^n \text{ in } \mathcal{Q}$$

are canonically isomorphic.

EXAMPLE 4.30. Fix $\phi \in k((t))^\times$ of positive winding number n. There exists a unique k-algebra homomorphism

$$(f \mapsto f \circ \phi) : k((t)) \to k((t))$$

and a unique multiplicative map

$$(f \mapsto \mathcal{N}_\phi f) : k((t)) \to k((t))$$

such that

$$Af(\mathbf{t}^n)A^{-1} = (f \circ \phi)(\mathbf{t})$$

and

$$\det((A^{-1}f(\mathbf{t})A)^{[n]}) = (\mathcal{N}_\phi f)(\mathbf{t})$$

for all $f \in k((t))$ and $A \in \mathcal{Q}^\times$ such that

$$\phi(\mathbf{t})A = A\mathbf{t}^n.$$

One can verify that

$$w(\mathcal{N}_\phi f) = w(f), \quad w(f \circ \phi) = nw(f)$$

for all $f \in k((t))^\times$. One can verify that via the map $f \mapsto f \circ \phi$ the ring $k((t))$ becomes a finite $k((t))$-algebra of rank n and that $\mathcal{N}_\phi$ is the norm mapping associated in the usual way to that ring extension.

5. Degree theory

DEFINITION 5.1. A k-submodule $P \subseteq \mathcal{H}$ will be called a *parallelotope* if P is a free k-module and

$$\mathcal{H}(\{n \in \mathbb{Z} \mid n \leq n_0\}) \subseteq P \subseteq \mathcal{H}(\{n \in \mathbb{Z} \mid \leq n_1\})$$

for some integers $n_0 \leq n_1$. By Example 4.14, for all parallelotopes $P \subset \mathcal{H}$ and $A \in \mathcal{Q}^\times$, again AP is a parallelotope.

PROPOSITION 5.2. *There exists a unique function*

$$((P_1, P_2) \mapsto [P_1 : P_2]) : \{\text{parallelotopes}\} \times \{\text{parallelotopes}\} \to \mathbb{Z}$$

with the following properties:

1. $[P_0 : P_1]$ *equals the k-rank of P_0/P_1 for all parallelotopes $P_0 \supseteq P_1$.*

2. $[P_0 : P_1] = -[P_1 : P_0]$ *for all parallelotopes P_0 and P_1.*

3. $[P_0 : P_2] = [P_0 : P_1] + [P_1 : P_2]$ *for all parallelotopes P_0, P_1, P_2.*

(We call $[\cdot : \cdot]$ the index function *associated to the family of parallelotopes in $\mathcal{H}$.)*

Proof. There exists a unique function

$$i : \{\text{parallelotopes}\} \times \{\text{parallelotopes}\} \to \mathbb{Z}$$

such that

$$i(P_0, P_1) = (k\text{-rank of } P_0/\mathcal{H}(\{n \le n_0\})) - (k\text{-rank of } P_1/\mathcal{H}(\{n \le n_0\}))$$

for all parallelotopes P_0 and P_1 and integers $n_0 \ll 0$. The function i has all the properties required of an index function. Therefore at least one index function exists. For all parallelotopes P_0, P_1 and P such that $P \subseteq P_0 \cap P_1$, and index functions $[\cdot, \cdot]$, one has

$$[P_0 : P_1] = [P_0 : P] + [P : P_1] = [P_0 : P] - [P_1 : P] = i(P_0, P_1).$$

Therefore at most one index function exists. $\square$

Definition 5.3. By Proposition 5.2 one has

$$[AP_1 : AP_2] = [P_1 : P_2]$$

for all $A \in \mathcal{Q}^\times$ and parallelotopes $P_1, P_2 \subset \mathcal{H}$. It follows that there exists a unique homomorphism

$$\deg : \mathcal{Q}^\times \to \mathbb{Z}$$

such that

$$\deg A = [AP : P]$$

for all $A \in \mathcal{Q}^\times$ and parallelotopes P. We call $\deg A$ the *degree* of A.

Example 5.4. For all $f \in k((t))^\times$ one has

$$\deg f(\mathbf{t}) = -w(f).$$

In particular,

$$\deg \mathbf{t} = -1.$$

For all $A \in \mathcal{Q}^\times$ such that the $\mathbb{N}$ by $1 - \mathbb{N}$ blocks of both A and A^{-1} vanish, one has $\deg A = 0$. For all $A \in \mathcal{Q}^\times$ such that the $1 - \mathbb{N}$ by $\mathbb{N}$ blocks of both A and A^{-1} vanish, one has $\deg A = 0$.

6. Quadratic forms

Definition 6.1. Let E be a free k-module. A *k-quadratic form*

$$q : E \to k$$

is a function such that for some k-bilinear function $b : E \times E \to k$ one has

$$q(e) = b(e, e)$$

for all $e \in E$. We put

$$q(e, f) := q(e + f) - q(e) - q(f) = b(e, f) + b(f, e)$$

for all $e, f \in E$, thus canonically associating a symmetric k-bilinear form $q(\cdot,\cdot)$ to the k-quadratic form $q(\cdot)$. We say that k-linear endomorphisms $\alpha, \beta : E \to E$ are *q-adjoint* if

$$q(\alpha e, f) = q(e, \beta f)$$

for all $e, f \in E$. We say that q is *nondegenerate* if for every $0 \neq e \in E$ there exists $0 \neq f \in E$ such that $q(e, f) \neq 0$.

Example 6.2. A free k-module E equipped with a basis $e, f \in E$ such that $q(xe + yf) = xy$ for all scalars x and y is called a *hyperbolic plane*. A hyperbolic plane is nondegenerate.

Example 6.3. The quadratic form

$$\left(h \mapsto \sum_{n \in \mathbb{N}} h_n h_{1-n}\right) : \mathcal{H} \to k$$

is nondegenerate. The quadratic form

$$\left(\left[\begin{array}{c} f \\ g \end{array}\right] \mapsto \sum_{n \in \mathbb{Z}} f_n g_{1-n}\right) : \left[\begin{array}{c} \mathcal{H} \\ \mathcal{H} \end{array}\right] \to k$$

is nondegenerate. Composition of the latter quadratic form with the k-linear isomorphism $(h \mapsto h^{[2]}) : \mathcal{H} \to \left[\begin{array}{c} \mathcal{H} \\ \mathcal{H} \end{array}\right]$ yields the former.

Example 6.4. Let E be a free k-module equipped with a nondegenerate k-quadratic form $q : E \to k$. Let $\alpha, \beta : E \to E$ be idempotent q-adjoint k-linear endomorphisms of E. Let k-linearly independent vectors $e_1, \ldots, e_n \in \alpha E$ be given. Then there exist vectors $f_1, \ldots, f_n \in \beta E$ such that $q(e_i, f_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$; necessarily $f_1, \ldots, f_n$ are k-linearly independent.

Definition 6.5. Let E be a free k-module equipped with a k-quadratic form q. A k-linear endomorphism $\pi : E \to E$ is called a *q-polarization* if

$$\pi^2 = \pi, \quad q \circ \pi = 0, \quad q \circ (1 - \pi) = 0.$$

For any q-polarization π, one has

$$\begin{array}{rcl} 0 & = & q((1-\pi)(e+f)) - q(\pi(e+f)) \\ & = & q((1-\pi)e, (1-\pi)f) - q(\pi e, \pi f) \\ & = & q(e,f) - q(\pi e, f) - q(e, \pi f) \\ & = & q((1-\pi)e, f) - q(e, \pi f) \end{array}$$

for all $e, f \in E$, i. e., π and $(1 - \pi)$ are q-adjoint.

EXAMPLE 6.6. Let $I \subset \mathbb{Z}$ be any subset such that $\mathbb{Z} = I \coprod (1-I)$. Let $\pi : \mathcal{H} \to \mathcal{H}$ be the k-linear endomorphism defined by the rule

$$(\pi h)_i := \begin{cases} h_i & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{cases}$$

for all $i \in \mathbb{Z}$. Then π is a polarization with respect to the quadratic form $h \mapsto \sum_{n\in\mathbb{N}} h_n h_{1-n} = \sum_{i\in I} h_i h_{1-i}$.

DEFINITION 6.7. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$. A k-linear endomorphism $\mu : E \to E$ is called is *q-projector* if $\mu^2 = \mu$ and μ is q-self-adjoint. For each q-projector μ one has $q(e) = q(\mu e) + q((1-\mu)e)$ for all $e \in E$.

EXAMPLE 6.8. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and a q-polarization π. Let $e_1, \dots, e_n \in \pi E$ and $f_1, \dots, f_n \in (1-\pi)E$ be vectors such that $q(e_i, f_j) = \delta_{ij}$ for $i, j = 1, \dots, n$. Then

$$\mu := \left(e \mapsto \sum_{i=1}^{n} (q(e_i, e) f_i + q(f_i, e) e_i) \right) : E \to E$$

is a q-projector such that $\mu\pi = \pi\mu$.

EXAMPLE 6.9. Let E be a free k-module equipped with a nondegenerate k-quadratic form q and a q-polarization π. Let $e_1, \dots, e_r \in \pi E$ and $f_1, \dots, f_r \in (1-\pi)E$ be vectors such that $q(e_i, f_j) = \delta_{ij}$ for $i, j = 1, \dots, r$. Let $V \subseteq E$ be any finitely generated k-submodule. Then there exist vectors $e_{r+1}, \dots, e_n \in \pi E$ and $f_{r+1}, \dots, f_n \in (1-\pi)E$ such that V is contained in the k-span of $e_1, \dots, e_n, f_1, \dots, f_n$ and $q(e_i, f_j) = \delta_{ij}$ for $i, j = 1, \dots, n$.

7. The big split orthogonal group

DEFINITION 7.1. We say that $A \in \mathcal{Q}$ is *split orthogonal* if $A \in \mathcal{Q}^\times$ and A preserves the quadratic form $(h \mapsto \sum_{i\in\mathbb{N}} h_i h_{1-i}) : \mathcal{H} \to k$. The set consisting of the split orthogonal matrices in $\mathcal{Q}$ forms a group under matrix multiplication called the *big split orthogonal group*.

EXAMPLE 7.2. Let a matrix $A \in \mathcal{Q}$ be given. Consider the following conditions:

- $\sum_{n\in\mathbb{Z}} A_{ni} A_{1-n,j} = \delta_{i,1-j}$ for all distinct $i, j \in \mathbb{Z}$.
- $\sum_{n\in\mathbb{N}} A_{ni} A_{1-n,i} = 0$ for all $i \in \mathbb{Z}$.
- $\sum_{n\in\mathbb{Z}} A_{in} A_{j,1-n} = \delta_{i,1-j}$ for all distinct $i, j \in \mathbb{Z}$.
- $\sum_{n\in\mathbb{N}} A_{in} A_{i,1-n} = 0$ for all $i \in \mathbb{Z}$.

These conditions are necessary and sufficient for A to be split orthogonal.

Definition 7.3. Given subsets $I, J \subseteq \mathbb{Z}$ and a matrix $A \in \mathcal{Q}(I, J)$, we define $A^\dagger \in \mathcal{Q}(1-J, 1-I)$ by the rule

$$A^\dagger_{ij} = A_{1-j,1-i}$$

for all $i \in 1-I$ and $j \in 1-J$. In other words, $A^\dagger$ is obtained from A by reflection through the anti-diagonal $\{i+j=1\}$. Just like the transpose operation, the dagger operation reverses matrix products. The dagger operation preserves almost upper triangularity, whereas the transpose operation does not.

Definition 7.4. Given a subset $I \subset \mathbb{Z}$ and a matrix $A \in \mathcal{Q}(I, 1-I)$, we say that A is *dagger-alternating* if $A^\dagger + A = 0$ and $A_{i,1-i} = 0$ for all $i \in I$. In other words, dagger-alternating matrices are anti-symmetric under reflection through the anti-diagonal $\{i+j=1\}$ and vanish along the anti-diagonal.

Example 7.5. Fix $A \in \mathcal{Q}$. Let $I \subset \mathbb{Z}$ be a subset such that

$$\mathbb{Z} = I \coprod (1-I),$$

e. g. $I = \mathbb{N}$ or $I = 2\mathbb{Z}$. The

$$\left[\begin{array}{c} 1-I \\ I \end{array}\right] \times \left[\begin{array}{c} 1-I \\ I \end{array}\right]^T$$

block decompositions of A and $A^\dagger$ take the form

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right], \quad A^\dagger = \left[\begin{array}{cc} d^\dagger & b^\dagger \\ c^\dagger & a^\dagger \end{array}\right].$$

The matrix A is split orthogonal if and only if

$$ad^\dagger + bc^\dagger = 1, \quad d^\dagger a + b^\dagger c = 1$$

and the matrices

$$d^\dagger b, \quad c^\dagger a, \quad ab^\dagger, \quad cd^\dagger$$

are dagger-alternating. In the special case

$$A = \left[\begin{array}{cc} 1 & b \\ 0 & 1 \end{array}\right],$$

the matrix A is split orthogonal if and only if the matrix b is dagger-alternating. In the special case

$$A = \left[\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right],$$

the matrix A is split orthogonal if and only if $ad^\dagger = d^\dagger a = 1$.

EXAMPLE 7.6. Let $\mathbf{p} \in \mathcal{Q}$ be defined by the rule

$$\mathbf{p}_{ij} := \begin{cases} 1 & \text{if } i = j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in \mathbb{Z}$. One has

$$\mathbf{p}^2 = \mathbf{p}, \qquad \mathbf{p} + \mathbf{p}^\dagger = 1.$$

We call $\mathbf{p}$ the *standard polarization*. The k-linear endomorphism

$$(h \mapsto \mathbf{p}h) : \mathcal{H} \to \mathcal{H}$$

is a $\left(h \mapsto \sum_{n \in \mathbb{N}} h_n h_{1-n}\right)$-polarization.

EXAMPLE 7.7. A matrix $A \in \mathcal{Q}$ is split orthogonal if and only if

$$AA^\dagger = A^\dagger A = 1$$

and

$$X^\dagger A^\dagger \mathbf{p} A X = X^\dagger \mathbf{p} X$$

for all $X \in \mathcal{Q}$ supported in a single column.

EXAMPLE 7.8. Let $X \in \mathcal{Q}$ be strictly upper triangular and supported in a single column. The matrices

$$1 - \mathbf{p}X + X^\dagger \mathbf{p}^\dagger, \quad 1 - \mathbf{p}^\dagger X + X^\dagger \mathbf{p}$$

and

$$1 - X + X^\dagger - X^\dagger \mathbf{p}^\dagger X = (1 - \mathbf{p}X + X^\dagger \mathbf{p}^\dagger)(1 - \mathbf{p}^\dagger X + X^\dagger \mathbf{p})$$

are split orthogonal.

EXAMPLE 7.9. The entrywise limit of a uniformly dominated entrywise convergent sequence of split orthogonal elements of $\mathcal{Q}$ is split orthogonal.

EXAMPLE 7.10. Let $A \in \mathcal{Q}$ be given and write

$$A^{[2]} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{Q}).$$

One has

$$(A^\dagger)^{[2]} = \begin{bmatrix} d^\dagger & b^\dagger \\ c^\dagger & a^\dagger \end{bmatrix}.$$

DEFINITION 7.11. The following conditions on a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{Q})$$

are equivalent:

- One has $A^{[2]} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some split orthogonal $A \in \mathcal{Q}$.

- The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to $\mathrm{GL}_2(\mathcal{Q})$ and preserves the quadratic form $\left(\begin{bmatrix} f \\ g \end{bmatrix} \mapsto \sum_{n\in\mathbb{Z}} f_n g_{1-n}\right) : \begin{bmatrix} \mathcal{H} \\ \mathcal{H} \end{bmatrix} \rightarrow k$.
- One has
$$ad^\dagger + bc^\dagger = 1, \quad d^\dagger a + b^\dagger c = 1$$
and the matrices
$$d^\dagger b, \quad c^\dagger a, \quad ab^\dagger, \quad cd^\dagger$$
are dagger-alternating.

Under the equivalent conditions above we say that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a split orthogonal element of $\mathrm{Mat}_2(\mathcal{Q})$.

8. Tame maps

Definition 8.1. Let subsets $I, J \subseteq \mathbb{Z}$ be given. Let A be an I by J matrix with scalar entries. We say that A is *almost diagonal* if both A and its transpose A^T are almost upper triangular. Let
$$\omega : J \rightarrow I$$
be a map and let W be the I by J matrix defined by the rule
$$W_{ij} := \delta_{i,\omega(j)}$$
for all $i \in I$ and $j \in J$. In this situation we say that W *represents* ω. The following conditions are equivalent:

- The matrix W is almost diagonal.

- $\forall n \in \mathbb{Z} \ \ |\{j \in J \mid \omega(j) \geq n \geq j\} \cup \{j \in J \mid j \geq n \geq \omega(j)\}| < \infty$.

- There exist gauges σ and τ such that
$$\sigma(j) \leq \omega(j) \leq \tau(j)$$
for all $j \in J$.

Under these equivalent conditions we say that the map ω is *tame*.

EXAMPLE 8.2. Let I and J be subsets of $\mathbb{Z}$. Let $\omega : J \to I$ be a map. Assuming that J is neither bounded above nor below, the map ω is tame if and only if

$$\lim_{j \to -\infty} \omega(j) = -\infty \text{ and } \lim_{j \to +\infty} \omega(j) = +\infty.$$

Assuming that J is either bounded above or bounded below, the map ω is tame if and only if the set $\{j \in J \mid \omega(j) = n\}$ is finite for all $n \in \mathbb{Z}$.

EXAMPLE 8.3. Let subsets $I, J \subseteq \mathbb{Z}$ be given. Let $I = \coprod_{i=1}^{n} I_i$ be a disjoint decomposition of I. A map $\omega : I \to J$ is tame if and only if the restriction of ω to I_i is tame for $i = 1, \ldots, n$.

EXAMPLE 8.4. Let subsets $I, J, K \subseteq \mathbb{Z}$ and tame maps $\eta : K \to J$ and $\omega : J \to I$ be given. Let W be the I by J matrix representing ω. Let Y be the J by K matrix representing η. The composite map $\omega \circ \eta : K \to I$ is tame and the product matrix WY represents the composition $\omega \circ \eta$. If ω is bijective, the inverse function $\omega^{-1} : I \to J$ is tame and the transposed matrix W^T represents ω^{-1}.

EXAMPLE 8.5. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame one-to-one map and let W be the almost diagonal $\mathbb{Z}$ by $\mathbb{Z}$ matrix representing ω. One has

$$(W^T W)_{ij} = \delta_{ij}, \quad (WW^T)_{ij} = \begin{cases} 1 & \text{if } i = j \in \omega(\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in \mathbb{Z}$. For each $A \in \mathcal{Q}^\times$, put

$$\omega_* A := 1 - WW^T + WAW^T.$$

One has

$$(\omega_* A)_{ij} = \begin{cases} A_{\omega^{-1}(i), \omega^{-1}(j)} & \text{if } i, j \in \omega(\mathbb{Z}) \\ \delta_{ij} & \text{otherwise} \end{cases}$$

for all $i, j \in \mathbb{Z}$. One has

$$\omega_* 1 = 1, \quad \omega_*(AB) = (\omega_* A)(\omega_* B), \quad (\omega \circ \eta)_* A = \omega_*(\eta_* A)$$

and

$$\omega(\mathbb{Z}) \cap \eta(\mathbb{Z}) = \emptyset \Rightarrow (\omega_* A)(\eta_* B) = (\eta_* B)(\omega_* A)$$

for all $A, B \in \mathcal{Q}^\times$ and tame one-to-one maps $\omega, \eta : \mathbb{Z} \to \mathbb{Z}$.

EXAMPLE 8.6. Fix a positive integer n. One has

$$((\ell \mapsto n\ell)_* A)^{[n]} = \begin{bmatrix} A & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

for all $A \in \mathcal{Q}^\times$. One has

$$g(\mathbf{t}^n)^{[n]} = \begin{bmatrix} g(\mathbf{t}) & & \\ & \ddots & \\ & & g(\mathbf{t}) \end{bmatrix}, \quad g(\mathbf{t}^n) = \prod_{i=1}^{n} (\ell \mapsto \ell n + 1 - i)_* g(\mathbf{t})$$

for all $g \in k((t))^\times$.

EXAMPLE 8.7. For all tame one-to-one maps $\omega : \mathbb{Z} \to \mathbb{Z}$ such that

$$\omega(1-n) = 1 - \omega(n)$$

for all $n \in \mathbb{Z}$ and split orthogonal matrices $A \in \mathcal{Q}$, again the matrix $\omega_* A$ is split orthogonal.

EXAMPLE 8.8. Let $f, \phi \in k((t))^\times$ be given. Assume that the winding number n of ϕ is positive. Let $A \in \mathcal{Q}^\times$ be given such that

$$\phi(\mathbf{t})A = A\mathbf{t}^n.$$

By Examples 1.6, 4.29 and 8.6, one has a factorization

$$A^{-1} f(\mathbf{t}) A = ((\ell \mapsto n\ell)_* (\mathcal{N}_\phi f)(\mathbf{t}))C$$

where C belongs to the commutator subgroup of the commutant of $\mathbf{t}^n$ in $\mathcal{Q}^\times$.

9. Partitions, wedge indices and diamond indices

DEFINITION 9.1. A *partition* λ is an infinite nonincreasing sequence

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

of nonnegative integers almost all terms of which vanish. Put

$$|\lambda| := \sum_i \lambda_i, \quad \ell(\lambda) := |\{i \in \mathbb{N} \mid \lambda_i > 0\}|,$$

thereby defining the *weight* and *length* of λ, respectively. See [**Macdonald SFHP**] for background concerning partitions.

DEFINITION 9.2. The *diagram* of partition λ is defined to be the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j \leq \lambda_i\}$ and (by abuse of notation) is again denoted by λ. We always visualize the diagram of a partition according to the row-column convention of matrix theory. The weight of a partition is the number of nodes in its diagram. The partition with diagram the transpose of the diagram of λ is denoted λ' and said to be *conjugate* to λ. One has has $|\lambda| = |\lambda'|$ and $\ell(\lambda) = \lambda'_1$.

DEFINITION 9.3. Given a partition λ, one writes

$$\lambda = (\alpha_1 > \cdots > \alpha_r \mid \beta_1 > \cdots > \beta_r)$$

under the following three conditions:

- The main diagonal of λ consists of r nodes.
- For $i = 1, \ldots, r$, there are α_i nodes of λ to the right of (i,i).
- For $i = 1, \ldots, r$, there are β_i nodes of λ below (i,i).

This notation for partitions is due to Frobenius.

EXAMPLE 9.4.

$$\begin{aligned}
(5 \geq 4 \geq 4 \geq 1 \geq 0 \geq \ldots) &= \begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \\ \square & & & & \end{array} \\
&= (4 > 2 > 1 \mid 3 > 1 > 0), \\
(5 \geq 4 \geq 4 \geq 1 \geq 0 \geq \ldots)' &= \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & \square & \\ \square & \square & \square & \\ \square & & & \end{array} \\
&= (3 > 1 > 0 \mid 4 > 2 > 1) \\
&= (4 \geq 3 \geq 3 \geq 3 \geq 1 \geq 0 \geq \ldots).
\end{aligned}$$

EXAMPLE 9.5. Let a $\mathbb{Z}$ by $\mathbb{N}$ matrix A with scalar entries be given such that $A_{ij} = \delta_{ij}$ for all $i, j \in \mathbb{N}$. Let a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) = (\alpha_1 > \dots > \alpha_r \mid \beta_1 > \dots > \beta_r)$$

be given. One has

$$\det_{i,j=1}^{N} A_{i-\lambda_i,j} = (-1)^{\sum_j \beta_j} \det_{i,j=1}^{r} A_{-\alpha_i,\beta_j+1}$$

for all $N \geq \ell(\lambda)$.

EXAMPLE 9.6. Let $x_1, \dots, x_N$ be independent variables and put

$$\left.\begin{array}{rcl} \sum_n h_n t^n & = & \prod_{i=1}^N (1 - x_i t)^{-1} \\ \sum_n (-1)^n e_n t^n & = & \prod_{i=1}^N (1 - x_i t) \end{array}\right\} \in \mathbb{Z}[x_1, \dots, x_N][[t]].$$

The h's and the e's are the so called *complete* and *elementary* symmetric functions of the x's, respectively. One has

$$\det_{i,j=1}^{\ell(\lambda)} h_{\lambda_i - i + j} = \frac{\det_{i,j=1}^{N} x_i^{\lambda_j + N - j}}{\det_{i,j=1}^{N} x_i^{N-j}} = \det_{i,j=1}^{\ell(\lambda')} e_{\lambda'_i - i + j}$$

for all partitions λ such that $N \geq \ell(\lambda)$. The determinant on the left is by definition the *S-function* of the x's indexed by λ. See [**Macdonald SFHP**] for background and proof.

EXAMPLE 9.7. Put

$$h_n := \begin{cases} 1/n! & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

One has

$$\det_{i,j=1}^{\ell(\lambda)} h_{\lambda_i - i + j} = 1 / \prod_{(i,j) \in \lambda} (\lambda_i + \lambda'_j - i - j + 1)$$

for all partitions λ. For example, for the partition

$$\lambda = \begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \\ \square & & & & \end{array}$$

one has

$$\det_{i,j=1}^{\ell(\lambda)} h_{\lambda_j + i - j} = \frac{1}{8 \cdot 6 \cdot 5 \cdot 4 \cdot 1 \cdot 6 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 1}.$$

See Macdonald [**Macdonald SFHP**] for background and proof.

DEFINITION 9.8. A subset $I \subset \mathbb{Z}$ such that

$$\sup I < \infty, \quad \inf(\mathbb{Z} \setminus I) > -\infty$$

will be called a *wedge index*. For each wedge index I, put

$$\deg I := |I \cap \mathbb{N}| - |(1 - \mathbb{N}) \setminus I|,$$

thereby defining the *degree* of I.

DEFINITION 9.9. A subset $I \subset \mathbb{Z}$ such that

$$\mathbb{Z} = I \coprod (1 - I), \quad \sup I < \infty$$

will be called a *diamond index*. The diamond indices are in bijective correspondence with the finite subsets of $\mathbb{N}$ under the map $I \mapsto I \cap \mathbb{N}$. Put

$$\text{parity}\, I \equiv |I \cap \mathbb{N}| \bmod 2$$

thereby defining the *parity* of I.

EXAMPLE 9.10. For each partition λ, the set $\{\lambda_i - 1 + i \mid i \in \mathbb{N}\}$ is a wedge index and one has

$$\mathbb{Z} = \{\lambda_i - 1 + i \mid i \in \mathbb{N}\} \coprod \{i - \lambda'_i \mid i \in \mathbb{N}\}.$$

The construction $\lambda \mapsto \{\lambda_i - 1 + i \mid i \in \mathbb{N}\}$ puts the partitions in bijective correspondence with the wedge indices of degree 0. More generally, the construction $(n, \lambda) \mapsto \{n + \lambda_i - 1 + i \mid i \in \mathbb{N}\}$ puts the cartesian product of $\mathbb{Z}$ and the set of partitions in bijective correspondence with the wedge indices.

EXAMPLE 9.11. Given any subset $I \subset \mathbb{Z}$, put

$$I^\diamond := \{2i \mid i \in I\} \cup \{1 - 2i \mid i \in \mathbb{Z} \setminus I\}.$$

One has

$$(\mathbb{Z} \setminus I)^\diamond = 1 - I^\diamond$$

for any subset $I \subset \mathbb{Z}$. The construction $I \mapsto I^\diamond$ puts the wedge indices in bijective correspondence with the diamond indices.

DEFINITION 9.12. We say that a diamond index I is *parity-balanced* if

$$|\{i \in I \cap \mathbb{N} \mid i \equiv 0 \bmod 2\}| = |\{i \in I \cap \mathbb{N} \mid i \equiv 1 \bmod 2\}|.$$

A diamond index I is parity-balanced if and only if $I = J^\diamond$ for some wedge index J of degree 0.

Example 9.13. Let $I \subset \mathbb{Z}$ be a parity-balanced diamond index and put

$$\begin{aligned} I \cap \mathbb{N} &= \{2\alpha_1 + 2 > \cdots > 2\alpha_r + 2\} \cup \{2\beta_1 + 1 > \cdots > 2\beta_r + 1\}, \\ I \cap 2\mathbb{Z} &= \{2\lambda_1 > 2(\lambda_2 - 1) > 2(\lambda_3 - 2) > \dots\}. \end{aligned}$$

One obtains in this way a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) = (\alpha_1 > \cdots > \alpha_r \mid \beta_1 > \cdots > \beta_r).$$

The construction $I \mapsto \lambda$ puts the parity-balanced diamond indices in bijective correspondence with the partitions.

10. The big Weyl group and its split orthogonal analogue

Definition 10.1. The *big Weyl group* is by definition the group of $\mathbb{Z}$ by $\mathbb{Z}$ almost diagonal permutation matrices. The *big split orthogonal Weyl group* is by definition the group of $\mathbb{Z}$ by $\mathbb{Z}$ almost diagonal split orthogonal permutation matrices.

Example 10.2. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a bijective map and let W be the $\mathbb{Z}$ by $\mathbb{Z}$ permutation matrix representing ω. The following properties are equivalent:

- The map ω is tame.
- The matrix W belongs to the big Weyl group.
- The matrices $\mathbf{p}W\mathbf{p}^\dagger$ and $\mathbf{p}^\dagger W\mathbf{p}$ are finitely supported, where $\mathbf{p} \in \mathcal{Q}$ is the standard polarization matrix.
- For some wedge index $I \subset \mathbb{Z}$, again $\omega(I)$ is a wedge index.
- For every wedge index $I \subset \mathbb{Z}$, again $\omega(I)$ is a wedge index.
- $\lim\limits_{j\to+\infty} \omega(j) = +\infty$ and $\lim\limits_{j\to-\infty} \omega(j) = -\infty$.

Under the equivalent conditions above we say that ω is a *tame permutation* of $\mathbb{Z}$. The tame permutations of $\mathbb{Z}$ form a group under composition isomorphic to the big Weyl group. The group of tame permutations of $\mathbb{Z}$ acts transitively on the family of wedge indices.

Example 10.3. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame permutation of $\mathbb{Z}$. One has

$$\deg I - \deg J = |I \setminus J| - |J \setminus I|$$

for all wedge indices I and J, and hence exists a unique integer $\deg \omega$ such that

$$\deg \omega(I) = \deg \omega + \deg I$$

for all wedge indices I. The group of tame permutations of $\mathbb{Z}$ is generated by the shift $n \mapsto n-1$ and the subgroup consisting of permutations stabilizing $\mathbb{N}$ and $1 - \mathbb{N}$.

EXAMPLE 10.4. The big Weyl group is generated by the bilateral shift $\mathbf{t}$ and elements W such that $\mathbf{p}W\mathbf{p}^\dagger = 0 = \mathbf{p}^\dagger W\mathbf{p}$ where $\mathbf{p}$ is the standard polarization. One has

$$\deg W = \left|\operatorname{supp} \mathbf{p}W\mathbf{p}^\dagger\right| - \left|\operatorname{supp} \mathbf{p}^\dagger W\mathbf{p}\right|$$

for all matrices W belonging to the big Weyl group.

EXAMPLE 10.5. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame permutation and let W be the almost diagonal permutation matrix representing ω. The following statements are equivalent:

- W is split orthogonal.
- $W^T = W^\dagger$.
- $\omega(1-n) = 1 - \omega(n)$ for all $n \in \mathbb{Z}$.

Under these equivalent conditions, we say that the tame permutation ω is *split orthogonal*. The split orthogonal tame permutations of $\mathbb{Z}$ form a group under composition isomorphic to the big split orthogonal Weyl group. The group of tame split orthogonal permutations acts transitively on the family of diamond indices. The group of tame split orthogonal permutations is generated by the transposition of 0 and 1 and the subgroup consisting of maps fixing $\mathbb{N}$ and $1 - \mathbb{N}$.

EXAMPLE 10.6. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame split orthogonal permutation of $\mathbb{Z}$. One has

$$\operatorname{parity} I - \operatorname{parity} J \equiv |I \setminus J| \bmod 2$$

for all diamond indices I and J, and hence there exists a unique congruence class $\operatorname{parity}\omega$ modulo 2 such that

$$\operatorname{parity}\omega(I) \equiv \operatorname{parity}\omega + \operatorname{parity} I \bmod 2$$

for all diamond indices I.

EXAMPLE 10.7. The big split orthogonal Weyl group is generated by the fundamental reflection $\mathbf{r}$ and the subgroup consisting of matrices W such that $\mathbf{p}^\dagger W\mathbf{p} = 0$ (and hence also $\mathbf{p}W\mathbf{p}^\dagger = 0$), where $\mathbf{p}$ is the standard polarization. The map

$$W \mapsto \left|\operatorname{supp}\ \mathbf{p}W\mathbf{p}^\dagger\right| \bmod 2$$

is a surjective homomorphism from the big split orthogonal Weyl group to $\mathbb{Z}/2\mathbb{Z}$.

Example 10.8. Temporarily let $\mathcal{W}$ denote the big Weyl group and let $\mathcal{W}_0$ denote subgroup consisting of matrices W such that $\mathbf{p}^\dagger W\mathbf{p} = 0$ and $\mathbf{p}W\mathbf{p}^\dagger = 0$. The map

$$W \mapsto \left(\left|\text{supp} \quad \mathbf{p}W\mathbf{p}^\dagger\right|, \left|\text{supp} \quad \mathbf{p}^\dagger W\mathbf{p}\right|\right)$$

puts the double coset space $\mathcal{W}_0 \setminus \mathcal{W}/\mathcal{W}_0$ in bijective correspondence with the set of pairs of nonnegative integers.

Example 10.9. Temporarily let $\mathcal{W}$ denote the big split orthogonal Weyl group and let $\mathcal{W}_0$ denote the subgroup consisting of matrices W such that $\mathbf{p}^\dagger W\mathbf{p} = 0$ (and hence also $\mathbf{p}W\mathbf{p}^\dagger = 0$). The map

$$W \mapsto \left|\text{supp} \quad \mathbf{p}W\mathbf{p}^\dagger\right|$$

puts the double coset space $\mathcal{W}_0 \setminus \mathcal{W}/\mathcal{W}_0$ in bijective correspondence with the nonnegative integers.

Example 10.10. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame one-to-one map. There exists a tame permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $\omega \circ \sigma$ is strictly increasing. If $\omega(1-n) = 1 - \omega(n)$ for all $n \in \mathbb{Z}$, there exists a unique tame split orthogonal permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $\omega \circ \sigma$ is strictly increasing.

11. Pivots

Throughout the discussion of pivots, unless explicitly noted otherwise, we assume that k is a field.

Definition 11.1. Let subsets $I, J \subseteq \mathbb{Z}$ and a matrix $A \in \mathcal{Q}(I, J)$ be given. Let $(i_0, j_0) \in I \times J$ be given. If the

$$\left[\begin{array}{c} I \cap \{i < i_0\} \\ \{i_0\} \\ I \cap \{i > i_0\} \end{array}\right] \times \left[\begin{array}{c} J \cap \{j < j_0\} \\ \{j_0\} \\ J \cap \{j > j_0\} \end{array}\right]^T$$

block decomposition of A takes the form

$$A = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{array}\right], \quad a_{22} \neq 0,$$

we say that (i_0, j_0) is a *pivot* of A. If this block decomposition takes the form

$$A = \left[\begin{array}{ccc} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{array}\right], \quad a_{22} \neq 0,$$

we say that the pivot (i_0, j_0) of A is *cleared*.

EXAMPLE 11.2. Fix $(i_0, j_0) \in \mathbb{Z}^2$ and let $\{A^{(n)}\}_{n=1}^{\infty}$ be a uniformly dominated sequence in $\mathcal{Q}$ converging entrywise to $A \in \mathcal{Q}$.

- There exists $n_0 \in \mathbb{N}$ such that (i_0, j_0) is a pivot of $A^{(n)}$ for all $n \geq n_0$ if and only if (i_0, j_0) is a pivot of A.
- If (i_0, j_0) is an uncleared pivot of A, then there exists $n_0 \in \mathbb{N}$ such that (i_0, j_0) is an uncleared pivot of $A^{(n)}$ for all $n \geq n_0$.
- If there exists $n_0 \in \mathbb{N}$ such that (i_0, j_0) is a cleared pivot of $A^{(n)}$ for all $n \geq n_0$, then (i_0, j_0) is a cleared pivot of A.

The converses of the latter two statements do not hold in general.

PROPOSITION 11.3. *Let $A \in \mathcal{Q}$ be given. Let (i_0, j_0) be a pivot of A. Let I (resp. J) be the set of integers indexing rows (resp. columns) of A wherein cleared pivots of A distinct from (i_0, j_0) appear. There exist unique matrices $X, Y \in \mathcal{Q}$ with the following properties:*

1. *X is strictly upper triangular and* $\operatorname{supp} X \subseteq (\mathbb{Z} \setminus I) \times \{i_0\}$.
2. *Y is strictly upper triangular and* $\operatorname{supp} Y \subseteq \{j_0\} \times (\mathbb{Z} \setminus J)$.
3. *(i_0, j_0) is a cleared pivot of $A' := (1 - X)A(1 - Y)$.*
4. *Every cleared pivot of A remains a cleared pivot of A'.*

(We call X the left pivot-clearing operator *and Y the* right pivot-clearing operator *associated to A and its pivot (i_0, j_0).)*

PROOF. By hypothesis, the

$$\left[\begin{array}{c} \{i < i_0\} \setminus I \\ \{i_0\} \\ \{i_0 < i\} \setminus I \\ I \end{array}\right] \times \left[\begin{array}{c} J \\ \{j < j_0\} \setminus J \\ \{j_0\} \\ \{j_0 < j\} \setminus J \end{array}\right]^T$$

block decomposition of A takes the form

$$A = \left[\begin{array}{cccc} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ a_{41} & 0 & 0 & 0 \end{array}\right], \quad a_{23} \neq 0.$$

To satisfy condition 1 the matrix X we seek must have a

$$\begin{bmatrix} \{i < i_0\} \setminus I \\ \{i_0\} \\ \{i_0 < i\} \setminus I \\ I \end{bmatrix} \times \begin{bmatrix} \{i < i_0\} \setminus I \\ \{i_0\} \\ \{i_0 < i\} \setminus I \\ I \end{bmatrix}^T$$

block decomposition of the form

$$X = \begin{bmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To satisfy condition 2 the matrix Y we seek must have a

$$\begin{bmatrix} J \\ \{j < j_0\} \setminus J \\ \{j_0\} \\ \{j_0 < j\} \setminus J \end{bmatrix} \times \begin{bmatrix} J \\ \{j < j_0\} \setminus J \\ \{j_0\} \\ \{j_0 < j\} \setminus J \end{bmatrix}^T$$

block decomposition of the form

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

One has

$$(1-X)A(1-Y) = \begin{bmatrix} 0 & a_{12} & a_{13} - xa_{23} & (a_{13} - xa_{23})y + a_{14} - xa_{24} \\ 0 & 0 & a_{23} & a_{24} - a_{23}y \\ 0 & a_{32} & 0 & a_{34} \\ a_{41} & 0 & 0 & 0 \end{bmatrix}.$$

Conditions 1 and 2 granted, condition 3 holds if and only if $x = a_{13}a_{23}^{-1}$ and $y = a_{23}^{-1}a_{24}$. Thus conditions 1–3 uniquely determine X and Y. Clearly conditions 1–3 imply condition 4. $\square$

LEMMA 11.4. *Let subsets $I, J \subseteq \mathbb{Z}$ be given. Let $A \in \mathcal{Q}(I, J)$ and a nonpivot $(i_0, j_0) \in \operatorname{supp} A$ be given. Let A' be the*

$$\left\{(i,j) \in I \times J \,\middle|\, \begin{array}{l} i \geq i_0 \\ j \leq j_0 \end{array}\right\}$$

block of A. There exists an uncleared pivot (i^*, j^*) of A'.

PROOF. For each $n \in \mathbb{N}$, put

$$j_n := \min\{j \in J \mid A_{i_{n-1}j} \neq 0\}, \quad i_n := \max\{i \in I \mid A_{ij_n} \neq 0\},$$

thereby defining a sequence of points in $\operatorname{supp} A'$ fitting into zigzag pattern thus:

$$\begin{array}{cccc} & & (i_0,j_1) & (i_0,j_0) \\ & (i_1,j_2) & (i_1,j_1) & \\ (i_2,j_3) & (i_2,j_2) & & \\ \cdot^{\cdot^{\cdot}} & & & \end{array}$$

Since the matrix A' is finitely supported there exists a unique pivot (i^*, j^*) of A' such that $i^* = i_n$ and $j^* = j_n$ for all $n \gg 0$. By hypothesis (i_0, j_0) is not a pivot of A, hence (i_0, j_0) is not a pivot of A', hence the zigzag sequence is nonconstant, and hence (i^*, j^*) is an uncleared pivot of A'. □

Example 11.5. Let $A \in \mathcal{Q}^\times$ be given such that every pivot of A is cleared. Then there exists a unique factorization of the form $A = DW$ where $D \in \mathcal{Q}^\times$ is diagonal and W is an element of the big Weyl group. Moreover, if A is split orthogonal, then the matrices D and W are likewise split orthogonal.

Example 11.6. Put

$$\lambda := ((i,j) \mapsto \max(-2i, 2j-1)) : \mathbb{Z}^2 \to \mathbb{Z}.$$

The $\mathbb{Z}$ by $\mathbb{Z}$ matrix with entry $\lambda(i,j)$ in position (i,j) looks like this:

$$\begin{bmatrix} & & & & \cdots & 4 & 5 \\ & & & \cdots & 2 & 3 & \vdots \\ & & \cdots & 0 & 1 & \vdots & \\ & \cdots & -2 & -1 & \vdots & & \\ \cdots & -4 & -3 & \vdots & & & \\ & & \vdots & & & & \end{bmatrix}$$

The level sets of λ thus fit together in a sort of herring bone pattern. For every $A \in \mathcal{Q}$, the restriction of λ to the set of pivots of A is one-to-one and bounded below. In other words, the function λ well orders the pivots of A.

Example 11.7. Let W and Y be elements of the big Weyl group. Let $B, C \in \mathcal{Q}^\times$ be upper triangular with nonzero diagonal entries. If $BWC = Y$, then $W = Y$.

Theorem 11.8. *Let $A \in \mathcal{Q}^\times$ be given. There exist $B, C, D, W \in \mathcal{Q}^\times$ with the following properties:*

- *$B - 1$ and $C - 1$ are strictly upper triangular.*
- *D is diagonal.*
- *W belongs to the big Weyl group.*
- *$BAC = DW$.*

Moreover, the matrix W thus arising is uniquely determined by A.

Proof. The uniqueness of W follows from Example 11.7. We turn now to the proof of the existence of B, C, D and W. We define sequences

$$\left\{A^{(n)}\right\}_{n=0}^{\infty}, \quad \left\{B^{(n)}\right\}_{n=0}^{\infty}, \quad \left\{C^{(n)}\right\}_{n=0}^{\infty}, \quad \left\{X^{(n)}\right\}_{n=1}^{\infty}, \quad \left\{Y^{(n)}\right\}_{n=1}^{\infty}$$

in $\mathcal{Q}$ by the following inductive procedure. Put

$$A^{(0)} := A, \quad B^{(0)} := 1, \quad C^{(0)} := 1.$$

For each $n \in \mathbb{N}$ such that $A^{(n)}$ has at least one uncleared pivot, let $X^{(n)}$ (resp. $Y^{(n)}$) be the left (resp. right) pivot-clearing operator associated to the matrix $A^{(n-1)}$ and the unique λ-minimal uncleared pivot of $A^{(n-1)}$ by Proposition 11.3, where λ is the function considered in Example 11.6. For each $n \in \mathbb{N}$ such that $A^{(n)}$ has no uncleared pivot, simply set $X^{(n)} := 0$ and $Y^{(n)} := 0$. For every $n \in \mathbb{N}$, set

$$\begin{array}{rcl} A^{(n)} & := & \left(1 - X^{(n)}\right) A^{(n-1)} \left(1 - Y^{(n)}\right), \\ B^{(n)} & := & \left(1 - X^{(n)}\right) B^{(n-1)}, \\ C^{(n)} & := & C^{(n-1)} \left(1 - Y^{(n)}\right). \end{array}$$

By Lemma 4.12 and Example 4.15, the sequences

$$\{A^{(n)}\}, \quad \{(A^{(n)})^{-1}\}, \quad \{B^{(n)}\}, \quad \{(B^{(n)})^{-1}\}, \quad \{C^{(n)}\}, \quad \{(C^{(n)})^{-1}\}$$

are uniformly dominated and entrywise convergent. Let M be the entrywise limit of the sequence $\{A^{(n)}\}$. Let B (resp. C) be the entrywise limit of the sequence $\{B^{(n)}\}$ (resp. $\{C^{(n)}\}$). One has $M, B, C \in \mathcal{Q}^\times$ by Example 4.26. The differences $B - 1$ and $C - 1$ are strictly upper triangular by Example 4.15. One has $BAC = M$ by Lemma 4.12. We claim that M has no uncleared pivots. Suppose to the contrary that M has an uncleared pivot (i^*, j^*). By Example 11.2 there exists $n_0 \in \mathbb{N}$ such that (i^*, j^*) is an uncleared pivot of $A^{(n)}$ for all $n \geq n_0$. For each $n \geq n_0$, let (i_n, j_n) be the unique λ-minimal uncleared pivot of $A^{(n)}$. By the definitions, for all $n' > n \geq n_0$, the pair (i_n, j_n)

is a cleared pivot of $A^{(n')}$ and in particular is distinct from the pair $(i_{n'}, j_{n'})$. By Example 11.2 it follows that each of the infinitely many points in the sequence $\{(i_n, j_n)\}_{n=n_0}^{\infty}$ is a cleared pivot of M. By construction one has $\lambda(i_n, j_n) \leq \lambda(i^*, j^*)$ for all $n \geq n_0$. But the set $\{(i,j) \in \operatorname{supp} M \mid \lambda(i,j) \leq \lambda(i^*, j^*)\}$ is finite because M is almost upper triangular. This contradiction proves the claim. By Example 11.5 one has a factorization $BAC = M = DW$ where D is diagonal and W is an element of the big Weyl group. □

Example 11.9. Let split orthogonal $A \in \mathcal{Q}$ and a pivot (i_0, j_0) of A be given. If $A_{i_0 j} = 0$ for all $j \notin \{j_0, 1-j_0\}$ and $A_{ij_0} = 0$ for all $i \notin \{i_0, 1-i_0\}$, then (i_0, j_0) is a cleared pivot of A. If (i_0, j_0) is a cleared pivot of A, then $(1-i_0, 1-j_0)$ is a cleared pivot of A, and $A_{i_0 j_0} A_{1-i_0, 1-j_0} = 1$. If $I \subseteq \mathbb{Z}$ is the set of integers indexing rows (resp. columns) wherein cleared pivots of A appear, then $I = 1 - I$.

Example 11.10. Let split orthogonal $A \in \mathcal{Q}$ be given. Let (i_0, j_0) be a pivot of A. Let I (resp. J) be the set of integers indexing rows (resp. columns) wherein cleared pivots of A distinct from (i_0, j_0) and $(1-i_0, 1-j_0)$ appear. Let X and Y be the left and right pivot-clearing operators associated to A and its pivot (i_0, j_0), respectively. Put

$$X^\star := X - X^\dagger + X^\dagger \mathbf{p}^\dagger X, \quad Y^\star := Y - Y^\dagger + Y \mathbf{p}^\dagger Y^\dagger$$

where $\mathbf{p}$ is the standard polarization. The matrices $X^\star$ and $Y^\star$ have the following properties:

- The matrices $X^\star$ and $Y^\star$ are strictly upper triangular.
- $\operatorname{supp} X^\star \subseteq ((\mathbb{Z} \setminus I) \times \{i_0\}) \cup (\{1 - i_0\} \times (\mathbb{Z} \setminus I))$.
- $\operatorname{supp} Y^\star \subseteq (\{j_0\} \times (\mathbb{Z} \setminus J)) \cup ((\mathbb{Z} \setminus J) \times \{1 - j_0\})$.
- The matrices $1 - X^\star$, $1 - Y^\star$ and $A' := (1 - X^\star) A (1 - Y^\star)$ are split orthogonal.
- The pairs (i_0, j_0) and $(1-i_0, 1-j_0)$ are cleared pivots of A'.
- Every cleared pivot of A remains a cleared pivot of A'.

This is the split orthogonal version of Proposition 11.3.

EXAMPLE 11.11. Let $A \in \mathcal{Q}$ be split orthogonal. There exist split orthogonal matrices $B, C, D, W \in \mathcal{Q}$ such that $B - 1$ and $C - 1$ are strictly upper triangular, D is diagonal and split orthogonal, W belongs to the big split orthogonal Weyl group, and $BAC = DW$. This is the split orthogonal version of Theorem 11.8.

EXAMPLE 11.12. In this example k is any artinian local ring. Let $\mathcal{B} \subseteq \mathcal{Q}^\times$ be the subgroup consisting of upper triangular matrices. Let $\mathcal{U} \subseteq \mathcal{Q}^\times$ be the subgroup consisting of matrices differing from the identity matrix by a strictly lower triangular matrix all entries of which belong to m. Let $\mathcal{W}$ be the big Weyl group. One has a disjoint union decomposition $\mathcal{Q}^\times = \coprod_{W \in \mathcal{W}} \mathcal{U}\mathcal{B}W\mathcal{B}$. Let $\mathcal{P} \subseteq \mathcal{Q}^\times$ be the subgroup consisting of matrices A such that the $\mathbb{N}$ by $1 - \mathbb{N}$ (lower left) blocks of both A and A^{-1} vanish. The group $\mathcal{Q}^\times$ is generated by $\mathcal{P}$ and $\mathbf{t}$.

EXAMPLE 11.13. Again k is any artinian local ring. Temporarily let $\mathcal{G}$ denote the group of split orthogonal elements of $\mathcal{Q}^\times$. Let $\mathcal{B}$ be the group of upper triangular elements of $\mathcal{G}$. Let $\mathcal{U}$ be the group of elements of $\mathcal{G}$ differing from the identity matrix by a strictly lower triangular matrix all entries of which belong to m. Let $\mathcal{W}$ be the big split orthogonal Weyl group. One has a disjoint union decomposition $\mathcal{G} = \coprod_{W \in \mathcal{W}} \mathcal{U}\mathcal{B}W\mathcal{B}$. Let $\mathcal{P} \subseteq \mathcal{G}$ be the subgroup consisting matrices with vanishing $\mathbb{N}$ by $1 - \mathbb{N}$ (lower left) block. The group $\mathcal{G}$ is generated by $\mathcal{P}$ and $\mathbf{r}$.

12. Clifford algebras

DEFINITION 12.1. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$. The *Clifford algebra* $\mathcal{C}(E, q)$ is by definition the quotient of the tensor algebra $\mathcal{T}(E)$ by the two-sided ideal $\mathcal{I}$ generated by the set $\{e \otimes e - q(e) \mid e \in E\}$. The Clifford algebra $\mathcal{C}(E, 0)$ is canonically isomorphic to the exterior algebra $\bigwedge(E)$. We denote the product in $\mathcal{C}(E, q)$ simply by juxtaposition, and we put

$$e^\sharp := e + \mathcal{I} \in \mathcal{C}(E, q)$$

for each $e \in E$. One has

$$(e^\sharp)^2 = q(e)$$

for all $e \in E$. The Clifford algebra $\mathcal{C}(E, q)$ has the following universal property:

- For all k-algebras $\mathcal{A}$ and k-linear maps $\phi : E \to \mathcal{A}$ such that

  $$\phi(e)^2 = q(e)$$

  for all $e \in E$, there exists a unique k-algebra homomorphism

  $$\phi^\natural : \mathcal{C}(E, q) \to \mathcal{A}$$

such that

$$\phi^\natural(e^\sharp) = \phi(e)$$

for all $e \in E$.

As will be explained presently, the natural map $(e \mapsto e^\sharp) : E \to \mathcal{C}(E,q)$ is always injective, but we do *not* identify E with a k-submodule of $\mathcal{C}(E,q)$ with respect to the map $e \mapsto e^\sharp$.

EXAMPLE 12.2. Let $\mathcal{A}$ be a k-algebra. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and a k-basis $\{e_i\}_{i\in S}$ indexed by a linearly ordered set S. Let $\phi : E \to \mathcal{A}$ be a k-linear map such that $\phi(e)^2 = q(e)$ for all $e \in E$. For each finite subset $I = \{i_1 < \cdots < i_r\} \subseteq S$, put

$$a_I := \phi(e_{i_1}) \cdots \phi(e_{i_r}) \in \mathcal{A}.$$

One has

$$a_{\{i\}} a_I = \sum_{j\in I} (-1)^{|\{\ell\in I|\ell<j\}|} \left\{ \begin{array}{ll} q(e_i,e_j) a_{I\setminus\{j\}} & \text{if } j < i \\ q(e_i) a_{I\setminus\{i\}} & \text{if } j = i \in I \\ a_{I\cup\{i\}} & \text{if } j = i \notin I \\ 0 & \text{if } j > i \end{array} \right.$$

for all $i \in S$ and finite subsets $I \subseteq S$. If the family $\{e_i\}$ indexed by $i \in S$ spans the k-module E, then the family $\{a_I\}$ indexed by finite subsets $I \subseteq S$ spans the $\mathcal{A}$ as k-module.

EXAMPLE 12.3. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and a k-basis $\{e_i\}_{i\in S}$ indexed by a linearly ordered set S. Let V be a free k-module equipped with a basis $\{v_I\}$ indexed by finite subsets $I \subseteq S$. For each $i \in S$, let $\partial_i, \delta_i : V \to V$ be the k-linear endomorphisms defined by the rules

$$\partial_i v_I := \left\{ \begin{array}{ll} (-1)^{|\{\ell\in I|\ell<i\}|} v_{I\setminus\{i\}} & \text{if } i \in I \\ 0 & \text{if } i \notin I \end{array} \right.$$

$$\delta_i v_I := \left\{ \begin{array}{ll} 0 & \text{if } i \in I \\ (-1)^{|\{\ell\in I|\ell<i\}|} v_{I\cup\{i\}} & \text{if } i \notin I \end{array} \right.$$

for all finite subsets $I \subseteq S$. One has

$$\partial_i^2 = 0, \quad \delta_i^2 = 0, \quad \partial_i\delta_i + \delta_i\partial_i = 1$$

for all $i \in S$, and

$$\partial_i\partial_j + \partial_j\partial_i = 0, \quad \delta_i\delta_j + \delta_j\delta_i = 0, \quad \partial_i\delta_j + \delta_j\partial_i = 0$$

for all distinct $i, j \in S$. Put

$$q_{ij} := \begin{cases} 0 & \text{if } i < j \\ q(e_i) & \text{if } i = j \\ q(e_i, e_j) & \text{if } i > j \end{cases}$$

for all $i, j \in S$. Let ϕ be the unique k-linear map from E to the k-algebra of k-linear endomorphisms of V such that

$$\phi(e_i)v_I := \delta_i v_I + \sum_{j \in I} q_{ij} \partial_j v_I$$

for all $i \in S$ and finite subsets $I \subseteq S$. One verifies by a straightforward calculation that $\phi(e)^2 = q(e)$ for all $e \in E$; let $\phi^\natural$ be the induced k-algebra homomorphism from the Clifford algebra $\mathcal{C}(E, q)$ to the k-algebra of k-linear endomorphisms of V. For each finite subset

$$I = \{i_1 < \cdots < i_n\} \subseteq S$$

put

$$e_I := e_{i_1}^\sharp \cdots e_{i_n}^\sharp \in \mathcal{C}(E, q),$$

and observe that

$$\phi^\natural(e_I)v_\emptyset = \phi(e_{i_1}) \cdots \phi(e_{i_n})v_\emptyset = \delta_{i_1} \cdots \delta_{i_n} v_\emptyset = v_I.$$

It follows that the family $\{e_I\}$ is k-linearly independent. In particular, the canonical map $(e \mapsto e^\sharp) : E \to \mathcal{C}(E, q)$ is injective.

THEOREM 12.4. *Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and let $\mathcal{C} = \mathcal{C}(E, q)$ be the associated Clifford algebra. Let E_0 be a free k-submodule of E. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the k-subalgebra generated by $E_0^\sharp$. Let $\{e_i\}_{i \in S}$ be a k-basis for E_0 indexed by a linearly ordered set S. For each finite subset*

$$I = \{i_1 < \cdots < i_n\} \subseteq S$$

put

$$e_I := e_{i_1}^\sharp \cdots e_{i_n}^\sharp \in \mathcal{C}_0.$$

The family $\{e_I\}$ indexed by finite subsets $I \subseteq S$ is a k-basis for $\mathcal{C}_0$ and the natural map $\mathcal{C}(E_0, q\,|_{E_0}) \to \mathcal{C}_0$ is bijective.

PROOF. Example 12.2 shows that the family $\{e_I\}$ spans $\mathcal{C}_0$ over k. To prove the k-linear independence of the family $\{e_I\}$ there is no loss of generality in assuming that $\{e_i\}_{i \in S}$ is a k-basis of E, in which case Example 12.3 does the job. Since the k-basis $\{e_I\}$ of $\mathcal{C}_0$ is the image of a k-basis for $\mathcal{C}(E_0, q\,|_{E_0})$, the natural map $\mathcal{C}(E_0, q\,|_{E_0}) \to \mathcal{C}_0$ is bijective as claimed. $\square$

EXAMPLE 12.5. Consider the *Pauli spin matrices*

$$\sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

of quantum mechanics. One has

$$(a\sigma_x + b\sigma_y + c\sigma_z)^2 = a^2 + b^2 + c^2$$

for any real numbers a, b and c. The natural $\mathbb{R}$-algebra homomorphism

$$\mathcal{C}(\mathbb{R}^3, (a,b,c) \mapsto a^2 + b^2 + c^2) \to \mathrm{Mat}_2(\mathbb{C})$$

is bijective.

DEFINITION 12.6. Let A be an n by n alternating matrix with scalar entries. (A matrix is said to be *alternating* if anti-symmetric and zero along the diagonal.) Put

$$\operatorname{pfaff} A := \begin{cases} 1 & \text{if } n = 0 \\ \\ 0 & \text{if } n \equiv 1 \bmod 2 \\ \\ \displaystyle\sum_{j=2}^{n} (-1)^j A_{1j} \operatorname{pfaff} A^{\{1,j\}} & \text{if } n \equiv 0 \bmod 2 \text{ and } n > 0 \end{cases}$$

where for each $I \subseteq \{1, \ldots, n\}$ we denote by A^I the alternating matrix obtained by striking row i and column i from A for all $i \in I$. One calls $\operatorname{pfaff} A$ the *pfaffian* of A. For example, one has

$$\operatorname{pfaff} A = A_{12}, \quad \operatorname{pfaff} A = A_{12}A_{24} - A_{13}A_{24} + A_{14}A_{23}$$

in the cases $n = 2$ and $n = 4$, respectively. It is well known that

$$(\operatorname{pfaff} A)^2 = \det A,$$

and

$$\operatorname{pfaff}(B^T AB) = (\det B)(\operatorname{pfaff} A)$$

for any n by n matrix B with scalar entries.

EXAMPLE 12.7. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and let $\mathcal{C} = \mathcal{C}(E, q)$ be the associated Clifford algebra. Let π be a q-polarization. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the k-subalgebra of $\mathcal{C}$ generated by $(\pi E)^\sharp$ (resp. $((1-\pi)E)^\sharp$). One has canonical k-algebra isomorphisms

$$\bigwedge(\pi E) = \mathcal{A}, \quad \bigwedge((1-\pi)E) = \mathcal{B}.$$

Let $\{e_i\}_{i\in S}$ be a k-basis for πE indexed by a linearly ordered set S. Let $\{f_j\}_{j\in T}$ be a k-basis for $(1-\pi)E$ indexed by a linearly ordered set T. Put

$$e_I := e_{i_1}^\sharp \cdots e_{i_r}^\sharp \in \mathcal{A}, \quad f_J := f_{j_1}^\sharp \cdots f_{j_s}^\sharp \in \mathcal{B}$$

for all finite subsets

$$I = \{i_1 < \cdots < i_r\} \subset S, \quad J = \{j_1 < \cdots < j_s\} \subset T.$$

Let $\mathcal{I}^\star$ be the right ideal of $\mathcal{C}$ generated by $(\pi E)^\sharp$. Let $\mathcal{J}$ be the left ideal of $\mathcal{C}$ generated by $((1-\pi)E)^\sharp$. The families

$$\{e_I\}, \quad \{f_J\}, \quad \{e_I f_J\}, \quad \{e_I f_J\}_{I\neq\emptyset}, \quad \{e_I f_J\}_{J\neq\emptyset}$$

are k-bases for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{I}^\star$ and $\mathcal{J}$, respectively. One has direct sum decompositions

$$\mathcal{C} = \mathcal{A} \oplus \mathcal{J} = \mathcal{I}^\star \oplus \mathcal{B} = k \oplus (\mathcal{I}^\star + \mathcal{J}).$$

One has

$$g_1^\sharp \cdots g_n^\sharp \equiv \operatorname*{pfaff}_{i,j=1}^{n} q((1-\pi)g_i, g_j) \bmod \mathcal{I}^\star + \mathcal{J}$$

for any vectors $g_1, \ldots, g_n \in E$ spanning a k-submodule on which the quadratic form q vanishes identically. Thus the theory of Clifford algebras is linked to the theory of pfaffians.

Example 12.8. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and let $\mathcal{C} = \mathcal{C}(E, q)$ be the associated Clifford algebra. Let π be a q-polarization. Let $e_1 \in \pi E$ and $f_1 \in (1-\pi)E$ be given such that $q(e_1, f_1) = 1$. The k-linear endomorphism

$$\mu := (e \mapsto q(e_1, e)f_1 + q(f_1, e)e_1) : E \to E$$

is a q-projector commuting with π. Let E_1 be the k-span of e_1 and f_1. One has $\mu E = E_1$. Put $E_0 := (1-\mu)E$. Then $E = E_0 \oplus E_1$. For $i = 0, 1$, let $\mathcal{C}_i$ be the k-subalgebra of $\mathcal{C}$ generated by $E_i^\sharp$. Let $\mathcal{I}^\star \subset \mathcal{C}$ be the right ideal generated by $(\pi E)^\sharp$. Let $\mathcal{I} \subset \mathcal{C}$ be the left ideal generated by $((1-\pi)E)^\sharp$. Let $\mathcal{I}_0^\star \subset \mathcal{C}_0$ be the right ideal generated by $(\pi E_0)^\sharp$. Let $\mathcal{I}_0 \subset \mathcal{C}_0$ be the left ideal generated by $((1-\pi)E_0)^\sharp$. Let $\phi : E \to \mathrm{Mat}_2(\mathcal{C}_0)$ be the unique k-linear map such that

$$\phi\,(e_0 + xe_1 + yf_1) = \begin{bmatrix} e_0^\sharp & y \\ x & -e_0^\sharp \end{bmatrix}$$

for all $e_0 \in E_0$ and scalars x and y. Then

$$\phi(e)^2 = \begin{bmatrix} q(e) & 0 \\ 0 & q(e) \end{bmatrix}$$

for all $e \in E$, the induced k-algebra homomorphism

$$\phi^\natural : \mathcal{C} \to \mathrm{Mat}_2(\mathcal{C}_0)$$

is bijective, and one has

$$\phi^\natural(\mathcal{I}^\star) = \begin{bmatrix} \mathcal{I}_0^\star & \mathcal{I}_0^\star \\ \mathcal{C}_0 & \mathcal{C}_0 \end{bmatrix}, \quad \phi^\natural(\mathcal{I}) = \begin{bmatrix} \mathcal{I}_0 & \mathcal{C}_0 \\ \mathcal{I}_0 & \mathcal{C}_0 \end{bmatrix}.$$

EXAMPLE 12.9. Let E be a free k-module equipped with a k-basis $\{e_i, f_i\}_{i=1}^n$ and a k-quadratic form $q : E \to k$ such that

$$q\left(\sum_{i=1}^n x_i e_i + y_i f_i\right) = \sum_{i=1}^n x_i y_i$$

for all scalars $x_1, \dots, x_n, y_1, \dots, y_n \in k$. Let $\mathcal{C} = \mathcal{C}(E, q)$ be the associated Clifford algebra and put $\mathcal{A} := \mathrm{Mat}_{2^n}(k)$. Let $\mathcal{J}^\star \subset \mathcal{A}$ be the right ideal consisting of matrices with vanishing first row. Let $\mathcal{J} \subset \mathcal{A}$ be the left ideal consisting of matrices with vanishing first column. Let $\mathcal{I}^\star \subset \mathcal{C}$ be the right ideal generated by $e_1^\sharp, \dots, e_n^\sharp$. Let $\mathcal{I} \subset \mathcal{C}$ be the left ideal generated by $f_1^\sharp, \dots, f_n^\sharp$. There exists a k-algebra isomorphism $\phi : \mathcal{C} \xrightarrow{\sim} \mathcal{A}$ such that $\phi(\mathcal{I}^\star) = \mathcal{J}^\star$ and $\phi(\mathcal{I}) = \mathcal{J}$.

EXAMPLE 12.10. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$. We say that an element x of the Clifford algebra $\mathcal{C} = \mathcal{C}(E, q)$ is *even* (resp. *odd*) if x belongs to the k-span of all products of even (resp. odd) numbers of elements of $E^\sharp$. We say that $x \in \mathcal{C}$ is *parity-homogeneous* if x is either even or odd. If $x \in \mathcal{C}$ is parity-homogeneous, we write $\deg x = 0, 1$ according as x is even or odd. Every $x \in \mathcal{C}$ has a unique decomposition of the form $x = y + z$ where $y \in \mathcal{C}$ is even and $z \in \mathcal{C}$ is odd.

EXAMPLE 12.11. Let E be a free k-module equipped with a k-quadratic form $q : E \to k$ and let $\mathcal{C} = \mathcal{C}(E, q)$ be the associated Clifford algebra. Let μ be a q-projector. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the k-subalgebra generated by $(\mu E)^\sharp$. Let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the k-subalgebra generated by $((1 - \mu)E)^\sharp$. The natural map

$$(x_0 \otimes x_1 \mapsto x_0 x_1) : \mathcal{C}_0 \otimes_k \mathcal{C}_1 \to \mathcal{C}$$

is a k-linear isomorphism but not quite a k-algebra isomorphism: one has

$$x_1 x_0 = (-1)^{(\deg x_0)(\deg x_1)} x_0 x_1$$

for all parity-homogeneous $x_0 \in \mathcal{C}_0$ and $x_1 \in \mathcal{C}_1$.

13. The infinite wedge model of fermionic Fock space

Definition 13.1. Let $\mathcal{F}$ be the free k-module on the basis

$$|I\rangle \quad (I: \text{ a wedge index})$$

For each wedge index I we define $\langle I| \in \mathcal{F}^*$ by the rule

$$\langle I\,|J\rangle = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

for all wedge indices J. The abbreviations

$$|\bullet\rangle = |\mathbb{Z} \setminus \mathbb{N}\rangle, \quad \langle\bullet| = \langle\mathbb{Z} \setminus \mathbb{N}|$$

will sometimes be employed. Given $f, g \in \mathcal{H}$, let

$$f^\sharp, g^\flat : \mathcal{F} \to \mathcal{F}$$

be the unique k-linear endomorphisms such that

$$f^\sharp|I\rangle = \sum_{n\in\mathbb{Z}\setminus I} (-1)^{|\{i\in I|i>n\}|} f_n |I \cup \{n\}\rangle$$

and

$$g^\flat|I\rangle = \sum_{n\in I} (-1)^{|\{i\in I|i>n\}|} g_{1-n} |I \setminus \{n\}\rangle$$

for all wedge indices I. One readily verifies that

$$(f^\sharp)^2 = 0, \quad (g^\flat)^2 = 0, \quad f^\sharp g^\flat + g^\flat f^\sharp = \sum_{n\in\mathbb{Z}} f_n g_{1-n}.$$

We call $\mathcal{F}$ the *infinite wedge* model of *fermionic Fock space*.

Example 13.2. We continue working with the infinite wedge model of fermionic Fock space. For each $n \in \mathbb{Z}$ let e_n be the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix. One has

$$e_n^\sharp|I\rangle = \begin{cases} (-1)^{|\{i\in I|i>n\}|}|I \cup \{n\}\rangle & \text{if } n \notin I \\ 0 & \text{otherwise} \end{cases}$$

$$e_{1-n}^\flat|I\rangle = \begin{cases} (-1)^{|\{i\in I|i>n\}|}|I \setminus \{n\}\rangle & \text{if } n \in I \\ 0 & \text{otherwise} \end{cases}$$

$$\langle I|e_n^\sharp = \begin{cases} (-1)^{|\{i\in I|i>n\}|}\langle I \setminus \{n\}| & \text{if } n \in I \\ 0 & \text{otherwise} \end{cases}$$

$$\langle I|e_{1-n}^\flat = \begin{cases} (-1)^{|\{i\in I|i>n\}|}\langle I \cup \{n\}| & \text{if } n \notin I \\ 0 & \text{otherwise} \end{cases}$$

for all wedge indices I and $n \in \mathbb{Z}$.

REMARK 13.3. Strictly speaking the equation

$$|\{i_1 > i_2 > i_3 > \dots\}\rangle = e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \dots$$

is nonsense, but it nonetheless provides a valuable guide to intuition. This bit of nonsense is the rationale behind the terminology.

EXAMPLE 13.4. For each $n \in \mathbb{Z}$ again let e_n denote the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix. One has

$$|I\rangle = (-1)^{j_1+\cdots+j_s} e^\sharp_{i_1} \cdots e^\sharp_{i_r} e^\flat_{1-j_1} \cdots e^\flat_{1-j_s} |\bullet\rangle$$

for all wedge indices I where

$$I \cap \mathbb{N} = \{i_1 > \cdots > i_r\}, \quad (\mathbb{Z} \setminus \mathbb{N}) \setminus I = \{j_1 > \cdots > j_s\}.$$

One has

$$\begin{array}{rcl} |I\rangle & = & \pm e^\sharp_{i_1} \cdots e^\sharp_{i_r} e^\flat_{1-j_1} \cdots e^\flat_{1-j_s} |J\rangle \\ \langle I| & = & \pm \langle J| e^\sharp_{j_s} \cdots e^\sharp_{j_1} e^\flat_{1-i_r} \cdots e^\flat_{1-i_1} \\ 1 = \langle I\,|J\rangle & = & \langle J| e^\sharp_{j_s} \cdots e^\sharp_{j_1} e^\flat_{1-i_r} \cdots e^\flat_{1-i_1} e^\sharp_{i_1} \cdots e^\sharp_{i_r} e^\flat_{1-j_1} \cdots e^\flat_{1-j_s} |J\rangle \end{array}$$

for all wedge indices I and J where $I \setminus J = \{i_1 > \cdots > i_r\}$ and $J \setminus I = \{j_1 > \cdots > j_s\}$.

PROPOSITION 13.5. *Fix a wedge index I. Let $L \subseteq \mathcal{F}$ be the k-submodule consisting of all vectors ψ such that $f^\sharp\psi = 0$ and $g^\flat\psi = 0$ for all $f \in \mathcal{H}(I)$ and $g \in \mathcal{H}(\mathbb{Z}\setminus(1-I))$. Let $V \subseteq \mathcal{F}$ be the intersection of all k-submodules of $\mathcal{F}$ containing L, stable under $f^\sharp$ for all $f \in \mathcal{H}$, and stable under $g^\flat$ for all $g \in \mathcal{H}$. Let $H \subseteq \mathcal{F}$ be the k-submodule spanned by all vectors of the form $f^\sharp\psi$ or $g^\flat\psi$ where $\psi \in \mathcal{F}$, $f \in \mathcal{H}(\mathbb{Z} \setminus I)$ and $g \in \mathcal{H}(1 - I)$. The k-module L is spanned over k by the vector $|I\rangle$. One has $V = \mathcal{F}$. A vector $\psi \in \mathcal{F}$ is annihilated by $\langle I|$ if and only if $\psi \in H$.*

PROOF. This follows directly from Examples 13.2 and 13.4. □

PROPOSITION 13.6. *The only k-linear endomorphisms of $\mathcal{F}$ commuting with all operators of the form $f^\sharp$ or $g^\flat$ with $f, g \in \mathcal{H}$ are the scalar multiplication operators.*

PROOF. This follows formally from the preceding proposition. □

EXAMPLE 13.7. Let $\mathcal{C}$ be the Clifford algebra associated to the free k-module $\begin{bmatrix} \mathcal{H} \\ \mathcal{H} \end{bmatrix}$ and the k-quadratic form $\begin{bmatrix} g \\ h \end{bmatrix} \mapsto \sum_{n\in\mathbb{Z}} g_n h_{1-n}$. By writing

$$\begin{bmatrix} g \\ 0 \end{bmatrix}^\sharp := g^\sharp, \quad \begin{bmatrix} 0 \\ h \end{bmatrix}^\sharp := h^\flat$$

for all $g, h \in \mathcal{H}$, we make the k-module $\mathcal{F}$ naturally into a left $\mathcal{C}$-module. Let $\mathcal{I}^\star \subset \mathcal{C}$ be the right ideal generated by

$$\mathcal{H}(\mathbb{N})^\sharp + \mathcal{H}(\mathbb{N})^\flat.$$

Let $\mathcal{J} \subset \mathcal{C}$ be the left ideal generated by

$$\mathcal{H}(\mathbb{Z} \setminus \mathbb{N})^\sharp + \mathcal{H}(\mathbb{Z} \setminus \mathbb{N})^\flat.$$

The sequences

$$0 \to \mathcal{J} \subset \mathcal{C} \overset{x \mapsto x|\bullet\rangle}{\to} \mathcal{F} \to 0$$

and

$$0 \to \mathcal{I}^\star + \mathcal{J} \subset \mathcal{C} \overset{x \mapsto \langle\bullet|x|\bullet\rangle}{\to} k \to 0$$

are exact.

14. The diamond model of fermionic Fock space

DEFINITION 14.1. Let $\mathcal{F}_\diamond$ be the free k-module on the basis

$$|I\rangle \quad (I : \text{ a diamond index}).$$

For each diamond index I let $\langle I|$ denote the unique k-linear functional on $\mathcal{F}_\diamond$ such that

$$\langle I\,|J\rangle = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

for all diamond indices J. The abbreviations

$$|\bullet\rangle = |1-\mathbb{N}\rangle, \quad \langle\bullet| = \langle 1-\mathbb{N}|$$

will sometimes be employed. For each $h \in \mathcal{H}$, let

$$h^\sharp : \mathcal{F}_\diamond \to \mathcal{F}_\diamond$$

be the unique k-linear endomorphism such that

$$h^\sharp|I\rangle := \sum_{n\in\mathbb{Z}\setminus I} (-1)^{|\{i\in I|\max(n,1-n)<i\}|} h_n |I \cup \{n\} \setminus \{1-n\}\rangle$$

for all diamond indices I. One readily verifies that

$$(h^\sharp)^2 = \sum_{n\in\mathbb{N}} h_n h_{1-n}.$$

We call $\mathcal{F}_\diamond$ the *diamond* model of fermionic Fock space.

EXAMPLE 14.2. For each $n \in \mathbb{Z}$ let $e_n \in \mathcal{H}$ be the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix. One has

$$e_n^\sharp|I\rangle = \begin{cases} (-1)^{|\{i\in I|\max(n,1-n)<i\}|}|I\cup\{n\}\setminus\{1-n\}\rangle & \text{if } n\notin I \\ 0 & \text{otherwise} \end{cases}$$

$$\langle I|e_n^\sharp = \begin{cases} (-1)^{|\{i\in I|\max(n,1-n)<i\}|}\langle I\setminus\{n\}\cup\{1-n\}| & \text{if } n\in I \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{Z}$ and diamond indices I. One has

$$\langle I| = \langle\bullet|e^\sharp_{1-i_r}\cdots e^\sharp_{1-i_1}, \qquad |I\rangle = e^\sharp_{i_1}\cdots e^\sharp_{i_r}|\bullet\rangle$$

for all diamond indices I, where

$$\{i_1 > \cdots > i_r\} = I \cap \mathbb{N}.$$

One has

$$\begin{aligned} \pm|I\rangle &= e^\sharp_{i_1}\cdots e^\sharp_{i_r}|J\rangle, \\ \pm\langle I| &= \langle J|e^\sharp_{1-i_r}\cdots e^\sharp_{1-i_1}, \\ 1 &= \langle J|e^\sharp_{1-i_r}\cdots e^\sharp_{1-i_1}e^\sharp_{i_1}\cdots e^\sharp_{i_r}|J\rangle \end{aligned}$$

for all diamond indices I and J, where

$$\{i_1 > \cdots > i_r\} = I \cap (1-J).$$

EXAMPLE 14.3. One naturally regards $\mathcal{F}_\diamond$ as a left module over the Clifford algebra $\mathcal{C}_\diamond$ associated to the free k-module $\mathcal{H}$ and the k-quadratic form $h \mapsto \sum_{n\in\mathbb{N}} h_n h_{1-n}$. Let $\mathcal{I}^\star_\diamond \subset \mathcal{C}_\diamond$ be the right ideal generated by

$$\mathcal{H}(\mathbb{N})^\sharp.$$

Let $\mathcal{J}_\diamond \subset \mathcal{C}_\diamond$ be the left ideal generated by

$$\mathcal{H}(1-\mathbb{N})^\sharp.$$

The sequences

$$0 \to \mathcal{J}_\diamond \to \mathcal{C}_\diamond \stackrel{x\mapsto x|\bullet\rangle}{\to} \mathcal{F}_\diamond \to 0$$

and

$$0 \to \mathcal{I}^\star_\diamond + \mathcal{J}_\diamond \subset \mathcal{C}_\diamond \stackrel{x\mapsto \langle\bullet|x|\bullet\rangle}{\to} k \to 0$$

are exact.

PROPOSITION 14.4. *Fix a diamond index I. Let $L \subseteq \mathcal{F}_\diamond$ be the k-submodule consisting of all vectors ψ such that $h^\sharp\psi = 0$ for all $h \in \mathcal{H}(I)$. Let $V \subseteq \mathcal{F}_\diamond$ be the intersection of all k-submdules of $\mathcal{F}$ containing L and stable under $h^\sharp$ for all $h \in \mathcal{H}$. Let $H \subseteq \mathcal{F}_\diamond$ be the k-submodule spanned by all vectors of the form $h^\sharp\psi$ where $\psi \in \mathcal{F}_\diamond$ and $h \in \mathcal{H}(1-I)$. The k-module L is spanned over k by the vector $|I\rangle$. One has $V = \mathcal{F}_\diamond$. A vector $\psi \in \mathcal{F}_\diamond$ is annihilated by the k-linear functional $\langle I|$ if and only if $\psi \in H$.*

PROOF. This follows directly from Example 14.2. $\square$

PROPOSITION 14.5. *The only k-linear endomorphisms of $\mathcal{F}_\diamond$ commuting with all operators of the form $h^\sharp$ with $h \in \mathcal{H}$ are the scalar multiplication operators.*

PROOF. This is a formal consequence of Proposition 14.4. $\square$

EXAMPLE 14.6. Temporarily put

$$\Phi := (h \mapsto h^{[2]}) : \mathcal{H} \xrightarrow{\sim} \begin{bmatrix} \mathcal{H} \\ \mathcal{H} \end{bmatrix}.$$

We keep the notation introduced in Examples 13.7 and 14.3. The map ϕ induces a k-algebra isomorphism

$$\Phi^\natural : \mathcal{C}_\diamond \xrightarrow{\sim} \mathcal{C}$$

such that

$$\Phi^\natural(\mathcal{I}^\star_\diamond) = \mathcal{I}^\star, \quad \Phi^\natural(\mathcal{J}_\diamond) = \mathcal{J}.$$

It follows that there exists a unique k-linear isomorphism

$$\tilde{\Phi} : \mathcal{F}_\diamond \xrightarrow{\sim} \mathcal{F}$$

such that

$$\tilde{\Phi}|\bullet\rangle = |\bullet\rangle, \quad \langle\bullet|\tilde{\Phi} = \langle\bullet|,$$

and

$$\tilde{\Phi}(h^\sharp\psi) = (f^\sharp + g^\flat)\tilde{\Phi}(\psi)$$

for all $f, g, h \in \mathcal{H}$ such that

$$\Phi(h) = \begin{bmatrix} f \\ g \end{bmatrix}$$

and $\psi \in \mathcal{F}_\diamond$. One can verify that

$$\tilde{\Phi}|I^\diamond\rangle = \pm|I\rangle$$

for all wedge indices I. (We omit the somewhat unpleasant description of the sign rule because fortunately we will not need to use it.) Thus the infinite wedge and diamond models of fermionic Fock space are equivalent.

15. The fundamental theorem

DEFINITION 15.1. Let $A \in \mathcal{Q}$ be a split orthogonal matrix. A k-linear automorphism $\tilde{A}$ of $\mathcal{F}_\diamond$ is called a *diamond representation* of A if

$$\tilde{A}(h^\sharp \psi) = (Ah)^\sharp \tilde{A}\psi$$

for all $h \in \mathcal{H}$ and $\psi \in \mathcal{F}_\diamond$. By Proposition 14.5 diamond representations are unique up to an invertible scalar multiple.

DEFINITION 15.2. Let $A \in \mathcal{Q}^\times$ be given. A k-linear automorphism $\tilde{A}$ of $\mathcal{F}$ is called an *infinite wedge representation* of A if

$$\tilde{A}(g^\sharp \psi) = (Ag)^\sharp \tilde{A}\psi, \quad \tilde{A}(h^\flat \psi) = (A^{-\dagger} h)^\flat \tilde{A}\psi$$

for all $g, h \in \mathcal{H}$ and $\psi \in \mathcal{F}$, where $A^{-\dagger} := (A^{-1})^\dagger$. By Proposition 13.6, infinite wedge representations are unique up to an invertible scalar multiple.

EXAMPLE 15.3. In fact the notion of diamond representation encompasses the notion of infinite wedge representation. Let $\tilde{\Phi} : \mathcal{F}_\diamond \xrightarrow{\sim} \mathcal{F}$ be the k-linear isomorphism constructed in Example 14.6. Let $A \in \mathcal{Q}^\times$ be given and let $B \in \mathcal{Q}$ be the unique split orthogonal matrix such that

$$B^{[2]} = \begin{bmatrix} A & 0 \\ 0 & A^{-\dagger} \end{bmatrix}.$$

(Look back at Definition 7.11.) Given a diamond representation $\tilde{B}$ of B, there exists a unique infinite wedge representation $\tilde{A}$ of A such that the diagram

$$\begin{array}{ccccc} & & \tilde{B} & & \\ & \mathcal{F}_\diamond & \rightarrow & \mathcal{F}_\diamond & \\ \tilde{\Phi} & \downarrow & & \downarrow & \tilde{\Phi} \\ & \mathcal{F} & \rightarrow & \mathcal{F} & \\ & & \tilde{A} & & \end{array}$$

commutes, and conversely, given an infinite wedge representation $\tilde{A}$ of A, there exists a unique diamond representation $\tilde{B}$ of B rendering the diagram above commutative.

EXAMPLE 15.4. As in Example 14.3, let $\mathcal{C}_\diamond$ be the Clifford algebra associated to the free k-module $\mathcal{H}$ and the k-quadratic form $h \mapsto \sum_{n\in\mathbb{N}} h_n h_{1-n}$ and let $\mathcal{J}_\diamond$ be the left ideal generated by $\mathcal{H}(1-\mathbb{N})^\sharp$. Let $A \in \mathcal{Q}$ be a split orthogonal matrix. Let

$$A^\natural : \mathcal{C}_\diamond \xrightarrow{\sim} \mathcal{C}_\diamond$$

be the induced k-algebra automorphism induced by the k-linear automorphism

$$(h \mapsto Ah) : \mathcal{H} \xrightarrow{\sim} \mathcal{H}.$$

There exist $x, y \in \mathcal{C}_\diamond$ such that

$$(A^\natural \mathcal{J}_\diamond)x \subseteq \mathcal{J}_\diamond, \quad \mathcal{J}_\diamond y \subseteq A^\natural \mathcal{J}_\diamond, \quad xy \equiv 1 \bmod A^\natural \mathcal{J}_\diamond, \quad yx \equiv 1 \bmod \mathcal{J}_\diamond$$

if and only if there exists a diamond representation $\tilde{A}$ of A such that

$$\tilde{A}|\bullet\rangle = x|\bullet\rangle.$$

EXAMPLE 15.5. As in Example 13.7, let $\mathcal{C}$ be the Clifford algebra associated to the free k-module $\left[\begin{array}{c}\mathcal{H}\\ \mathcal{H}\end{array}\right]$ and the quadratic form

$$\left[\begin{array}{c} g \\ h \end{array}\right] \mapsto \sum_{n\in\mathbb{Z}} g_n h_{1-n},$$

write

$$\left[\begin{array}{c} g \\ 0 \end{array}\right]^\sharp = g^\sharp, \quad \left[\begin{array}{c} 0 \\ h \end{array}\right]^\sharp = h^\flat,$$

and let $\mathcal{J} \subset \mathcal{C}$ be the left ideal generated by

$$\mathcal{H}(\mathbb{Z}\setminus\mathbb{N})^\sharp + \mathcal{H}(\mathbb{Z}\setminus\mathbb{N})^\flat.$$

Let $A \in \mathcal{Q}^\times$ be given. Let

$$A^\natural : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

be the k-algebra automorphism induced by the k-linear automorphism

$$\left(\left[\begin{array}{c} g \\ h \end{array}\right] \mapsto \left[\begin{array}{c} Ag \\ A^{-\dagger}h \end{array}\right]\right) : \left[\begin{array}{c} \mathcal{H} \\ \mathcal{H} \end{array}\right] \xrightarrow{\sim} \left[\begin{array}{c} \mathcal{H} \\ \mathcal{H} \end{array}\right].$$

There exist $x, y \in \mathcal{C}$ such that

$$(A^\natural \mathcal{J})x \subseteq \mathcal{J}, \quad \mathcal{J} y \subseteq A^\natural \mathcal{J}, \quad xy \equiv 1 \bmod A^\natural \mathcal{J}, \quad yx \equiv 1 \bmod \mathcal{J}$$

if and only if there exists an infinite wedge representation $\tilde{A}$ of A such that

$$\tilde{A}|\bullet\rangle = x|\bullet\rangle.$$

EXAMPLE 15.6. The triple

$$A = \mathbf{r}, \quad x = e_1^\sharp, \quad y = e_0^\sharp$$

satisfies the conditions enunciated in Example 15.4, and hence there exists a unique diamond representation $\tilde{\mathbf{r}}$ of $\mathbf{r}$ such that

$$\tilde{\mathbf{r}}|\bullet\rangle = e_1^\sharp|\bullet\rangle.$$

EXAMPLE 15.7. If $A \in \mathcal{Q}$ is a split orthogonal matrix with vanishing $\mathbb{N}$ by $1-\mathbb{N}$ block, the triple

$$A, \quad x = 1, \quad y = 1$$

satisfies the conditions enunciated in Example 15.4, and hence there exists a unique diamond representation $\tilde{A}$ of A such that

$$\tilde{A}|\bullet\rangle = |\bullet\rangle.$$

EXAMPLE 15.8. The triple

$$A = \mathbf{t}, \quad x = e_1^\flat, \quad y = e_0^\sharp$$

satisfies the conditions enunciated in Example 15.5 and hence there exists a unique infinite wedge representation $\tilde{\mathbf{t}}$ of $\mathbf{t}$ such that

$$\tilde{\mathbf{t}}|\bullet\rangle = e_1^\flat|\bullet\rangle.$$

EXAMPLE 15.9. If $A \in \mathcal{Q}^\times$ is such that the $\mathbb{N}$ by $1-\mathbb{N}$ blocks of both A and A^{-1} vanish, the triple

$$A, \quad x = 1, \quad y = 1$$

satisfies the conditions enunciated in Example 15.5 and hence there exists a unique infinite wedge representation $\tilde{A}$ of A such that

$$\tilde{A}|\bullet\rangle = |\bullet\rangle.$$

THEOREM 15.10.

1. *Let A be an almost upper triangular split orthogonal $\mathbb{Z}$ by $\mathbb{Z}$ matrix with scalar entries. There exists a diamond representation of A unique up to invertible scalar multiple.*

2. *Let $A \in \mathcal{Q}^\times$ be given. There exists an infinite wedge representation of A unique up to invertible scalar multiple.*

PROOF. In both diamond and infinite wedge cases, uniqueness has already been noted. Existence in the diamond case follows from Examples 11.13, 15.6 and 15.7. Existence in the infinite wedge case follows from Examples 11.12, 15.8, and 15.9. $\square$

16. Matrix coefficient calculations in the diamond model

EXAMPLE 16.1. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame split orthogonal permutation. There exists a diamond representation $\tilde{W}$ of the matrix W representing ω such that

$$\tilde{W}|I\rangle = \pm|\omega(I)\rangle$$

for all diamond indices I.

EXAMPLE 16.2. Fix diamond indices I and J . Let a split orthogonal matrix $A \in \mathcal{Q}$ be given with a

$$\left[\begin{array}{c} I \\ 1-I \end{array}\right] \times \left[\begin{array}{c} J \\ 1-J \end{array}\right]^T$$

block decomposition of the form

$$A = \left[\begin{array}{cc} a & b \\ 0 & d \end{array}\right] \text{ or } A = \left[\begin{array}{cc} a & 0 \\ c & d \end{array}\right].$$

Let $\tilde{A}$ be a diamond representation of A and put

$$x := \langle I|\tilde{A}|J\rangle.$$

Under the first assumption one has

$$A\mathcal{H}(J) = \mathcal{H}(I)$$

and hence

$$\tilde{A}|J\rangle = x|I\rangle.$$

Under the second assumption one has

$$A\mathcal{H}(1-J) = \mathcal{H}(1-I)$$

and hence

$$\langle J|\tilde{A} = x\langle I|.$$

Both conclusions are justified by Fundamental Theorem combined with Proposition 14.4.

EXAMPLE 16.3. Let $A \in \mathcal{Q}^\times$ be split orthogonal and assume that the

$$\begin{bmatrix} 1-\mathbb{N} \\ \mathbb{N} \end{bmatrix} \times \begin{bmatrix} 1-\mathbb{N} \\ \mathbb{N} \end{bmatrix}^T$$

block decomposition of A takes the form

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Let I be any diamond index and write

$$I \cap \mathbb{N} = \{i_1 > \cdots > i_r\}.$$

For each $n \in \mathbb{Z}$ let e_n be the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix, let a_n be the n^{th} column of A, and let $\bar{a}_n$ be the n^{th} column of $A^\dagger$. Let $\tilde{A}$ be any diamond representation of A. One has

$$\begin{aligned}
\langle \bullet | \tilde{A} | I \rangle &= \langle \bullet | \tilde{A} e_{i_1}^\sharp \cdots e_{i_r}^\sharp | \bullet \rangle \\
&= \langle \bullet | a_{i_1}^\sharp \cdots a_{i_r}^\sharp | \bullet \rangle \langle \bullet | \tilde{A} | \bullet \rangle \\
&= \left(\operatorname*{pfaff}_{\mu,\nu=1}^{r} \sum_{n \in \mathbb{N}} A_{1-n,i_\mu} A_{n,i_\nu} \right) \langle \bullet | \tilde{A} | \bullet \rangle.
\end{aligned}$$

under the first assumption. One has

$$\begin{aligned}
\langle I | \tilde{A} | \bullet \rangle &= \langle \bullet | e_{1-i_r}^\sharp \cdots e_{1-i_1}^\sharp \tilde{A} | \bullet \rangle \\
&= \langle \bullet | \bar{a}_{1-i_r}^\sharp \cdots \bar{a}_{1-i_1}^\sharp | \bullet \rangle \langle \bullet | \tilde{A} | \bullet \rangle \\
&= \left(\operatorname*{pfaff}_{\mu,\nu=1}^{r} \sum_{n \in \mathbb{N}} A^\dagger_{n,1-i_\mu} A^\dagger_{1-n,1-i_\nu} \right) \langle \bullet | \tilde{A} | \bullet \rangle \\
&= \left(\operatorname*{pfaff}_{\mu,\nu=1}^{r} \sum_{n \in \mathbb{N}} A_{i_\mu,1-n} A_{i_\nu,n} \right) \langle \bullet | \tilde{A} | \bullet \rangle
\end{aligned}$$

under the second assumption.

EXAMPLE 16.4. Let split orthogonal $A \in \mathcal{Q}$ be given. Let $\tilde{A}$ be a diamond representation of A. Fix a diamond index J and put

$$x := \langle J|\tilde{A}|J\rangle.$$

Make one of the following assumptions:

- $\tilde{A}|J\rangle = x|J\rangle$.
- $\langle J|\tilde{A} = x\langle J|$.

Let I be another diamond index and write

$$I \setminus J = \{i_1 > \cdots > i_r\}, \quad J \setminus I = \{1 - i_1 > \cdots > 1 - i_r\}.$$

For each $n \in \mathbb{Z}$, let e_n denote the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix, let a_n denote the n^{th} column of A, and let $\bar{a}_n$ denote the n^{th} column of $A^\dagger$. One has

$$\begin{aligned} \langle I|\tilde{A}|I\rangle &= \langle J|e^\sharp_{1-i_r} \cdots e^\sharp_{1-i_1} \tilde{A} e^\sharp_{i_1} \cdots e^\sharp_{i_r}|J\rangle \\ &= \langle J|e^\sharp_{1-i_r} \cdots e^\sharp_{1-i_1} a^\sharp_{i_1} \cdots a^\sharp_{i_r}|J\rangle\langle J|\tilde{A}|J\rangle \\ &= \left(\det_{i,j\in I\setminus J} A_{ij}\right)\langle J|\tilde{A}|J\rangle \end{aligned}$$

under the first assumption. One has

$$\begin{aligned} \langle I|\tilde{A}|I\rangle &= \langle J|e^\sharp_{1-i_r} \cdots e^\sharp_{1-i_1} \tilde{A} e^\sharp_{i_1} \cdots e^\sharp_{i_r}|J\rangle \\ &= \langle J|\bar{a}^\sharp_{1-i_r} \cdots \bar{a}^\sharp_{1-i_1} e^\sharp_{i_1} \cdots e_{i_r}|J\rangle\langle J|\tilde{A}|J\rangle \\ &= \left(\det_{i,j\in I\setminus J} A^\dagger_{1-j,1-i}\right)\langle J|\tilde{A}|J\rangle \\ &= \left(\det_{i,j\in I\setminus J} A_{ij}\right)\langle J|\tilde{A}|J\rangle \end{aligned}$$

under the second assumption.

17. The transversality theorem

THEOREM 17.1. *Let split orthogonal $A \in \mathcal{Q}$ be given. Let $\tilde{A}$ be any diamond representation of A. The following assertions are equivalent:*

- $\langle \bullet | \tilde{A} | \bullet \rangle \not\equiv 0 \bmod m$.
- *The* $1-\mathbb{N}$ *by* $1-\mathbb{N}$ *block of* A *is invertible.*
- *The* $\mathbb{N}$ *by* $\mathbb{N}$ *block of* A^{-1} *is invertible.*
- $\mathcal{H} = \mathcal{H}(\mathbb{N}) \oplus A\mathcal{H}(1-\mathbb{N})$.
- $\mathcal{H} = A^{-1}\mathcal{H}(\mathbb{N}) \oplus \mathcal{H}(1-\mathbb{N})$.

PROOF. The equivalence of the last four conditions is trivial. Without loss of generality we may assume that $m = 0$ and hence that k is a field. After making the evident reductions based on Examples 10.9, 11.13 and 16.3, we may assume that for some positive integer n one has a factorization

$$A = BW$$

where $B, W \in \mathcal{Q}$ have the following properties:

- B and W are split orthogonal.
- $B - 1$ vanishes outside the $1-\mathbb{N}$ by $\mathbb{N}$ block.
- The tame split orthogonal permutation

  $$\omega := \left(i \mapsto \left\{ \begin{array}{ll} 1-i & \text{if } -n < i \le n \\ i & \text{otherwise} \end{array} \right. \right) : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$$

  is represented by W.

Let $\tilde{B}$ and $\tilde{W}$ be Fock representations of B and W. Without loss of generality we may assume that $\tilde{A} = \tilde{B}\tilde{W}$. Put

$$I := \omega(1-\mathbb{N}) = \{1, \ldots, n\} \cup \{-n, -n-1, -n-2, \ldots, \}.$$

Given $x, y \in k$ we write $x \sim y$ if x and y generate the same ideal of k. We now calculate with

$$\left[\begin{array}{c} \{i \le -n\} \\ \{-n < i \le 0\} \\ \{0 < i \le n\} \\ \{n < i\} \end{array} \right] \times \left[\begin{array}{c} \{j \le -n\} \\ \{-n < j \le 0\} \\ \{0 < j \le n\} \\ \{n < j\} \end{array} \right]^T$$

block decompositions. Write

$$B = \left[\begin{array}{cccc} 1 & 0 & b_{13} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right], \quad W = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & w_{23} & 0 \\ 0 & w_{32} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

Then

$$A = \begin{bmatrix} 1 & b_{13}w_{32} & 0 & b_{14} \\ 0 & b_{23}w_{23} & w_{23} & b_{24} \\ 0 & w_{32} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad w_{23} = \underbrace{\begin{bmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{bmatrix}}_{n}.$$

The matrix b_{23} is dagger-alternating, hence the matrix $b_{23}w_{23}$ alternating. One has

$$\operatorname{pfaff} b_{23}w_{23} \sim \langle \bullet | \tilde{B} | I \rangle \sim \langle \bullet | \tilde{B}\tilde{W} | \bullet \rangle = \langle \bullet | \tilde{A} | \bullet \rangle$$

by Examples 16.1 and 16.3. Finally, $\det b_{23}w_{23} = (\operatorname{pfaff} b_{23}w_{23})^2$ is a unit of k if and only if the $1-\mathbb{N}$ by $1-\mathbb{N}$ block of A is invertible. We are done. □

Corollary 17.2. *Let $A \in \mathcal{Q}^\times$ be given. Let*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$$

be the

$$\begin{bmatrix} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{bmatrix} \times \begin{bmatrix} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{bmatrix}^T$$

block decompositions of A and A^{-1}, respectively. Let $\tilde{A}$ be any infinite wedge representation of $A^\diamond$. The following assertions are equivalent:

- $\langle \bullet | \tilde{A} | \bullet \rangle \not\equiv 0 \bmod m$.
- *The block a is invertible.*
- *The block $\bar{d}$ is invertible.*
- $\mathcal{H} = \mathcal{H}(\mathbb{N}) \oplus A\mathcal{H}(1-\mathbb{N})$.
- $\mathcal{H} = A^{-1}\mathcal{H}(\mathbb{N}) \oplus \mathcal{H}(1-\mathbb{N})$.

Proof. The equivalence of the last four conditions is trivial. Let $B \in \mathcal{Q}$ be the unique split orthogonal matrix such that

$$B^{[2]} = \begin{bmatrix} A & 0 \\ 0 & A^{-\dagger} \end{bmatrix}.$$

By Example 15.3 and the theorem, invertibility of the $1-\mathbb{N}$ by $1-\mathbb{N}$ block of B is equivalent to condition 1. By direct calculation one verifies that the $1-\mathbb{N}$ by $1-\mathbb{N}$ block of B is invertible if and only if both a and $\bar{d}$ are invertible. □

18. Matrix coefficient calculations in the infinite wedge model

EXAMPLE 18.1. Let $\omega : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$ be a tame permutation. Let $W \in \mathcal{Q}^\times$ be the permutation matrix representing ω. There exists an infinite wedge representation $\tilde{W}$ of W such that

$$\tilde{W}|I\rangle = \pm|\omega(I)\rangle$$

for all wedge indices I.

EXAMPLE 18.2. Fix wedge indices I and J . Let $A \in \mathcal{Q}^\times$ be given such that the

$$\left[\begin{array}{c} I \\ \mathbb{Z}\setminus I \end{array}\right] \times \left[\begin{array}{c} J \\ \mathbb{Z}\setminus J \end{array}\right]^T$$

block decompositions of A and A^{-1} are of one of the following two forms:

- $A = \left[\begin{array}{cc} a & b \\ 0 & d \end{array}\right], \quad A^{-1} = \left[\begin{array}{cc} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{array}\right]$
- $A = \left[\begin{array}{cc} a & 0 \\ c & d \end{array}\right], \quad A^{-1} = \left[\begin{array}{cc} \bar{a} & 0 \\ \bar{c} & \bar{d} \end{array}\right]$

Let $\tilde{A}$ be an infinite wedge representation of A and put

$$x := \langle I|\tilde{A}|J\rangle.$$

Under the first assumption one has

$$A\mathcal{H}(J) = \mathcal{H}(I)$$

and hence

$$\tilde{A}|J\rangle = x|I\rangle.$$

Under the second assumption one has

$$A\mathcal{H}(\mathbb{Z}\setminus J) = \mathcal{H}(\mathbb{Z}\setminus I)$$

and hence

$$\langle J|\tilde{A} = x\langle I|.$$

Both conclusions are justified by Fundamental Theorem combined with Proposition 13.5.

EXAMPLE 18.3. Let $A \in \mathcal{Q}^\times$ and a wedge index J be given. Let $\tilde{A}$ be an infinite wedge representation of A and put

$$x := \langle J|\tilde{A}|J\rangle.$$

Assume that

$$\tilde{A}|J\rangle = x|J\rangle.$$

Let I be another wedge index and write

$$I \setminus J = \{i_1 > \cdots > i_r\}, \quad J \setminus I = \{j_1 > \cdots > j_s\}.$$

For each $n \in \mathbb{Z}$, let e_n denote the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix, let a_n denote the n^{th} column of A, and let $\bar{a}_n$ denote the n^{th} column of $A^{-\dagger}$. One has

$$\begin{array}{rcl}
\langle I|\tilde{A}|I\rangle & = & \langle J|e^\sharp_{j_s}\cdots e^\sharp_{j_1}e^\flat_{1-i_r}\cdots e^\flat_{1-i_1}\tilde{A}e^\sharp_{i_1}\cdots e^\sharp_{i_r}e^\flat_{1-j_1}\cdots e^\flat_{1-j_s}|J\rangle \\
 & = & \langle J|e^\sharp_{j_s}\cdots e^\sharp_{j_1}e^\flat_{1-i_r}\cdots e^\flat_{1-i_1}a^\sharp_{i_1}\cdots a^\sharp_{i_r}\bar{a}^\flat_{1-j_1}\cdots \bar{a}^\flat_{1-j_s}|J\rangle\langle J|\tilde{A}|J\rangle \\
 & = & \left(\det\limits_{i,j\in I\setminus J} A_{ij}\right)\left(\det\limits_{i,j\in J\setminus I} A^{-\dagger}_{1-j,1-i}\right)\langle J|\tilde{A}|J\rangle \\
 & = & \left(\det\limits_{i,j\in I\setminus J} A_{ij}\right)\left(\det\limits_{i,j\in J\setminus I} A^{-1}_{ij}\right)\langle J|\tilde{A}|J\rangle.
\end{array}$$

This formula links fermionic Fock space to the Kyoto school notion of τ-function.

DEFINITION 18.4. We introduce the abbreviated notation

$$\begin{array}{rcl}
\langle N| & := & \langle\{n \in \mathbb{Z} \mid n \leq N\}|, \\
|N\rangle & := & |\{n \in \mathbb{Z} \mid n \leq N\}\rangle
\end{array}$$

where N is any integer. In particular, one has $\langle\bullet| = \langle 0|$ and $|\bullet\rangle = |0\rangle$.

DEFINITION 18.5. We say that $A \in \mathcal{Q}$ belongs to the *deformation class* if $A - 1$ is strictly lower triangular with entries in m. The set of deformation class matrices forms a group under matrix multiplication. For each $A \in \mathcal{Q}^\times$ of the deformation class, there exists a unique infinite wedge representation $\tilde{A}$ of A such that

$$\langle N|\tilde{A} = \langle N|$$

for all $N \in \mathbb{Z}$; in this situation we say that $\tilde{A}$ is *left-normalized*.

DEFINITION 18.6. Let matrix $A \in \mathcal{Q}$ be given. We say that A *belongs to the test class* if for some positive integer N the

$$\left[\begin{array}{c} \{i \leq -N\} \\ \{-N < i \leq N\} \\ \{N < i\} \end{array}\right] \times \left[\begin{array}{c} \{j \leq -N\} \\ \{-N < j \leq N\} \\ \{N < j\} \end{array}\right]^T$$

block decomposition of A takes the form

$$A = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{array}\right]$$

where the blocks have the following properties:

- $a_{11} - 1$ and $a_{33} - 1$ are strictly upper triangular.
- The matrix a_{22} is invertible.

The matrices of the test class form a group under matrix multiplication. For each $A \in \mathcal{Q}^\times$ of the test class there exists a unique infinite wedge representation $\tilde{A}$ of A such that

$$\tilde{A}|N\rangle = |N\rangle$$

for all integers $N \ll 0$; in this situation we say that $\tilde{A}$ is *right-normalized.*

EXAMPLE 18.7. If a matrix $A \in \mathcal{Q}^\times$ belongs both to the test class and to the deformation class, an infinite wedge representation of A is right-normalized if and only if left-normalized.

EXAMPLE 18.8. Let $A \in \mathcal{Q}^\times$ of the deformation class and $B \in \mathcal{Q}^\times$ of the test class be given. Let

$$A = \left[\begin{array}{cc} a_{11} & 0 \\ a_{21} & a_{22} \end{array}\right]$$

be the

$$\left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right] \times \left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right]$$

block decomposition of A and put

$$U := \left[\begin{array}{cc} a_{11} & 0 \\ 0 & a_{22} \end{array}\right], \quad C := AU^{-1} = \left[\begin{array}{cc} 1 & 0 \\ a_{21}a_{11}^{-1} & 1 \end{array}\right].$$

Let $\tilde{B}$ be the right-normalized infinite wedge representation of B. Let $\tilde{A}$, $\tilde{U}$ and $\tilde{C}$ be the left-normalized infinite wedge representations of A, U and C, respectively. Necessarily one has $\tilde{A} = \tilde{C}\tilde{U}$. Moreover the matrix C belongs to the test class and hence $\tilde{C}$ is also right-normalized. The upshot is that

$$\langle \bullet | \tilde{B}\tilde{A} | \bullet \rangle = \langle \bullet | \tilde{B}\tilde{C} | \bullet \rangle = \det_{i,j=1}^{N} (BC)_{1-i,1-j} = \det_{i,j=1}^{N} (BA)_{1-i,1-j}$$

for all $N \gg 0$ by Example 18.3.

Example 18.9. Let A, B, C, $\tilde{A}$ and $\tilde{B}$ be as in Example 18.8. Suppose also that for some positive integer N the

$$\left[\begin{array}{c} \{i \leq -N\} \\ \{-N < i \leq N\} \\ \{N < i\} \end{array}\right] \times \left[\begin{array}{c} \{j \leq -N\} \\ \{-N < j \leq N\} \\ \{N < j\} \end{array}\right]^T$$

block decompositions of B and C take the form

$$B = \left[\begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & b_{33} \end{array}\right], \quad C = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & 1 \end{array}\right],$$

respectively. Then $\langle \bullet | \tilde{B}\tilde{A} | \bullet \rangle = 1$.

Definition 18.10. Let $A \in \mathcal{Q}$ be given. Let c be a scalar. Let I and J be wedge indices. Write

$$I = \{i_1 > i_2 > \dots\}, \quad J = \{j_1 > j_2 > \dots\}.$$

We write

$$c = A_{IJ}$$

if one of the following conditions hold:

- $|I \setminus J| \neq |J \setminus I|$ and $c = 0$.
- $|I \setminus J| = |J \setminus I|$ and

  $$c = \det_{\mu,\nu=1}^{N} A_{i_\mu, j_\nu}$$

  for all but finitely many positive integers N.

Otherwise the expression A_{IJ} is not defined.

Example 18.11. If a matrix $A \in \mathcal{Q}^\times$ belongs either to the test class or to the deformation class, the minor A_{IJ} is defined for all wedge indices I and J.

EXAMPLE 18.12. Let $A \in \mathcal{Q}$ of the deformation class be given, and let $\tilde{A}$ be the left-normalized infinite wedge representation of A. Let I and J be wedge indices. We claim that

$$\langle I|\tilde{A}|J\rangle = A_{IJ}.$$

To prove the claim, write

$$I = \{i_1 > \cdots > i_r\} \coprod \{n \leq N\}, \quad J = \{j_1 > \cdots > j_s\} \coprod \{n \leq N\}$$

where N is any sufficiently negative integer. For each $n \in \mathbb{Z}$, let e_n denote the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix, and let $\bar{a}_n$ denote the n^{th} column of $A^\dagger$. One has

$$\begin{aligned}
\langle I|\tilde{A}|J\rangle &= \langle N|e^\flat_{1-i_r} \cdots e^\flat_{1-i_1} \tilde{A} e^\sharp_{j_1} \cdots e^\sharp_{j_s}|N\rangle \\
&= \langle N|\tilde{A}((A^{-1})^{-\dagger} e_{1-i_r})^\flat \cdots ((A^{-1})^{-\dagger} e_{1-i_1})^\flat e^\sharp_{j_1} \cdots e^\sharp_{j_s}|N\rangle \\
&= \langle N|\bar{a}^\flat_{1-i_r} \cdots \bar{a}^\flat_{1-i_1} e^\sharp_{j_1} \cdots e^\sharp_{j_s}|N\rangle \\
&= A_{IJ}.
\end{aligned}$$

The claim is proved.

EXAMPLE 18.13. Let $A \in \mathcal{Q}$ of the test class be given, and let $\tilde{A}$ be the right-normalized infinite wedge representation of A. Let I and J be wedge indices. We claim that

$$\langle I|\tilde{A}|J\rangle = A_{IJ}.$$

As above, to prove the claim, write

$$I = \{i_1 > \cdots > i_r\} \coprod \{n \leq N\}, \quad J = \{j_1 > \cdots > j_s\} \coprod \{n \leq N\}$$

where N is any sufficiently negative integer. For each $n \in \mathbb{Z}$ let e_n be the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix and let a_n be the n^{th} column of A. One has

$$\begin{aligned}
\langle I|\tilde{A}|J\rangle &= \langle N|e^\flat_{1-i_r} \cdots e^\flat_{1-i_1} \tilde{A} e^\sharp_{j_1} \cdots e^\sharp_{j_s}|N\rangle \\
&= \langle N|e^\flat_{1-i_r} \cdots e^\flat_{1-i_1} a^\sharp_{j_1} \cdots a^\sharp_{j_s}|N\rangle \\
&= A_{IJ}.
\end{aligned}$$

The claim is proved.

EXAMPLE 18.14. If $A \in \mathcal{Q}^\times$ belongs to the deformation class or to the test class, then for each wedge index J there exist only finitely many wedge indices I such that $A_{IJ} \neq 0$.

EXAMPLE 18.15. Let

$$h = \sum_i h_i t^i \in 1 + tk[[t]]$$

be given and put

$$H := h(\mathbf{t}) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & h_1 & h_2 & h_3 & \\ \ddots & 0 & 1 & h_1 & h_2 & \ddots \\ \ddots & 0 & 0 & 1 & h_1 & \ddots \\ & 0 & 0 & 0 & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

thereby defining a matrix of the test class, and let $\tilde{H}$ be the right-normalized infinite wedge representation of H. Let an integer N and partitions λ and μ be given. Put

$$I := \{N + 1 - i + \lambda_i \mid i \in \mathbb{N}\}, \quad J := \{N + 1 - j + \mu_j \mid j \in \mathbb{N}\},$$

thereby defining wedge indices such that $|I \setminus J| = |J \setminus I|$. One has

$$\det_{i,j=1}^{\max(\ell(\lambda),\ell(\mu))} h_{\mu_j - \lambda_i + i - j} = H_{IJ} = \langle I|\tilde{H}|J\rangle.$$

The determinant on the left is the so called *skew S-function* associated to λ and μ, taking the h's as the complete symmetric function. One has

$$\det_{i,j=1}^{\ell(\mu)} h_{\mu_i - i + j} = \langle \bullet|\tilde{H}|J\rangle$$

in the special case $N = 0$ and $\lambda = 0$; thus Schur functions are recovered as matrix coefficients.

EXAMPLE 18.16. Let

$$h = \sum_i h_i t^{-i} \in 1 + t^{-1}m[t^{-1}]$$

be given and put

$$H := h(\mathbf{t}) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & 0 & 0 & 0 & \\ \ddots & h_1 & 1 & 0 & 0 & \ddots \\ \ddots & h_2 & h_1 & 1 & 0 & \ddots \\ & h_3 & h_2 & h_1 & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

thereby defining a matrix of the deformation class, and let $\tilde{H}$ be the left-normalized infinite wedge representation of H. Let an integer N and partitions λ and μ be given. Put

$$I := \{N + 1 - i + \lambda_i \mid i \in \mathbb{N}\}, \quad J := \{N + 1 - j + \mu_j \mid j \in \mathbb{N}\},$$

thereby defining wedge indices such that $|I \setminus J| = |J \setminus I|$. One has

$$\det_{i,j=1}^{\max(\ell(\lambda),\ell(\mu))} h_{\lambda_i - i - \mu_j + j} = H_{IJ} = \langle I|\tilde{H}|J\rangle.$$

Again skew S-functions appear as matrix coefficients. One has

$$\det_{i,j=1}^{\ell(\lambda)} h_{\lambda_i - i + j} = \langle I|\tilde{H}|\bullet\rangle$$

in the special case $\mu = 0$ and $N = 0$; thus again Schur functions appear as matrix coefficients.

EXAMPLE 18.17. Let k_0 be an artinian local ring and assume that

$$k = k_0[\epsilon]/(\epsilon^N),$$

where ϵ is a variable and N is a large positive integer which eventually we let tend to infinity. Let $\mathcal{Q}_0 \subseteq \mathcal{Q}$ be the k_0-subalgebra consisting of matrices with entries in k_0. Fix $A \in \mathcal{Q}_0$ of the test class and put

$$H = \sum_{i=0}^{\infty} \epsilon^i \mathbf{t}^{-i} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & 0 & 0 & 0 & \\ \ddots & \epsilon & 1 & 0 & 0 & \ddots \\ \ddots & \epsilon^2 & \epsilon & 1 & 0 & \ddots \\ & \epsilon^3 & \epsilon^2 & \epsilon & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

which is a matrix of the deformation class. Let $\tilde{A}$ be the right-normalized infinite wedge representation of A, and let $\tilde{H}$ be the left-normalized infinite wedge representation of H. Put

$$b_n := (-1)^n A_{I_n, I_0}, \quad c_n := A_{J_0, J_n}$$

where

$$I_n := \mathbb{Z} \setminus \mathbb{N} \setminus \{0\} \setminus \{n\}, \quad J_n := \mathbb{Z} \setminus \mathbb{N} \setminus \{0\} \cup \{n\}$$

for all nonnegative integers n. Note that $b_n = 0$ for all $n \gg 0$. One has

$$\langle \bullet | \tilde{A}\tilde{H} | \bullet \rangle = \sum_{n=0}^{N-1} c_n \epsilon^n \quad (c_n \in k_0).$$

Abusing notation in what is we hope an understandable fashion, one has

$$\begin{array}{ccc} \begin{bmatrix} \dots & b_2 & b_1 & b_0 & 0 & \dots \end{bmatrix} A & = & \begin{bmatrix} \dots & 0 & c_0 & c_1 & c_2 & \dots \end{bmatrix} \\ \uparrow \text{ position } 0 \text{ at } b_0 & & \uparrow \text{ position } 0 \text{ at } c_0 \end{array}$$

This is the essence of the Kyoto school method for recovering the Baker function from the τ-function. See [**Segal-Wilson 1985**] for background.

Example 18.18. Let $A \in \mathcal{Q}$ of the deformation class and $B \in \mathcal{Q}$ of the test class be given. Choose any positive integer N such that the

$$\left[\begin{array}{c} \{i \leq -N\} \\ \{-N < i \leq N\} \\ \{N < i\} \end{array}\right] \times \left[\begin{array}{c} \{j \leq -N\} \\ \{-N < j \leq N\} \\ \{N < j\} \end{array}\right]^T$$

block decomposition of $B \in \mathcal{Q}$ takes the form

$$B = \left[\begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{array}\right]$$

where $b_{11} - 1$ and $b_{33} - 1$ are strictly upper triangular and put

$$V := \left[\begin{array}{ccc} b_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b_{33} \end{array}\right].$$

Suppose that the

$$\left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right] \times \left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right]^T$$

block decomposition of A take the form

$$A = \left[\begin{array}{cc} a_{11} & 0 \\ a_{21} & a_{22} \end{array}\right],$$

and put

$$U := \left[\begin{array}{cc} a_{11} & 0 \\ 0 & a_{22} \end{array}\right].$$

Then the matrix $U^{-1}V^{-1}BA$ belongs to the test class.

CHAPTER 2

Commuting differential operators and τ-functions

In chapter (except in the very last section) the scalar ring k is assumed to be a field of characteristic 0.

1. Commuting differential operators

Put

$$K := k((x)), \quad D := \frac{d}{dx}.$$

Let $K[D]$ denote the ring of differential operators. The typical element of $K[D]$ can be written in the form

$$\sum_{i=0}^{n} a_i D^i \quad (a_i \in K, \quad a_n \neq 0)$$

and multiplication of such expressions is effected by repeatedly applying the *Leibniz rule*

$$D^n a = \sum_i \left(\begin{array}{c} n \\ i \end{array} \right) a^{(i)} D^{n-i}$$

from freshman calculus.

The problem that interests us is that of constructing commuting pairs

$$L = D^p + \sum_{i=0}^{p-1} a_i D^i, \quad M = D^q + \sum_{j=0}^{q-1} b_j D^j$$

of differential operators where $(p, q) = 1$. This problem was originally investigated in the 1920's by Burchnall, Chaundy and Baker, later and independently in the 1970's by Krichever. See [**Mumford 1978**] for background.

EXAMPLE 1.1. Consider the case $p = 2$ and $q = 3$. Write

$$L = D^2 - 2u, \quad M = D^3 - 3aD - 3b/2 \quad (u, a, b \in K)$$

One has

$$\begin{aligned} LM &= D^5 - 3a''D - 6a'D^2 - 3aD^3 - 3b''/2 - 3b'D \\ &\quad -3bD^2/2 - 2uD^3 + 6auD + 3bu \\ &= D^5 + (-3a - 2u)D^3 + (-6a' - 3b/2)D^2 \\ &\quad +(-3a'' - 3b' + 6au)D + (-3b''/2 + 3bu), \\ ML &= D^5 - 2u''' - 6u''D - 6u'D^2 - 2uD^3 - 3aD^3 + \\ &\quad 6au' + 6auD - 3bD^2/2 + 3bu \\ &= D^5 + (-2u - 3a)D^3 + (-6u' - 3b/2)D^2 \\ &\quad +(-6u'' + 6au)D + (-2u''' + 6au' + 3bu) \end{aligned}$$

and hence

$$LM = ML \Leftrightarrow \begin{cases} u' = a' \\ b' = a'' \\ a''' = 12aa' \quad \leftarrow \text{stationary Korteweg de Vries.} \end{cases}$$

The stationary Korteweg de Vries equation has the Weierstrass $\wp$-function as a solution, as well as the rational function $1/x^2$.

2. Dressing operators

The Leibniz rule makes sense also for negative values of n and thus one can equip the set of expressions

$$\sum_{i=-\infty}^{n} a_i D^i \quad (a_i \in K, \quad a_n \neq 0)$$

with the structure of associative k-algebra; this larger ring is denoted $K((D^{-1}))$ and its elements are called *pseudo-differential operators*. Let $k((D))^{-1}$ be the subring consisting of pseudo-differential operators with constant coefficients. Let $K[[D^{-1}]]$ be the subring of pseudo differential operators of order ≤ 0 and let $k[[D^{-1}]]$ be the subring consisting of such with constant coefficients.

EXAMPLE 2.1. Each pseudo-differential operator

$$\Psi = 1 + \sum_{i=1}^{\infty} c_i D^{-i} \quad (c_i \in K)$$

of degree zero with leading coefficient 1 is invertible. One has

$$\begin{aligned}
\Psi D \Psi^{-1} &= & D &+O(D^{-1}) \\
\Psi D^2 \Psi^{-1} &= & D^2 - 2c_1' &+O(D^{-1}) \\
\Psi D^3 \Psi^{-1} &= & D^3 - 3c_1' D + 3c_1' c_1 - 3c_1'' - 3c_2' &+O(D^{-1})
\end{aligned}$$

Elements of $K[D]$ of the form

$$\text{“differential operator part” of } \Psi D^n \Psi^{-1}$$

have an extremely important role to play in the theory of the Kadomtsev-Petviashvili (KP) hierarchy. See [**Segal-Wilson 1985**] for background.

A superior reformulation of the original “commuting differential operators” problem is as follows. Let

$$A \subseteq k((D))^{-1}$$

be a k-subalgebra such that

$$A \cap k[[D^{-1}]] = k, \quad \dim_k \frac{k((D^{-1}))}{A + k[[D^{-1}]]} < \infty.$$

Such a k-algebra A is the coordinate ring of an affine curve over k with a unique nonsingular k-rational point at infinity. We ask for a pseudodifferential operator

$$\Psi = 1 + \sum_{i=1}^{\infty} c_i D^{-i} \quad (c_i \in K)$$

of order zero such that

$$\Psi A \Psi^{-1} \subseteq K[D].$$

The operator Ψ is called a *dressing operator* for the ring A. From the dressing operator Ψ and constant coefficient pseudodifferential operators

$$f = D^p + O(D^{p-1}), \quad g = D^q + O(D^{q-1})$$

of relatively prime order one obtains commuting differential operators

$$\Psi f \Psi^{-1} = D^p + O(D^{p-1}), \quad \Psi g \Psi^{-1} = D^q + O(D^{q-1})$$

of the sort we originally set out to study. Hereafter we focus our attention on the construction of dressing operators.

EXAMPLE 2.2. Let $A \subset k((D^{-1}))$ be a k-subalgebra generated by constant coefficient pseudo-differential operators

$$f = D^2 + O(D^{-1}), \quad g = D^3 + O(D^{-1})$$

for which there exists a dressing operator

$$\Psi = 1 + \sum_{i=1}^{\infty} c_i D^{-i} \quad (c_i \in K).$$

One has

$$\Psi f \Psi^{-1} = D^2 - 2c_1', \quad \Psi g \Psi^{-1} = D^3 - 3c_1' D + 3c_1' c_1 - 3c_1'' - 3c_2'$$

by Example 2.1 and hence

$$c_1'''' = 12 c_1'' c_1'$$

by Example 1.1, i. e., c_1' is in this situation a solution of the stationary Korteweg de Vries equation.

3. Baker functions

Let $K\{\{z\}\}$ be the ring of series

$$\sum_{i \in \mathbb{Z}} c_i z^i \quad (c_i \in K)$$

where c_i is x-adically bounded and $c_i \to 0$ as $i \to -\infty$. In a more or less obvious way this is an ultrametric Banach algebra over K. The function

$$e^{xz^{-1}} = \sum_{n=0}^{\infty} \frac{x^n z^{-n}}{n!}$$

lives in $K\{\{z\}\}$. The Laurent series field $k((z))$ lives in $K\{\{z\}\}$.

Now fix a k-subspace

$$W \subset k((z))$$

such that

$$\dim_k W \cap k[[z^{-1}]] = \dim_k \frac{k((z))}{W + k[[z^{-1}]]} < \infty;$$

in this situation we say that W is of *index zero*. The collection of such subspaces is the famous *Sato Grassmannian*. Put

$$\tilde{W} := \Big(\text{closure of } e^{-xz^{-1}}(K \otimes W) \text{ in } K\{\{z\}\}\Big) \cap K((z))$$

where $K \otimes W$ denotes the K-span in $K\{\{z\}\}$ of W. Facts:

- The natural map
$$\tilde{W} \to \frac{K((z))}{K[[z^{-1}]]}$$
is bijective.

- One has
$$\frac{\partial}{\partial x}\left(e^{xz^{-1}}\tilde{W}\right) \subseteq e^{xz^{-1}}\tilde{W},$$
and the rule
$$\left(\sum_i a_i D^i\right) w := \sum_i a_i \frac{\partial^i w}{\partial x^i}$$
makes $e^{xz^{-1}}\tilde{W}$ into a free left $K[D]$-module of rank 1.

In particular, there exists unique
$$\psi_W := 1 + \sum_{i>0} c_i z^i \in z\tilde{W}.$$
The function ψ_W is called the *Baker function* associated to the index zero subspace $W \subset k((z))$. Put
$$\Psi_W = 1 + \sum_{i>0} c_i D^{-i} \in K[[D^{-1}]].$$
It follows that
$$\Psi_W \left(\sum_i a_i D^{-i}\right) \Psi_W^{-1} \in K[D] \Leftrightarrow \left(\sum_i a_i z^i\right) W \subseteq W$$
for all $\sum_i a_i z^i \in k((z))$. In particular, if there exist functions
$$f = \sum_i a_i z^i = z^p + O(z^{p-1}), \quad g = \sum_j b_j z^j = z^q + O(z^{q-1})$$
in $k((z))$ of relatively prime order such that
$$fW, gW \subseteq W,$$
the corresponding differential operators
$$\Psi\left(\sum_i a_i D^{-i}\right)\Psi^{-1} = D^p + O(D^{p-1}), \quad \Psi\left(\sum_j b_j D^{-j}\right)\Psi^{-1} = D^q + O(D^{q-1})$$
commute and are of the form we originally interested ourselves in. Thus the highly nonlinear problem of constructing of commuting differential operators reduces to the essentially linear problem of constructing Baker functions.

4. "Bare-handed" construction of Baker functions

We continue in the setting of the previous section. Put

$$\mathcal{H} := \left(\begin{array}{l} \text{the space of column vectors} \\ h \text{ with entries } h_i \text{ in } k \text{ indexed} \\ \text{by } \mathbb{Z} \text{ vanishing for } i \gg 0. \end{array} \right).$$

Identify $k((z))$ with $\mathcal{H}$ by the rule

$$\sum_i a_i z^i \leftrightarrow \begin{bmatrix} \vdots \\ a_2 \\ a_1 \\ a_0 \\ a_{-1} \\ a_{-2} \\ \vdots \end{bmatrix} \leftarrow \text{position } 0.$$

There exists a k-basis

$$\{w_n\}_{n=1}^{\infty}$$

of W, an integer N and a partition

$$\lambda : \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0 = \lambda_{\ell+1} \geq \dots$$

such that

$$w_n = z^{-n+\lambda_n} + O(z^{-n+\lambda_n+1}) = z^{-n} + O(z^N)$$

for all n; this follows from the index zero condition. We arrange the sequence $w_1, w_2, \dots$ into a $\mathbb{Z}$ by $\mathbb{N}$ matrix with entries in k which we denote again by W. The

$$\begin{bmatrix} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{bmatrix} \times \mathbb{N}$$

block decomposition of W takes the form

$$W = \begin{bmatrix} P \\ 1 + Q \end{bmatrix}$$

where Q is an $\mathbb{N}$ by $\mathbb{N}$ matrix with only finitely many nonvanishing rows.

EXAMPLE 4.1. Let $A \subset k((z))$ be the k-algebra generated by functions

$$f = z^{-2}, \quad g = z^{-3}(1 - (g_2/4)z^4 - (g_3/4)z^6)^{1/2}.$$

If

$$g_2^3 - 27g_3^2 \neq 0,$$

the k-algebra A is a copy of the affine coordinate ring of the elliptic curve

$$E: \quad Y^2 = X^3 - g_2X/4 - g_3/4$$

defined over k. There exists a unique k-basis $\{a_n\}_{n=1}^{\infty}$ of A such that

$$a_n = \begin{cases} 1 & \text{if } n = 1 \\ z^{-n} + O(z) & \text{if } n > 1 \end{cases}$$

for all $n \in \mathbb{N}$. Let A denote also the $\mathbb{Z}$ by $\mathbb{N}$ matrix obtained by arranging the vectors $\{a_n\}_{n=1}^{\infty}$ in the manner described above. The matrix

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{128}\,{g_2}^2 & 0 & -\frac{1}{64}\,g_2\,g_3 & 0 & -\frac{1}{512}\,{g_2}^3 - \frac{1}{128}\,{g_3}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{8}g_3 & 0 & -\frac{1}{128}\,{g_2}^2 & 0 & -\frac{1}{32}\,g_2\,g_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{8}g_2 & 0 & -\frac{1}{8}g_3 & 0 & -\frac{3}{128}\,{g_2}^2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{8}g_2 & 0 & -\frac{1}{8}g_3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

is the $\{-6, \ldots, 7\} \times \{1, \ldots, 7\}$ block of A.

Let $\mathbf{t}$ be the $\mathbb{Z}$ by $\mathbb{Z}$ matrix with 1's along the superdiagonal $j-i=1$ and 0's elsewhere. One has

$$\mathbf{t} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 0 & 1 & 0 & 0 & \\ \ddots & 0 & 0 & 1 & 0 & \ddots \\ \ddots & 0 & 0 & 0 & 1 & \ddots \\ & 0 & 0 & 0 & 0 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$\nwarrow$ main diagonal

and

$$e^{-x\mathbf{t}^{-1}} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & 0 & 0 & 0 & \\ \ddots & -x & 1 & 0 & 0 & \ddots \\ \ddots & x^2/2 & -x & 1 & 0 & \ddots \\ & -x^3/6 & x^2/2 & -x & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$\nwarrow$ main diagonal.

The problem of constructing the Baker function

$$\psi_W = 1 + \sum_{i=1}^{\infty} c_i z^i$$

comes down to constructing sequences

$$\{b_i\}_{i=1}^{\infty}, \quad \{c_i\}_{i=1}^{\infty}$$

in K tending x-adically to 0 such that

$$e^{-x\mathbf{t}^{-1}}W\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ c_3 \\ c_2 \\ c_1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \leftarrow \text{position } 1$$

Now we are going to cut to the chase. Put

$$\tau_W(x,z) = \lim_{N\to\infty} \det_{i,j=1}^{N} (e^{x\mathbf{t}^{-1}}(1 - z\mathbf{t}^{-1})^{-1}W)_{ij}$$

where

$$(1 - z\mathbf{t}^{-1}) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & 0 & 0 & 0 & \\ \ddots & z & 1 & 0 & 0 & \ddots \\ \ddots & z^2 & z & 1 & 0 & \ddots \\ & z^3 & z^2 & z & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$\nwarrow$

main diagonal.

The limit exists (x,z)-adically in $k[[x,z]]$. If one replaces W by another matrix representing the same index zero subspace of $k((z))$, $\tau_W(x,z)$ is multiplied by a nonzero constant factor. Put

$$\tau_W(x) := \tau_W(x,0).$$

The power series $\tau_W(x)$ is the determinant of the system of linear equations we are trying to solve. The great discovery of the Kyoto school is that

$$\tau_W(x) \propto x^{\sum_i \lambda_i}(1 + O(x)), \quad \psi_W = \frac{\tau_W(x,z)}{\tau_W(x)}, \quad c_1 = -\frac{d}{dx}\log \tau_W(x).$$

The order of vanishing of $\tau_W(x)$ was (in a different language) also calculated by Fay; for background see [**Segal-Wilson 1985**].

EXAMPLE 4.2. As in Example 4.1, assume that $A \subset k((z))$ is generated as a k-algebra by functions

$$f = z^{-2}, \quad g = z^{-3}(1 - (g_2/4)z^4 - (g_3/4)z^6)^{1/2}.$$

Let A be the matrix constructed in Example 4.1. One has

$$\tau_A(x) = -\left(x - \frac{1}{240}\, g_2\, x^5 - \frac{1}{840}\, g_3\, x^7\right) + O(x^8)$$

and

$$c_1' = -\frac{d^2}{dx^2}\log \tau_A(x) = \frac{1}{x^2} + \frac{g_2 x^2}{20} + \frac{g_3 x^4}{28} + O(x^5)$$

Since c_1' satisfies the stationary Korteweg-deVries equation by Example 2.2, it follows that

$$(c_1'')^2 = 4(c_1')^3 - g_2 c_1' - g_3.$$

Finally, if

$$g_2^3 - 27 g_3^2 \neq 0,$$

then A is the affine coordinate ring of the elliptic curve

$$E:\ y^2 = x^3 - g_2 x/4 - g_3/4$$

and

$$c_1'(x) = \wp(x),$$

where $\wp$ is the Weierstrass $\wp$-function attached to E.

5. Complements in characteristic $p > 0$

Versions of the preceding game can be played in characteristic p and also p-adically. I attempted to play it in the papers [**Anderson 1994a**] and [**Anderson 1994b**]; of the latter paper I will not speak here. Assume now that k is a finite field of q elements, but otherwise leave everything the same: fix a subspace $W \subset k((z))$, basis $\{w_n\}_{n=1}^{\infty}$ and partition λ as above. Consider the matrix

$$E(x, y, z) := (1 - z\mathbf{t})^{-1} \prod_{i=0}^{\infty} (1 - y^{q^i}\mathbf{t}^{-1})(1 - x^{q^i}\mathbf{t}^{-1})^{-1}$$

The limit

$$\tau_W(x, y, z) := \lim_{N\to\infty} \det_{i,j=1}^{N} (E(x, y, z)W)_{1-i,1-j}$$

exists (x, y, z)-adically in $k[[x, y, z]]$. One has

$$\tau_W(x, y, 0) \sim \prod_{(i,j)\in\lambda} (x^{q^i} - y^{q^j})$$

where $\sim$ means that the left side is equal to the right side times a power series in x and y with constant term 1, and $(i, j) \in \lambda$ means that i and j are positive integers such that $j \leq \lambda_i$; in particular,

$$\tau_W(x, 0, 0) \neq 0.$$

Now suppose that $A \subseteq k((z))$ is a k-subalgebra such that

$$A \cap k[[z]] = k, \quad \dim_k \frac{k((z))}{A + k[[z]]} < \infty.$$

Suppose that W is a rank one projective A-module. Then the quotient

$$\psi_W(x, z) = \frac{\tau_W(x, 0, z)}{\tau_W(x, 0, 0)}$$

admits interpretation as the Baker function associated to a rank one elliptic A-module. See [**Anderson 1994a**] for background and details.

CHAPTER 3

Reciprocity

We work exclusively with the infinite wedge model of fermionic Fock space in this chapter. As usual, k denotes an artinian local ring and m the maximal ideal thereof. The main result of this section is Theorem 3.12.

1. Commutator calculations

DEFINITION 1.1. Given $A, B \in \mathcal{Q}^\times$ such that $AB = BA$, there exists a unique invertible scalar $\{A, B\}$ such that

$$\tilde{B}\tilde{A} = \{A, B\}\tilde{A}\tilde{B}$$

for all infinite wedge representations $\tilde{A}$ and $\tilde{B}$ of A and B, respectively.

EXAMPLE 1.2. For all $A, B, C \in \mathcal{Q}^\times$ the following assertions hold:

- $\{A, A\} = 1$.
- If $AB = BA$, then $\{A, B\} = \{B, A\}^{-1}$.
- If $AC = CA$ and $BC = CB$, then $\{A, C\}\{B, C\} = \{AB, C\}$.
- If $AB = BA$, then $\{A, B\} = \{CAC^{-1}, CBC^{-1}\}$.

EXAMPLE 1.3. Let $A, B \in \mathcal{Q}^\times$ be commuting matrices. If the

$$\left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right] \times \left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right]^T$$

block decompositions of A, A^{-1}, B and B^{-1} are

$$\text{all of the form } \left[\begin{array}{cc} * & * \\ 0 & * \end{array}\right], \quad \text{or all of the form } \left[\begin{array}{cc} * & 0 \\ * & * \end{array}\right],$$

then $\{A, B\} = 1$.

EXAMPLE 1.4. Let $\omega : \mathbb{Z} \to \mathbb{Z}$ be a tame one-to-one map. One has

$$\{A, B\} = \{\omega_* A, \omega_* B\}$$

for all commuting matrices $A, B \in \mathcal{Q}^\times$. Example 15.5 of Chapter 1 can be exploited to give a proof of this fact.

EXAMPLE 1.5. Let $\omega, \eta : \mathbb{Z} \to \mathbb{Z}$ be tame one-to-one maps with disjoint images. One has

$$\{\omega_* A, \eta_* B\} = (-1)^{\deg A \cdot \deg B}$$

for all $A, B \in \mathcal{Q}^\times$. Again Example 15.5 of Chapter 1 can be exploited to give a proof of this fact.

EXAMPLE 1.6. Let

$$f = \sum_i a_i t^i \in 1 + t^{-1} m[t^{-1}], \quad g = \sum_j b_j t^j \in 1 + tk[[t]]$$

be given. Put

$$A := f(\mathbf{t}) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & 0 & 0 & 0 & \\ \ddots & a_{-1} & 1 & 0 & 0 & \ddots \\ \ddots & a_{-2} & a_{-1} & 1 & 0 & \ddots \\ & a_{-3} & a_{-2} & a_{-1} & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathcal{Q}^\times$$

and

$$B := g(\mathbf{t}) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & 1 & b_1 & b_2 & b_3 & \\ \ddots & 0 & 1 & b_1 & b_2 & \ddots \\ \ddots & 0 & 0 & 1 & b_1 & \ddots \\ & 0 & 0 & 0 & 1 & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathcal{Q}^\times.$$

One has

$$\{A, B\} = \frac{\langle \bullet | \tilde{B}\tilde{A} | \bullet \rangle}{\langle \bullet | \tilde{A}\tilde{B} | \bullet \rangle} = \det_{i,j=1}^{n} (BA)_{1-i,1-j} = \det_{i,j=1}^{n} (A^{-1}B^{-1})_{ij}$$

for all integers $n \gg 0$ by Example 18.8. Also by Examples 18.16 and ? one has

$$\{A, B\} = \sum_\lambda \left(\det_{i,j=1}^{\ell(\lambda)} a_{i-\lambda_i-j} \right) \left(\det_{i,j=1}^{\ell(\lambda)} b_{\lambda_i-i+j} \right)$$

where the sum is

EXAMPLE 1.7. Fix scalars x and y. Assume that x is nilpotent. Put

$$A \quad := \quad 1 - x\mathbf{t}^{-1} \quad = \quad \begin{bmatrix} \ddots & & & & \\ \ddots & 1 & & & \\ & -x & 1 & & \\ & & -x & 1 & \\ & & & \ddots & \ddots \end{bmatrix} \in \mathcal{Q}^{\times},$$

$$B \quad := \quad 1 - y\mathbf{t} \quad = \quad \begin{bmatrix} \ddots & \ddots & & & \\ & 1 & -y & & \\ & & 1 & -y & \\ & & & 1 & \ddots \\ & & & & \ddots \end{bmatrix} \in \mathcal{Q}^{\times}.$$

One has

$$\underbrace{\begin{vmatrix} 1+xy & -y & & & \\ -x & 1+xy & -y & & \\ & -x & 1+xy & \ddots & \\ & & \ddots & \ddots & -y \\ & & & -x & 1+xy \end{vmatrix}}_{n} = 1 + xy + \cdots + (xy)^n.$$

We conclude that

$$\{A, B\} = \{1 - x\mathbf{t}^{-1}, 1 - y\mathbf{t}\} = (1 - xy)^{-1}$$

via Example 1.6.

EXAMPLE 1.8. Fix scalars $x, y \in k$. Assume that x is nilpotent. By a calculation similar to that presented in Example 1.7, one can verify that

$$\{1 - x\mathbf{t}^{-p}, 1 - y\mathbf{t}^{q}\} = (1 - x^{q/(p,q)}y^{p/(p,q)})^{-(p,q)}$$

for all positive integers p and q, where (p, q) denotes the greatest common divisor of p and q. It is also possible to deduce this formula by factoring $1 - xt^{-p}$ and $1 - yt^{q}$ in $k'((t))$ where k'/k is a suitably constructed finite flat k-algebra.

Example 1.9. Let $f \in k[[t]]^\times$ be given. Put

$$A := f(\mathbf{t}), \quad B := \mathbf{t}.$$

We claim that

$$\{A, B\} = c := \text{constant term of } f.$$

By Example 18.2 of Chapter 1, there exist unique infinite wedge representations $\tilde{A}$ and $\tilde{B}$ of A and B, respectively, such that

$$\tilde{A}|0\rangle = |0\rangle, \quad \tilde{B}|0\rangle = |{-1}\rangle = e_1^\flat|0\rangle,$$

where e_n denotes the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix. One has

$$\tilde{B}\tilde{A}|0\rangle = \tilde{B}|0\rangle = e_1^\flat|0\rangle = c\tilde{A}e_1^\flat|0\rangle = c\tilde{A}\tilde{B}|0\rangle.$$

The claim is proved.

Example 1.10. Let $f \in k[t]^\times$ be given. Put

$$A := f(\mathbf{t}), \quad B := \mathbf{t}.$$

We claim that

$$\{A, B\} = c := \text{constant term of } f.$$

By Example 18.2 of Chapter 1, there exist unique infinite wedge representations $\tilde{A}$ and $\tilde{B}$ of A and B, respectively, such that

$$\langle 0|\tilde{A} = \langle 0|, \quad \langle 0|\tilde{B} = \langle 1| = \langle 0|e_0^\flat$$

where e_n denotes the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix. One has

$$\langle 0|\tilde{B}\tilde{A} = \langle 0|e_0^\flat\tilde{A} = c\langle 0|\tilde{A}e_0^\flat = c\langle 0|\tilde{A}\tilde{B}.$$

The claim is proved.

Example 1.11. Let

$$f \in 1 + t^{-1}m[t^{-1}]$$

be given. By Example 18.8 there exists a positive integer N such that one has

$$\{f(\mathbf{t}), g(\mathbf{t})\} = 1$$

for all $g \in 1 + t^N k[[t]]$.

2. The Contou-Carrère symbol

DEFINITION 2.1. Given $f, g \in k((t))^\times$, put

$$\{f, g\} = (-1)^{w(f)w(g)} \frac{a_0^{w(g)}}{b_0^{w(f)}} \frac{\prod_{i=1}^\infty \prod_{j=1}^\infty \left(1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)}\right)^{(i,j)}}{\prod_{i=1}^\infty \prod_{j=1}^\infty \left(1 - a_{-i}^{j/(i,j)} b_j^{i/(i,j)}\right)^{(i,j)}},$$

where $\{a_i\}$ and $\{b_j\}$ are the systems of Witt parameters associated to f and g, respectively. All but finitely many of the terms appearing in the products differ from 1 and thus $\{f, g\}$ is a well defined invertible scalar. The function

$$\{\cdot, \cdot\} : k((t))^\times \times k((t))^\times \to k^\times$$

is an elementary version of the *Contou-Carrère symbol* [**Contou-Carrère 1994**].

EXAMPLE 2.2. By combining all the calculations carried out above, one finds that

$$\{\omega_* f(\mathbf{t}), \omega_* g(\mathbf{t})\} = (-1)^{w(f)w(g)}\{f, g\}$$

for all $f, g \in k((t))^\times$ and any tame one-to-one map $\omega : \mathbb{Z} \to \mathbb{Z}$. One also has

$$\{\omega_* f(\mathbf{t}), \eta_* g(\mathbf{t})\} = (-1)^{w(f)w(g)}$$

for all $f, g \in k((t))^\times$ and any tame one-to-one maps $\omega, \eta : \mathbb{Z} \to \mathbb{Z}$ with disjoint images.

EXAMPLE 2.3. One has

$$\begin{aligned} \{f, f\} &= (-1)^{w(f)} \\ \{f, g\} &= \{g, f\}^{-1} \\ \{fg, h\} &= \{f, h\}\{g, h\} \end{aligned}$$

for all $f, g, h \in k((t))^\times$.

EXAMPLE 2.4. If k is a field, the Contou-Carrère symbol coincides with the tame symbol.

EXAMPLE 2.5. If $k = k_0[\epsilon]$ where k_0 is a field and $\epsilon^3 = 0$, one has

$$\{1 - \epsilon f, 1 - \epsilon g\} = 1 - \epsilon^2 \operatorname{Res}(f' g dt)$$

for all $f, g \in k_0((t))$. Thus one recovers the residue from the Contou-Carrère symbol.

EXAMPLE 2.6. We claim that the Contou-Carrère symbol satisfies the *adjunction formula*

$$\{\mathcal{N}_\phi f, g\} = \{f, g \circ \phi\}$$

for all $f, g, \phi \in k((t))^\times$ such that ϕ has a positive winding number. In particular, the Contou-Carrère symbol is reparameterization invariant. Let n be the winding number of ϕ. By Example 4.27, there exists $A \in \mathcal{Q}^\times$ such that

$$\phi(\mathbf{t})A = A\mathbf{t}^n.$$

By definition (see Example 4.30) one has

$$(\mathcal{N}_\phi f)(\mathbf{t}) = \det((A^{-1}f(\mathbf{t})A)^{[n]}), \quad (g \circ \phi)(\mathbf{t}) = Ag(\mathbf{t}^n)A^{-1}.$$

By Example 8.8 one has a factorization

$$A^{-1}f(\mathbf{t})A = ((\ell \mapsto n\ell)_*(\mathcal{N}_\phi f)(\mathbf{t}))C$$

where C is an element of the commutator subgroup of the subgroup of $\mathcal{Q}^\times$ consisting of matrices commuting with $\mathbf{t}^n$. Exploiting all the previous calculations, we have

$$\begin{aligned}
& (-1)^{w(f)w(g\circ\phi)}\{f, g \circ \phi\} \\
= \quad & \{f(\mathbf{t}), (g \circ \phi)(\mathbf{t})\} \\
= \quad & \{f(\mathbf{t}), A(g(\mathbf{t}^n))A^{-1}\} \\
= \quad & \{A^{-1}f(\mathbf{t})A, g(\mathbf{t}^n)\} \\
= \quad & \left\{(\ell \mapsto n\ell)_*(\mathcal{N}_\phi f)(\mathbf{t}), \prod_{i=1}^{n}(\ell \mapsto n\ell + 1 - i)_* g(\mathbf{t})\right\} \\
= \quad & (-1)^{w(\mathcal{N}_\phi f)nw(g)}\{\mathcal{N}_\phi f, g\}.
\end{aligned}$$

Finally, the signs cancel. The claim is proved.

3. An adelic formalism and a reciprocity law

Under this heading we fix a map

$$\tau : \mathbb{Z} \to \mathbb{Z}$$

with the following properties:

- τ is bijective.
- $\tau(n) > n$ for all $n \in \mathbb{Z}$.
- For each integer n_0 maximal among the nonpositive elements of its τ-orbit one has
$$\tau^{1-i}(n_0) = 1 - \tau^i(n_0)$$
for all $i \in \mathbb{Z}$.

We call such a map τ a *multi-Toeplitz structure*. All the constructions under this heading depend on the choice of τ, but for brevity's sake we usually suppress reference to τ in the notation. We usually refer to τ-orbits as *standard places*.

Example 3.1. For each positive integer N, the map

$$(n \mapsto n + N) : \mathbb{Z} \to \mathbb{Z}$$

is a multi-Toeplitz structure with N orbits.

Example 3.2. For each $n \in \mathbb{Z}$, let $e(n)$ be the largest power of 2 dividing $\max(n, 1 - n)$. Put

$$\alpha(n) := \begin{cases} n + 2e(n) & \text{if } n \neq 1 - e(n) \\ e(n) & \text{if } n = 1 - e(n) \end{cases}$$

for all $n \in \mathbb{Z}$, thereby defining a map $\alpha : \mathbb{Z} \to \mathbb{Z}$ that turns out to be a multi-Toeplitz structure. Here's the orbit structure of α:

$$\begin{array}{ccccccccccccccc}
\dots & \to & -4 & \to & -2 & \to & 0 & \to & 1 & \to & 3 & \to & 5 & \to & \dots \\
\dots & \to & -9 & \to & -5 & \to & -1 & \to & 2 & \to & 6 & \to & 10 & \to & \dots \\
\dots & \to & -19 & \to & -11 & \to & -3 & \to & 4 & \to & 12 & \to & 20 & \to & \dots \\
 & & & & & & \vdots & & & & & & & &
\end{array}$$

The sequence $\{1 - 2^n\}_{n=0}^{\infty}$ meets each α-orbit in $\mathbb{Z}$ exactly once.

DEFINITION 3.3. We say that a matrix $A \in \mathcal{Q}$ is *multi-Toeplitz* under the following conditions:

- $A_{\tau(i),\tau(j)} = A_{ij}$ for all $i, j \in \mathbb{Z}$.
- $A_{ij} = 0$ for all $i, j \in \mathbb{Z}$ belonging to distinct τ-orbits.

We define the *universal adele ring* $\mathbb{A} \subset \mathcal{Q}$ to be the k-subalgebra consisting of multi-Toeplitz matrices. We define $\mathcal{O} \subset \mathbb{A}$ to be the k-subalgebra of consisting of upper triangular matrices. Every element of $\mathbb{A}$ is fixed by the dagger involution and consequently $\mathbb{A}$ is a commutative k-algebra.

DEFINITION 3.4. Given $A \in \mathbb{A}$ and a standard place P there exists a unique Laurent series

$$A_P = \sum_n a_n t^n \in k((t))$$

such that

$$A_{i,\tau^n(i)} = a_n$$

for all $i \in P$ and $n \in \mathbb{Z}$. The map

$$(A \mapsto A_P) : \mathbb{A} \to k((t))$$

is a surjective k-algebra homomorphism. For each $A \in \mathbb{A}$ and standard place P, one has $A = 0$ (resp. $A \in \mathcal{O}$) if and only $A_P = 0$ (resp. $A_P \in k[[t]]$) for all standard places P. For each collection $\{f_P\}$ of elements of $k((t))$ indexed by standard places P, there exists $A \in \mathbb{A}$ such that $A_P = f_P$ for all standard places P if and only if $f_P \in k[[t]]$ for all but finitely many standard places P.

EXAMPLE 3.5. One has

$$\deg f = \sum_P w(f_P)$$

for all $f \in \mathbb{A}^\times$ where the sum is extended over standard places P.

DEFINITION 3.6. Let $\mathbf{e} \in \mathcal{H}$ be defined by the rule

$$\mathbf{e}_n := \begin{cases} 1 & \text{if } n = \max\left(\{\tau^i(n) \mid i \in \mathbb{Z}\} \setminus \mathbb{N}\right) \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{Z}$. The maps

$$(f \mapsto f\mathbf{e}) : \mathbb{A} \to \mathcal{H}, \quad (f \mapsto f\mathbf{e}) : \mathcal{O} \to \mathcal{H}(\mathbb{Z} \setminus \mathbb{N})$$

are bijective. Put

$$\mathrm{Res} := \left(\sum_i a_i t^i \mapsto a_{-1} \right) : k((t)) \to k.$$

One has

$$\sum_{n \in \mathbb{Z}} (f\mathbf{e})_n (g\mathbf{e})_{1-n} = \sum_P \mathrm{Res}(f_P g_P)$$

for all $f, g \in \mathbb{A}$. We introduce the abbreviated notation

$$f^\sharp := (f\mathbf{e})^\sharp, \quad g^\flat := (g\mathbf{e})^\flat$$

for all $f, g \in \mathbb{A}$. One then has

$$(f^\sharp)^2 = 0, \quad (g^\flat)^2 = 0, \quad f^\sharp g^\flat + g^\flat f^\sharp = \sum_P \mathrm{Res}(f_P g_P)$$

for all $f, g \in \mathbb{A}$. For each $a \in \mathbb{A}^\times$, the corresponding infinite wedge representations $\tilde{a}$ of a are characterized by the identities

$$\tilde{a}(f^\sharp \psi) = (af)^\sharp (\tilde{a}\psi), \quad \tilde{a}(g^\flat \psi) = (a^{-1}g)^\flat (\tilde{a}\psi)$$

for all $f, g \in \mathbb{A}$ and $\psi \in \mathcal{F}$.

DEFINITION 3.7. For any infinite subset $P \subseteq \mathbb{Z}$ such that $P = 1 - P$, e. g., any standard place P, there exists a unique strictly increasing bijective map $[P] : \mathbb{Z} \to P$ such that

$$[P](1 - n) = 1 - [P](n)$$

for all $n \in \mathbb{Z}$.

EXAMPLE 3.8. Fix $f \in \mathbb{A}^\times$ and let $P_1, \ldots, P_n$ be any finite collection distinct standard places such that $f_P \in k[[t]]^\times$ for all $P \notin \{P_1, \ldots, P_n\}$. Then one has a unique factorization of the form

$$f = [Q]_* A \cdot [P_1]_* f_1(\mathbf{t}) \cdots [P_n]_* f_n(\mathbf{t})$$

where $f_1, \ldots, f_n \in k((t))^\times$, $Q := \mathbb{Z} \setminus (P_1 \cup \cdots \cup P_n)$, and $A \in \mathcal{Q}^\times$ is a matrix such that both A and A^{-1} are upper triangular.

EXAMPLE 3.9. Let $f, g \in \mathbb{A}^\times$ be given. We claim that

$$\{f, g\} = (-1)^{\deg f \cdot \deg g} \prod_P \{f_P, g_P\}$$

where the product is extended over all standard places. The claim is verified by factoring f and g in the fashion described in Example 3.8 and calculating according to the rules worked out above.

EXAMPLE 3.10. Let $f, g \in \mathbb{A}^\times$ and a matrix $\Omega \in \mathcal{Q}^\times$ be given such that the $\mathbb{Z} \setminus \mathbb{N}$ by $\mathbb{N}$ blocks of the matrices

$$\Omega^{-1} f \Omega, \quad \Omega^{-1} f^{-1} \Omega, \quad \Omega^{-1} g \Omega, \quad \Omega^{-1} g^{-1} \Omega$$

vanish. Then

$$1 = \prod_P \{f_P, g_P\} \in k^\times$$

where the product is extended over all standard places P. Again this is verified by a straightforward calculation based upon the rules worked out above.

EXAMPLE 3.11. Let $K \subset \mathbb{A}$ be a flat k-subalgebra such that the k-modules $K \cap \mathcal{O}$ and $\mathbb{A}/(K + \mathcal{O})$ are finitely generated. By Example 4.4 of Chapter 1 and the hypothesis, one has

$$\mathcal{H} = K\mathbf{e} \oplus \mathcal{H}(I)$$

for some wedge index $I \subset \mathbb{Z}$. It follows that for each $n \in \mathbb{Z} \setminus I$ there exists unique $f^{(n)} \in K$ such that

$$f^{(n)}\mathbf{e} - e_n \in \mathcal{H}(I),$$

where e_n denotes the n^{th} column of the $\mathbb{Z}$ by $\mathbb{Z}$ identity matrix. Let $A \in \mathcal{Q}$ be defined by the rule

$$Ae_n = \begin{cases} f^{(n)}\mathbf{e} & \text{if } n \in \mathbb{Z} \setminus I \\ e_n & \text{if } n \in I \end{cases}$$

for all $n \in \mathbb{Z}$. By construction the

$$\left[\begin{array}{c} I \\ \mathbb{Z} \setminus I \end{array} \right] \times \left[\begin{array}{c} I \\ \mathbb{Z} \setminus I \end{array} \right]^T$$

block decomposition of the matrix A is of the form

$$A = \left[\begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right],$$

and hence $A \in \mathcal{Q}^\times$. Choose now a tame permutation $\omega : \mathbb{Z} \to \mathbb{Z}$ such that $\omega(\mathbb{N}) = \mathbb{Z} \setminus I$ and let W be the element of the big Weyl group representing ω. Put

$$\Omega := AW.$$

The matrix Ω the following properties:

- $\Omega\mathcal{H}(\mathbb{N}) = K\mathbf{e}$.
- For each $f \in K$ the $\mathbb{Z} \setminus \mathbb{N}$ by $\mathbb{N}$ block of $\Omega^{-1} f \Omega$ vanishes.
- For each $b \in \mathbb{A}^\times$, the $\mathbb{N}$ by $\mathbb{N}$ block of $b^{-1}\Omega$ is invertible if and only if $\mathbb{A} = b\mathcal{O} \oplus K$.

We call Ω a *Riemann matrix* for K.

THEOREM 3.12. *Let $K \subset \mathbb{A}$ be a flat k-subalgebra such that the k-modules $K \cap \mathcal{O}$ and $\mathbb{A}/(K + \mathcal{O})$ are finitely generated. Then for all $f, g \in K^\times$ one has $\prod_P \{f_P, g_P\} = 1$.*

PROOF. There is nothing left to prove. □

4. Recovery of some classical reciprocity laws

We assume under this heading that k is a countable algebraically closed field and we suppose that a multi-Toeplitz structure with infinitely many orbits has been fixed.

DEFINITION 4.1. A finitely generated field extension K/k of transcendence degree 1 will be called a *function field*.

DEFINITION 4.2. An *adelization* of a function field K/k is a k-algebra embedding

$$f \mapsto f^\iota : K \to \mathbb{A}$$

such that

$$\mathbb{A} = a\mathcal{O} \oplus K^\iota$$

for some $a \in \mathbb{A}^\times$.

DEFINITION 4.3. Let K/k be a function field. A function

$$v : K \to \mathbb{Z} \coprod \{+\infty\}$$

is called a *normalized additive valuation* if it has the following properties:

$$\begin{array}{rcl} v^{-1}(+\infty) & = & \{0\} \\ v(k^\times) & = & \{0\} \\ v(fg) & = & v(f) + v(g) \\ v(f+g) & \geq & \min(v(f), v(g)) \end{array}$$

Under our hypothesis that k is countable, the set of normalized additive valuations of k is a countably infinite set.

Example 4.4. Let K/k be a function field. Let $\iota : K \to \mathbb{A}$ be a k-algebra embedding. Then ι is an adelization if and only if the following two conditions hold:

- For each standard place P, the map
$$(f \mapsto w(f_P^\iota)) : K \to \mathbb{Z} \coprod \{+\infty\}$$
is a normalized additive valuation of K/k.

- The map $P \mapsto (f \mapsto w(f_P^\iota))$ puts the standard places in bijective correspondence with the normalized additive valuations of K/k.

Since we are assuming that k is a countable algebraically closed field, every function field K/k can be adelized.

Example 4.5. Let K/k be a function field. Let $\iota_1, \iota_2 : K \to \mathbb{A}$ be adelizations. Then there exists a matrix $A \in \mathcal{Q}^\times$ such that the
$$\left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right] \times \left[\begin{array}{c} \mathbb{Z} \setminus \mathbb{N} \\ \mathbb{N} \end{array}\right]^T$$
block decompositions of A and A^{-1} take the form
$$\left[\begin{array}{cc} * & * \\ 0 & * \end{array}\right]$$
and
$$Af^{\iota_1}A^{-1} = f^{\iota_2}$$
for all $f \in K$. Thus all adelizations of K/k are conjugate.

Example 4.6. Let K/k be a function field equipped with an adelization ι. Let $k[\epsilon]$ be a (commutative) k-algebra generated by a finite collection $\{\epsilon_i\}$ of nilpotent elements; then $k[\epsilon]$ is an artinian local ring. Put $K[\epsilon] := K \otimes_k k[\epsilon]$. For each standard place P, we extend the embedding $(f \mapsto f_P^\iota)$ by $k[\epsilon]$-linearity to an embedding $K[\epsilon] \to k[\epsilon]((t))$. Let $\{\cdot, \cdot\}$ denote the version of the Contou-Carrère symbol defined over $k[\epsilon]$ rather than over k. By Theorem 3.12, for any $f, g \in K[\epsilon]^\times$, one has
$$\prod_v \{f_P^\iota, g_P^\iota\} = 1 \in k[\epsilon]^\times$$
where the product is extended over all standard places P; the product is well defined because all but finitely many terms are equal to 1.

Example 4.7. When all the ϵ's vanish, Example 4.6 reduces to the Weil reciprocity law obeyed by the tame symbol; thus the general reciprocity law for the Contou-Carrère symbol can be viewed as a "deformation" of the Weil reciprocity law.

EXAMPLE 4.8. Assume that the collection $\{\epsilon_i\}$ consists of just one element, which we denote now by ϵ. Assume that $\epsilon^3 = 0$ and that $1, \epsilon, \epsilon^2$ are k-linearly independent. Assume that $f = 1 - \epsilon f_0$ and $g = 1 - \epsilon g_0$ where $f_0, g_0 \in K$. In this special case Example 4.6 reduces to the assertion that the residues of the differential $g_0 df_0$ sum to 0.

EXAMPLE 4.9. The relationship of the Contou-Carrère symbol to the ring of Witt vectors in characteristic p deserves further discussion; we would like to include such a discussion in a later draft of the notes.

CHAPTER 4

Calculation of genus 1 Jacobians

Under this heading k is assumed to be a field of characteristic 0.

1. The logarithmic derivative yoga

Let V be a vector space over k and let $\mathbf{1}$ denote the identity map $V \to V$. Let $\mathfrak{g} \subseteq \mathrm{End}_k(V)$ be a Lie k-subalgebra and let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie k-subalgebras. Assume that

$$\mathfrak{g},\ \mathfrak{a},\ \mathfrak{b} \supseteq k \cdot \mathbf{1}, \quad [\mathfrak{g}, \mathfrak{g}] \subseteq k \cdot \mathbf{1}.$$

Let $L \subset V$ be a one-dimensional k-subspace (a line). Let $H \subset V$ be a k-subspace of codimension 1 (a hyperplane). Assume that

$$\mathfrak{a}H \subseteq H, \quad \mathfrak{b}L \subseteq L.$$

Fix $0 \neq \ell \in L$ and $0 \neq h^* \in H^\perp \subseteq V^*$. For any endomorphism $X : V \to V$, put

$$e^{tX} = \sum_{n=0}^{\infty} \frac{X^n}{n!} x^n \in \mathrm{End}_k(V)[[x]].$$

For all $X, Y \in \mathfrak{g}$ one has

$$e^{x(X+Y)} = e^{-\lambda x^2/2} e^{xX} e^{xY}$$

where

$$[X, Y] = \lambda \cdot \mathbf{1}.$$

The identity is proved by verifying that both sides are equal if $x = 0$, and satisfy the same first order linear differential equation. (I thank Dennis Stanton for explaining this to me.) For each $X \in \mathfrak{g}$ put

$$\tau_X := h^* e^{xX} \ell \in k[[x]].$$

One has

$$\tau_{A+X} = e^{-\lambda x^2/2 + \alpha x} \tau_X$$

for all $A \in \mathfrak{a}$ and $X \in \mathfrak{g}$ where

$$[A, X] = \lambda \cdot \mathbf{1}, \quad A\ell = \alpha\ell \quad (\lambda, \alpha \in k).$$

One has

$$\tau_{X+B} = e^{-\mu x^2/2 + \beta x} \tau_X$$

for all $X \in \mathfrak{g}$ and $B \in \mathfrak{b}$ where

$$[X, B] = \mu \cdot \mathbf{1}, \quad h^* B = \beta h^* \quad (\mu, \beta \in k).$$

Assume now that the coset $X + \mathfrak{a} + \mathfrak{b} \in \mathfrak{g}/(\mathfrak{a} + \mathfrak{b})$ contains some Y such that $\tau_Y \neq 0$. The Laurent series

$$-\frac{d^3}{dx^3} \log \tau_X \in k((x))$$

is then well-defined, depends only on the coset $X + \mathfrak{a} + \mathfrak{b}$, and is also independent of the choice of ℓ and h^*.

2. The basic construction

Let K/k be a genus 1 function field and let ω be a nonzero k-rational everywhere regular differential of K/k. Let $\mathbb{A} = \mathbb{A}_{K/k}$ denote the adèle ring of K/k and let $\mathcal{O} = \mathcal{O}_{K/k}$ be the subring consisting of integeral ad'eles. Let $\mathcal{C} = \mathcal{C}(K/k, \omega)$ be the Clifford algebra associated to the quadratic space

$$\left(\begin{bmatrix} \mathbb{A} \\ \mathbb{A} \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} \mapsto \operatorname{Res}(fg\omega) \right)$$

and write

$$\begin{bmatrix} f \\ 0 \end{bmatrix}^\sharp = f^\sharp, \quad \begin{bmatrix} 0 \\ g \end{bmatrix}^\sharp = g^\flat$$

for all $f, g \in \mathbb{A}$. Let $\mathcal{J} = \mathcal{J}(K/k, \omega) \subset \mathcal{C}$ be the left ideal generated by $\mathcal{O}_{K/k}^\sharp + \mathcal{O}_{K/k}^\flat$. Let $\mathcal{I} = \mathcal{I}(K/k, \omega) \subset \mathcal{C}$ be the right ideal generated by $K^\sharp + K^\flat$. It can be shown that the quotient $\mathcal{C}/(\mathcal{I} + \mathcal{J})$ is one-dimensional over k and is generated by $f^\sharp$ where $f \in \mathbb{A}$ is any adele such that $\operatorname{Res}(f\omega) = 1$. Finally, put

$$\mathcal{F} = \mathcal{F}(K/k, \omega) = \mathcal{C}/\mathcal{J},$$

thereby defining a left $\mathcal{C}$-module. Put $|\bullet\rangle = 1 + \mathcal{J} \in \mathcal{F}$, and choose any nonzero k-linear function $\langle\bullet|$ on $\mathcal{F}$ killing $\mathcal{I}\mathcal{F}$. For each $X \in \mathbb{A}$ there exists a unique derivation $X^\natural$ of $\mathcal{C}$ such that

$$X^\natural f^\sharp = (Xf)^\sharp, \quad X^\natural g^\flat = -(Xg)^\flat$$

for all $f, g \in \mathbb{A}$. It can be shown that for each $X \in \mathbb{A}$ there exists a k-linear endomorphism $\tilde{X}$ of $\mathcal{F}$ well defined up to a scalar addend such that

$$\tilde{X}(x\psi) = (X^\natural x)\psi + x\tilde{X}\psi$$

for all adeles $x \in \mathcal{C}$ and $\psi \in \mathcal{F}$. Now choose any $X \in \mathbb{A}$ such that $\mathrm{Res}(X\omega) = 1$, lifting $\tilde{X}$ as above, and put

$$\tau(x) = \sum_{i=0}^{\infty} \langle \bullet | \tilde{X}^n | \bullet \rangle \frac{x^n}{n!} \in k[[x]].$$

Presently it will be explained that $\tau \neq 0$; this granted, it follows that the Laurent series

$$\wp'_{K/k,\omega} := -\frac{d^3}{dx^3} \log \tau(x) \in k((x))$$

is well defined, i. e., independent of all the choices made to define it.

The question now arises as to just what interpretation the power series $\wp'_{K/k,\omega}$ has. Let $\tilde{K}/\bar{k}$ be the base-change of the extension K/k to an algebraic closure $\bar{k}$ of k and let $\bar{\omega}$ be the differential of $\tilde{K}/\bar{k}$ obtained from ω by base-change. Let L/k be another genus one function field equipped with a nonzero k-rational everywhere regular differential η. Suppose that we can identify the extensions $\bar{L}/\bar{k}$ and $\bar{K}/\bar{k}$ in such a way that $\bar{\omega} = \bar{\eta}$. Then it is clear that $\wp'_{K/k,\omega} = \wp'_{L/k,\omega}$. If L/k happens to be an elliptic function field, one can verify by the methods developed above that $\wp'_{L/k,\eta}$ is the derivative the Weierstrass $\wp$-function associated to L/k and η. In short, calculation of $\wp'_{K/k,\omega}$ is tantamount to computing the Jacobian of the genus one curve whose function field is K/k.

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